Planar $\mathcal{N} = 4$ gauge theory and integrability

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Summary

- what is the $\mathcal{N}$=4 gauge theory?
- why $\mathcal{N}$=4 gauge theory?
- gauge/string correspondence
- Maldacena's conjecture (AdS/CFT)
- integrability and the exact solution of AdS/CFT
- conclusion

What is the $\mathcal{N}$=4 planar gauge theory?

$\mathcal{N}$= 4 gauge theory (in 4D) is a supersymmetric version of QCD

Field content
SU(N) matrices: $A_\mu, \Phi_I \ (I = 1, ..., 6), \ \Psi_\alpha(\alpha = 1, ..., 4)$ and derivatives

$$S = \int d^4x \Big(\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathrm{Tr}\, \mathcal{D}_\mu \Phi_i\, \mathcal{D}^\mu \Phi^i - \frac{g_{\mathrm{YM}}^2}{4} \mathrm{Tr}\, [\Phi_i, \Phi_j][\Phi^i, \Phi^j] +$$
$$\mathrm{Tr}\, \bar{\psi}^a\, \sigma^\mu \mathcal{D}_\mu\, \psi_a - \frac{i g_{\mathrm{YM}}}{2} \mathrm{Tr}\, \sigma_i^{ab} \psi_a [\Phi^i, \psi_b] - \frac{i g_{\mathrm{YM}}}{2} \mathrm{Tr}\, \sigma^i_{ab} \bar{\psi}^a [\Phi_i, \bar{\psi}^b] \Big)$$

it can be obtained by dimensional reduction from D=10 $\mathcal{N}$= 1 SYM

$$S = \int d^{10}x \left(\frac{1}{4} \mathrm{Tr}\, F_{MN} F^{MN} + \frac{1}{2} \mathrm{Tr}\, \psi\, \Gamma^M \mathcal{D}_M\, \psi \right)$$

planar ('t Hooft) limit: $N \to \infty$ **with** $\lambda = g_{YM}^2 N$ **fixed**

Symmetries

- the beta function is zero, presumably at all orders $\longrightarrow$ the Poincaré group gets promoted to the conformal group SO(4,2) $\cong$ SU(2,2)

the coupling constant is not running!

- there are four copies of supersymmetry generators, which are rotated into one another by the R-symmetry SO(6) $\cong$ SU(4)

- the total symmetry super-group is PSU(2,2|4)

$$\begin{pmatrix} L,\ \bar{L},\ P,\ K,\ D & Q,\ \bar{S} \\ \bar{Q},\ S & R \end{pmatrix}$$

Why the $\mathcal{N}$=4 planar gauge theory?

"more is less": it is simpler than QCD and may be exactly solvable due to the high amount of symmetry

first example of precise duality with a string theory (type IIB on $AdS_5 \times S^5$)

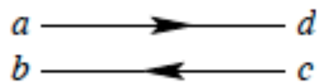
[Maldacena 97; Witten 98; Gubser, Klebanov, Polyakov 98]

although not realized in nature, it can help our understanding of strongly coupled gauge theories

String/gauge duality

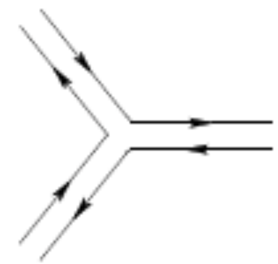
Feynman graphs for large N gauge theories ~ string expansion **['t Hooft]**

propagators:



$\sim g_{\rm YM}^2$

vertices:



