Higher order integrals of motion in a gauge covariant Hamiltonian framework

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Second HEP School in Mgurele
Măgurele, Bucharest, 22-23 October 2009

Outline

Conditions for a conserved quantity (1)

Let $(\mathcal{M}, \mathbf{g})$ a N-dimensional manifold with the metric tensor $\mathbf{g}$. Classical dynamics of a point charge q of mass M in the external Abelian gauge field A_i and a scalar potential $V(x^i)$

$$H = \frac{1}{2M} g^{ij}(p_i - qA_i)(p_j - qA_j) + V \,.$$

Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} \,.$$

Hamilton equations of motion are not manifestly gauge covariant.

Conditions for a conserved quantity (2)

Gauge covariant formulation

$$\mathbf{\Pi} = \mathbf{p} - q\mathbf{A} = M\dot{\mathbf{x}}\,.$$

Hamiltonian becomes

$$H = \frac{1}{2M} g^{ij}\Pi_i \Pi_j + V\,,$$

Covariant Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial x^i} + qF_{ij}\frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial \Pi_j}\,.$$

where $F_{ij} = A_{j;i} - A_{j;i}$ is the field strength.

Conditions for a conserved quantity (3)

Fundamental Poisson brackets

$$\{x^i, x^j\} = 0\,,\ \{x^i, \Pi_j\} = \delta^i_j\,,\ \{\Pi_i, \Pi_j\} = qF_{ij}\,,$$

Momenta **Π** are not canonical.
Hamilton's equations:

$$\dot{x}^i = \{x^i, H\} = \frac{1}{M} g^{ij}\Pi_j\,,$$

$$\dot{\Pi}_i = \{\Pi_i, H\} = qF_{ij}\dot{x}^j - V_{,i}\,.$$

Conditions for a conserved quantity (5)

Conserved quantities of motion in terms of phase-space variables (x^i, Π_i)

$$K = K_0 + \sum_{n=1}^{p} \frac{1}{n!} K_n^{i_1 \cdots i_n}(x) \cdots \Pi_{i_1} \Pi_{i_n} \,,$$

Bracket

$$\{K, H\} = 0 \,.$$

vanishes.

Conditions for a conserved quantity (6)

Series of constraints:

$$K_1^i V_{,i} = 0\,,$$

$$K_{0,i} + qF_{ji} K_1^j = M K_{2i}^j V_{,j}\,.$$

$$K_n^{(i_1 \cdots i_n ; i_{n+1})} + qF_j{}^{(i_{n+1}} K_{n+1}^{i_1 \cdots i_n) j} = \frac{M}{(n+1)} K_{n+2}^{i_1 \cdots i_{n+1} j} V_{,j}$$
$$\text{for} \quad n = 1\,, \cdots (p-2)\,,$$

$$K_{p-1}^{(i_1 \cdots i_{p-1} ; i_p)} + qF_j{}^{(i_p} K_p^{i_1 \cdots i_{p-1}) j} = 0\,,$$

$$K_p^{(i_1 \cdots i_p ; i_{p+1})} = 0\,.$$

Role of Killing-Yano tensors (1)

Stäckel Killing tensor is totally symmetric

$$K_p^{(i_1\cdots i_p;i_{p+1})} = 0\,.$$

A differential p -form f is called a KY tensor if its covariant derivative $f_{\mu_1\cdots\mu_p;\lambda}$ is totally antisymmetric.

$$f_{\mu_1\cdots(\mu_p;\lambda)} = 0\,.$$

These two generalization of Killing vectors could be related. Let $f_{\mu_1\cdots\mu_p}$ be a KY tensor, then the tensor field

$$K_{2\mu\nu} = f_{\mu\mu_2\cdots\mu_p} f^{\mu_2\cdots\mu_p}{}_{\nu}\,,$$

is a Stäckel-Killing tensor associated with Killing-Yano tensor f.

Role of Killing-Yano tensors (2)

In pseudo-classical spinning particles models the condition of the electromagnetic field $F_{\mu\nu}$ to maintain the non generic supersymmetry associated with a KY tensor f of rank p is

$$F_{\nu[\mu_p} f_{\mu_1 \cdots \mu_{p-1}]}{}^{\nu} = 0\,,$$

Consequences of this condition for the series of constraints
Assume that the Stäckel-Killing tensor $K_{2\mu\nu}$ is associated with a Killing-Yano tensor $f_{\mu\nu}$

$$K_{2\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\ \nu}\,.$$

In this case, condition for the electromagnetic field $F_{\mu\nu}$ reads

$$F_{\lambda[\mu} f_{\nu]}{}^{\lambda} = 0\,.$$

Role of Killing-Yano tensors (3)

We get

$$F_j{}^{i_2} K_2^{i_1 j} = 0\,.$$

Therefore Killing-Yano tensors prove to produce significant simplifications in the series of constraints for the higher order integrals of motion.

Examples (1)

Consider $\mathcal{M}$ to be a 3-dimensional Euclidean space $\mathbb{E}^3$

We investigate the constant of motion in a Kepler-Coulomb potential adding different types of electric and magnetic fields

We consider the motion of a point charge q of mass M in the Coulomb potential Q/r produce by a charge Q when some external electric or magnetic fields are also present.

Non relativistic Kepler-Coulomb problem admits two vector constants of motion

- angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{\Pi}\,,$$

- Runge-Lenz vector

$$\mathbf{K} = \mathbf{\Pi} \times \mathbf{L} + MqQ\frac{\mathbf{r}}{r}\,.$$

Electric charge q moves in the Coulomb potential with a constant electric field $\mathbf{E}$ present.