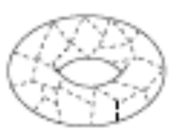
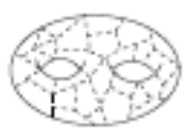
$\sim \frac{1}{g_{\rm YM}^2}$

loops: factor of N from Tr 1

diagram with E edges, V vertices and F loops:

$$N^F\,(g_{\rm YM}^2)^{E-V} = N^{V-E+F}\,(g_{\rm YM}^2 N)^{E-V}$$

genus: $\chi = V - E + F = 2 - 2g$

$\mathcal{F} = N^2$  $+ 1$  $+ \frac{1}{N^2}$  $+ \ldots = \sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \sum_{l=0}^{\infty} c_{g,l}\,\lambda^l$

string expansion: *$g_S \sim 1/N^2$*

low-dimensional example: duality between matrix models and non-critical string theories and 2d quantum gravity **[Kazakov, Migdal, Kostov,...]**

AdS/CFT correspondence

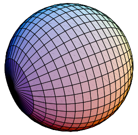
Correspondence between string theory on an AdS space and a conformal field theory on the boundary of AdS [Maldacena 97]

- context: string theory in presence of D-branes **[Polchinski; Witten; Klebanov; Horowitz, Strominger,...]**

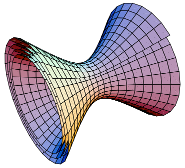
- rejoins early ideas about gauge/string correspondence **[Wilson; Polyakov; 't Hooft]**

First example: $\mathcal{N} = 4$ gauge theory in 4D is dual to type IIB string theory on $AdS_5 \times S^5$ **[Maldacena 97; Witten 98; Gubser, Klebanov, Polyakov 98]**

S^5: $$x_1^2 + x_2^2 + x_3^2 + \ldots + x_6^2 = 1$$



AdS_5: $$-y_{-1}^2 - y_0^2 + y_1^2 + y_2^2 + \ldots y_4^2 = -1$$



More recently: $\mathcal{N} = 6$ Chern-Simons theory in 3D is dual to type IIB string theory on $AdS_4 \times CP^3$ **[Aharony, Bergman, Jafferis, Maldacena, 08]**

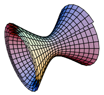
Dictionary of the correspondence

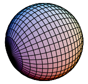
't Hooft coupling	$g^2 = \frac{g_{\mathrm{YM}}^2 N}{16\,\pi^2}$	**string tension**	$T = 2g$
number of colors	N	**string coupling**	$g_{\mathrm{s}} = \frac{g}{N}$
planar limit		free strings	
local operators	$\mathrm{Tr}\,(\Phi_{I_1}\Phi_{I_2}\ldots\Phi_{I_L})$	**string states**	

scaling dimension — $E(g)$ — energy of the string

R-charges — J_a — angular momenta

the same global symmetry PSU(2,2|4)





$E(g), S_1, S_2, J_1, J_2, J_3$

[Lipatov, 98]

Integrability

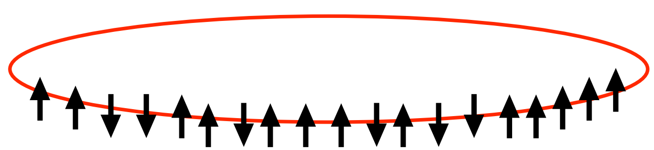
One loop dilatation operator
=
integrable spin chain

[Minahan, Zarembo, 02]

$$Z = \Phi_1 + i\Phi_2$$

$$W = \Phi_3 + i\Phi_4$$

$$\mathrm{tr}\, ZZZWWZZZWWWZWZZZZ\ldots$$



$$\hat{D}_1 = 2\sum_{l=1}^{L}(1 - P_{l,l+1})$$

solution in terms of Bethe Ansatz equations ⟷ solution of the classical sigma model in terms of an algebraic curve

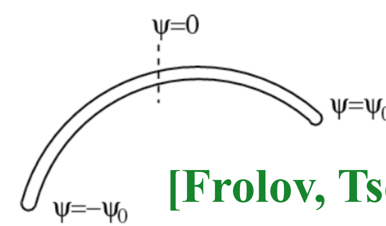
String sigma model
is
classically integrable

[Bena, Polchinski, Roiban, 02]

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \left[G^{(AdS_5)}_{mn}\,\partial_a X^m\,\partial^a X^n + G^{(S^5)}_{mn}\,\partial_a Y^m\,\partial^a Y^n\right]$$

$$+\ \text{fermions}$$

⟷ string solution, *e.g.*



[Frolov, Tseytlin, 02]

[Kazakov, Marshakov, Minahan, Zarembo, 04]

Integrability in AdS/CFT

extends to the whole PSL(2,2|4) group

[Beisert, Staudacher 03]

survives at higher loops

[Beisert, Kristjansen, Staudacher 03]
[Beisert 03-04]

↓ **[conjecture]**

There exists a model which is integrable for any value of the coupling constant g

?

spin chain at $g \to 0$

perturbative $\mathcal{N}$=4 SYM

sigma model at $g \to \infty$

perturbative string theory on AdS_5 x S^5

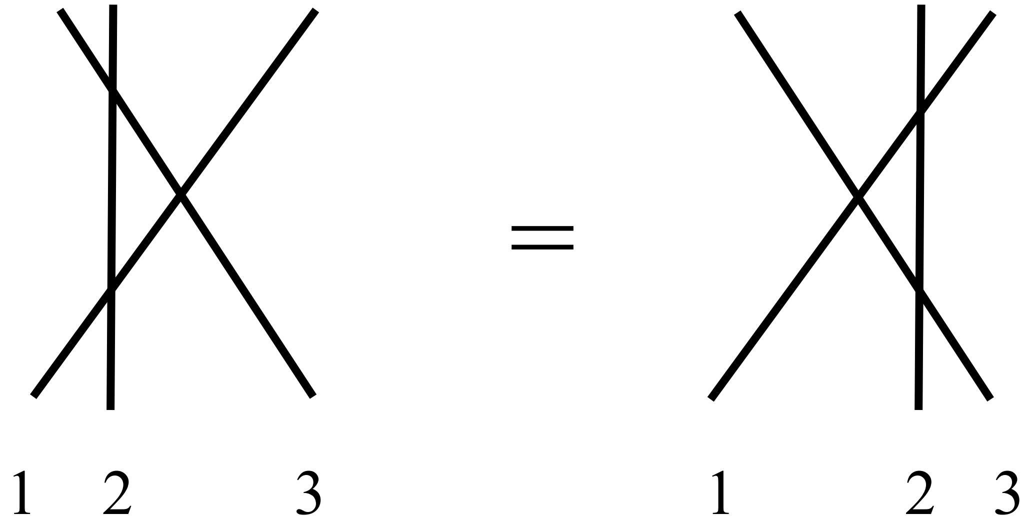
Integrability

existence of an infinite number of integrals of motion $[I_m, I_n]=0$

factorized scattering (no particle production)

Yang-Baxter equation

$$S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$$



exact solution for some 2d field theories

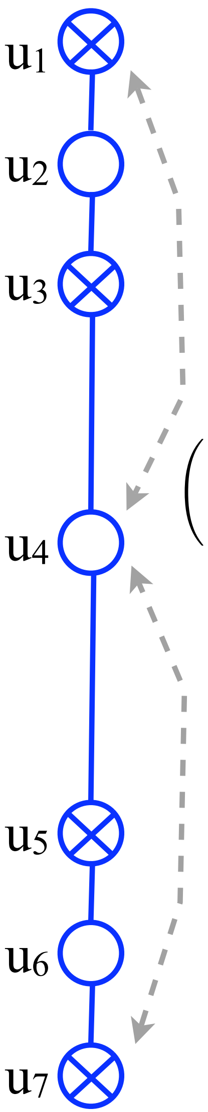
[Zamolodchikov, Zamolodchikov, 70ties]

The all-loop Bethe Ansatz equations

psu(2,2|4)

[Beisert, Staudacher, 05]
[Beisert, 05]

[Arutynov, Frolov, Zamaklar, 06]