Hamiltonian:

$$H = \frac{1}{2M}\mathbf{\Pi}^2 + q\frac{Q}{r} - q\mathbf{E}\cdot\mathbf{r}\,,$$

with $\mathbf{\Pi} = M\dot{\mathbf{r}}$ in spherical coordinates of $\mathbb{E}^3$.

Looking for a constant of motion of the form

$$K = K_0 + K_{1i}\Pi_i + \frac{1}{2}K_{2ij}\Pi_i\Pi_j \,.$$

Components K_{2ij} are Stäckel-Killing tensors, of rank $p = 2$

$$K_{2ij} = 2\delta_{ij}\mathbf{n}\cdot\mathbf{r} - (n_i r_j + n_j n_i)\,,$$

written in spherical coordinates with $\mathbf{n}$ an arbitrary constant vector.

Choose $\mathbf{n}$ along $\mathbf{E}$

Examples (4)
I. Constant electric field

Solution of the series of constraints for a first integral of motion

$$K_0 = \frac{MqQ}{r}\mathbf{E}\cdot\mathbf{r} - \frac{Mq}{2}\mathbf{E}\cdot[\mathbf{r}\times(\mathbf{r}\times\mathbf{E})]\,.$$

$$\mathbf{K}_1 = \mathbf{r}\times\mathbf{E}\,,$$

modulo an arbitrary constant factor. This vector $\mathbf{K}_1$ contribute to a conserved quantity with a term proportional to the angular momentum $\mathbf{L}$ along the direction of the electric field $\mathbf{E}$. In conclusion, when a uniform constant electric field is present, the KC system admits two constants of motion $\mathbf{L}\cdot\mathbf{E}$ and $\mathbf{C}\cdot\mathbf{E}$ where $\mathbf{C}$ is a generalization of the Runge-Lenz vector

$$\mathbf{C} = \mathbf{K} - \frac{Mq}{2}\mathbf{r}\times(\mathbf{r}\times\mathbf{E})\,.$$

Spherically symmetric magnetic field

$$\mathbf{B} = f(r)\mathbf{r}\,,$$

$$F_{ij} = \epsilon_{ijk} B_k = \epsilon_{ijk} r_k f(r)\,,$$

+ Coulomb potential acting on a electric charge q.
Start with a Stäckel-Killing K_{2ij} of rank 2 as in in the previous example.
From the hierarchy of constraints we get

$$K_{1i} = q\left[\int rf(r)dr\right](\mathbf{n}\times\mathbf{r})_i\,,$$

Examples (6)

II. Spherically symmetric magnetic field

Equation for K_0 can be solely solved making choice of a definite form for the function $f(r)$

$$f(r) = \frac{g}{r^{5/2}} \,,$$

with g a constant connected with the strength of the magnetic field.

With this special form of the function $f(r)$ we get

$$K_0 = \left[\frac{MqQ}{r} - \frac{2g^2q^2}{r} \right] (\mathbf{n} \cdot \mathbf{r}) \,,$$

and

$$K_{1i} = -\frac{2gq}{r^{1/2}} (\mathbf{r} \times \mathbf{n})_i \,.$$

Collecting the terms K_0, K_{1i}, K_{2ij} the constant of motion becomes

$$K = \mathbf{n} \cdot \left(\mathbf{K} + \frac{2gq}{r^{1/2}} \mathbf{L} - 2g^2 q^2 \frac{\mathbf{r}}{r} \right) ,$$

with $\mathbf{n}$ an arbitrary constant unit vector and $\mathbf{K}, \mathbf{L}$ as in the pure Coulomb problem.

Magnetic field along a fixed direction $\mathbf{n}$

$$\mathbf{B} = B(\mathbf{r} \cdot \mathbf{n})\mathbf{n}\,,$$

where, for the beginning, $B(\mathbf{r} \cdot \mathbf{n})$ is an arbitrary function. Again start with a Stäckel-Killing K_{2ij} of rank 2 and we get

$$K_{1i} = q\left[\int rB(\mathbf{r} \cdot \mathbf{n})d(\mathbf{r} \cdot \mathbf{n})\right](\mathbf{r} \times \mathbf{n})_i\,.$$

Equation for K_0 proves to be solvable for a particular form of the magnetic field

$$\mathbf{B} = \frac{\alpha}{\sqrt{\alpha\mathbf{r} \cdot \mathbf{n} + \beta}}\,\mathbf{n}\,,$$

with α, β two arbitrary constants.

Finally we get for K_0 and K_{1i}

$$K_0 = \frac{MqQ}{r}(\mathbf{r}\cdot\mathbf{n}) + \alpha q^2 (\mathbf{r}\times\mathbf{n})^2 \,,$$

$$K_{1i} = -2q\sqrt{\alpha \mathbf{r}\cdot\mathbf{n} + \beta}\,(\mathbf{r}\times\mathbf{n})_i \,.$$

Constant of motion for this configuration of the magnetic field superposed on the Coulomb potential becomes:

$$K = \mathbf{n}\cdot\left[\mathbf{K} + 2q\sqrt{\alpha \mathbf{r}\cdot\mathbf{n} + \beta}\;\mathbf{L}\right] + \alpha q^2 (\mathbf{r}\times\mathbf{n})^2 \,.$$

As in the previous example the angular momentum $\mathbf{L}$ is no longer conserved, forming part of the constant of motion K .

Outlook

- Non-Abelian dynamics
- N-dimensional curved spaces
- Higher order Killing tensors (rank ≥ 3)
- .....

References

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