$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k} x_{4,j}^+}{1 - 1/x_{1,k} x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-},$$

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L = \prod_{j\neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \sigma^2(x_{4,k}, x_{4,j})$$

$$\times \prod_{j=1}^{K_1} \frac{1 - 1/x_{4,k}^- x_{1,j}}{1 - 1/x_{4,k}^+ x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - 1/x_{4,k}^- x_{7,j}}{1 - 1/x_{4,k}^+ x_{7,j}},$$

Dressing factor

[Janik'06;
Beisert-Hernandez-Lopez'06;
Beisert-Eden-Staudacher'06]

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}}{u_{6,k} - u_{7,j} - \frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{7,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{7,k} x_{4,j}^+}{1 - 1/x_{7,k} x_{4,j}^-}.$$

$$x + \frac{1}{x} = \frac{u}{g}$$

$$x^\pm + \frac{1}{x^\pm} = \frac{1}{g}\left(u \pm \frac{i}{2}\right)$$

the form of these equations is almost entirely dictated by symmetry

Connection(s) with the Hubbard model

2 seemingly unrelated connections with the 1d Hubbard model

- su(2) sector reproducible from the Hubbard model at half filling (except for the dressing phase) **[Rej, Serban, Staudacher, 06]**

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}\,,$$

[Beisert, Dippel, Staudacher, 04]

$$E(p) = \sqrt{1+16g^2\sin^2\frac{p}{2}} - 1$$

- Beisert's su(2|2) symmetric **S-matrix** ~ Hubbard Shastry's **R-matrix**

$\Rightarrow$ hidden supersymmetry in the Hubbard model **[Beisert, 06]**

$$H = \frac{1}{2\,g}\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left(e^{i\phi_\sigma}\, c^\dagger_{i,\sigma}c_{i+1,\sigma} + e^{-i\phi_\sigma}\, c^\dagger_{i+1,\sigma}c_{i,\sigma}\right) - \frac{1}{2g^2}\sum_{i=1}^{L} c^\dagger_{i,\uparrow}c_{i,\uparrow}c^\dagger_{i,\downarrow}c_{i,\downarrow}$$

Checks: the anomalous dimension of the *L*-twist operator

twist *L* operators: $\mathrm{Tr}\ D^{k_1} Z \ldots D^{k_L} Z$ $\quad D = D_0 + iD_1$

described by a sl(2) spin chain $\quad k_1 + \ldots k_L = M \quad$ *M* is Lorentz spin

[Korchemsky, 89]
[Belitsky, Gorsky, Korchemsky, 06]
[Alday, Maldacena, 07]

$$\Delta = M + f(g,L) \ln M + \ldots$$

when M → ∞, BA equations become integral equations

***L* finite:** $f(g,L) = f(g)$ universal (does not depend on *L*)

The universal scaling function

the most advanced test of the AdS/CFT correspondence via integrability:

perturbatively in gauge theory:

$$f(g) = 8\,g^2 - \frac{8}{3}\,\pi^2 g^4 + \frac{88}{45}\,\pi^4 g^6 - 16\left(\frac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots .$$

[Moch, Vermaseren, Vogt, 04]
[Lipatov et al., 04]

[Bern et al, 06]
[Cachazo et al, 06]

perturbatively in string theory
at strong coupling:

$$f(g) = 4\,g - \frac{3\,\log 2}{\pi} - \frac{\mathrm{K}}{4\,\pi^2}\frac{1}{g} + \ldots$$

[Gubser, Klebanov, Polyakov, 98] → **[Klebanov et al, 06]** **[Kotikov, Lipatov, 06]** **[Alday et al. 07]** **[Kostov, D.S., Volin, 07]**

[Frolov, Tseytlin, 02] → **[Casteill, Kristjansen, 07]** **[Belitsky, 07]**

[Roiban, Tseytlin, 07] → **[Basso, Korchemsky, Kotanski, 07]** **[Kostov, D.S., Volin, 08]**

both expansions can be
reproduced from BES equation:

weak coupling: **[Beisert, Eden, Staudacher, 06]**

non-perturbative corrections at strong coupling **[Basso, Korchemsky, 08]** **[Alday, Maldacena, 07]**

Finite size corrections: the TBA program

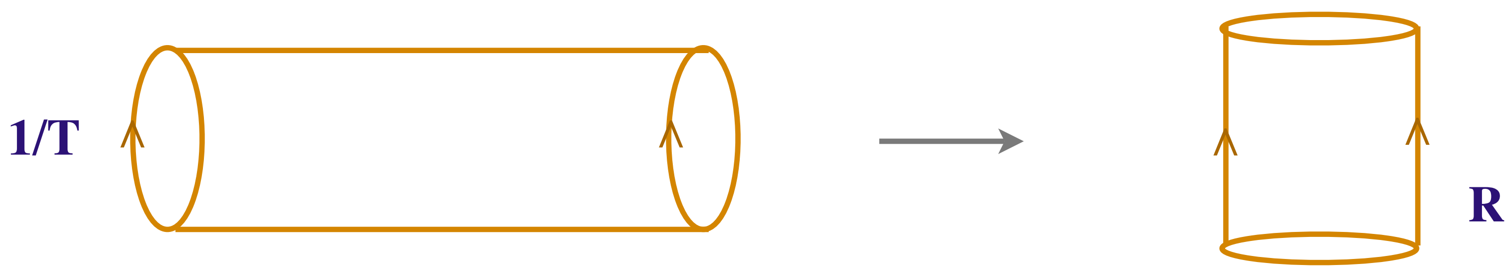
Use the field-theoretical methods to compute finite-size corrections:

[Ambjorn, Janik, Kristjansen 05]

- Lüscher terms **[Janik, Lukowski 07,...]**

- put the theory on the cylinder and make a "double Wick rotation" **$1/T \to R$**
  [Arutynov, Frolov 07; Bajnok, Janik,08]

- **difficulty**: the rotated theory is not equivalent to the original one ("mirror theory")



simplest wrapping correction: the four loop L=4 (Konishi operator)

[Fiamberti, Santambrogio, Sieg, Zanon ,08]: perturbative computation in N=4 SYM

=

[Bajnok, Janik,08]: from the Thermodynamic Bethe Ansatz

- incorporate finite size effects into the Bethe Ansatz via the Hirota equation:
[Gromov, Kazakov, Vieira, 08-09]

The origin of integrability?

There is more in $\mathcal{N}$=4 SYM than the dilatation operator...

- the multigluon amplitudes have a particular structure at higher loops - >
  BDS conjecture **[Bern, Dixon, Smirnov 05]** (fails for n>5)

- this structure was checked at strong coupling for 4 (and many) gluons
  [Alday, Maldacena 07]

- dual superconformal symmetry **[Drummond, Henn, Korchemsky, Sokatchev, 07-08]**
  (and duality between multigluon amplitudes and the Wilson loops with lightlike cusps)

[the structure of the $\mathcal{N}$=4 SYM amplitudes is a whole subject in itself]
[Witten, Cachazo, Britto, Feng, Spradlin, Volovich, Arkani-Hamed, ... 03-09]

Connection between this structure and integrability? **[Berkovits, Maldacena, 08]**
[Beisert, Ricci, Tseytlin, Wolf, 08]
[Drummond, Henn, Plefka, 09],...

Integrable open spin chain for gluon amplitudes [Lipatov, 08]

Conclusion

- the AdS/CFT correspondence can be explored in detail using integrability
- integrability constitutes an extremely powerful non-perturbative tool
- the duality implies a non-trivial “change of variables” but it is not yet clear if there are some “more fundamental” variables
- 1/N corrections may also be accessible with the integrability tools
- less supersymmetric, finite N?