Algorithmic advances in NSPT

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Based on work done in collaboration with
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Motivation

Why perturbation theory?

Renormalization

$$\textbf{Ex.:}\ \ O(q) \overset{q\to\infty}{\approx} \alpha_{\overline{\mathrm{MS}}}(q) + k_1\,\alpha_{\overline{\mathrm{MS}}}(q)^2 + k_2\,\alpha_{\overline{\mathrm{MS}}}(q)^3 + \dots$$

$$\approx 10\% \qquad \approx 1\%$$

but also

- quark-masses
- effective electro-weak Hamiltonians
- small-flow time expansions
- quasi-PDF
- ...

Improvement

$$\textbf{Ex.:}\ \ \tilde{g}_0^2 = g_0^2[1 + b_{\mathrm{g}}(g_0)am_q], \quad b_{\mathrm{g}} = \mathrm{O}(g_0^2)$$

and in general

$$O^{\mathrm{lat}}(q,a) - O(q) \overset{a\to 0}{\approx} g_0\,\delta_1(a,q) + g_0^2\,\delta_2(a,q) + \dots$$

Motivation

Need for automation

Perturbative **lattice** calculations are particularly **difficult** and human **time consuming** ...

Why?

- Feynman rules can be very complicated
- In gauge-theories new vertices appear at each order
  $\Rightarrow$ the number of diagrams grows very rapidly
- Requires numerical evaluation even for simple diagrams
  $\Rightarrow$ naive computational cost $\propto V_{\mathrm{4d}}^{N}$ @ N loops

A possible **solution** ...

Numerical stochastic perturbation theory (NSPT)

✓ Fully automated

✓ Easy to set-up and very flexible tool

✓ Allows for high-order computations

✗ Systematic and statistical errors

Introduction

The Parisi and Wu's way

(Parisi, Wu '81; Zwanziger '81; Batrouni et. al. '85; Zinn-Justin '86; Damgaard, Hüffel '87; Zinn-Justin, Zwanziger '88)

Ex.: lattice φ^4-theory

$$S(\varphi) = \sum_x \left\{ \frac{1}{2}\varphi(x)\Delta\varphi(x) + \frac{g}{4!}\varphi(x)^4 \right\}, \quad \Delta = (-\partial_\mu^*\partial_\mu + m^2) > 0$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathrm{d}\varphi\, \mathcal{O}(\varphi)\, e^{-S(\varphi)}$$

Langevin equation

$$\partial_t \phi(t,x) = -\frac{\delta S(\phi)}{\delta \phi(t,x)} + \eta(t,x)$$

$$\frac{\delta S(\phi)}{\delta \phi(t,x)} = \Delta\phi(t,x) + \frac{g}{3!}\phi(t,x)^3, \quad \langle \eta(t,x)\eta(s,y)\rangle_\eta = 2\delta_{xy}\delta(t-s)$$

Stochastic quantization

$$\lim_{t\to\infty} \langle \phi(t,x_1)\ldots\phi(t,x_n)\rangle_\eta = \langle \varphi(x_1)\ldots\varphi(x_n)\rangle$$

$$\lim_{t\to\infty} P(t,\phi) = P_{\mathrm{eq}}(\phi) \propto e^{-S(\phi)}$$

Introduction

Stochastic perturbation theory

(Parisi, Wu '81; Damgaard, Hüffel '87)

Perturbative field

$$\phi = \phi_0 + g\,\phi_1 + g^2\,\phi_2 + \ldots$$

Forces

$$\frac{\delta S(\phi)}{\delta\phi} = F_0(\phi_0) + g\,F_1(\phi_0,\phi_1) + \ldots$$

Dynamics

$$\partial_t\phi_0 = -\Delta\phi_0 + \eta$$

$$\partial_t\phi_1 = -\Delta\phi_1 - \tfrac{1}{3!}\phi_0^3$$

$$\ldots$$

$$\partial_t\phi_r = -F_r(\phi_0,\ldots,\phi_r) + \delta_{r0}\,\eta \qquad [F_r(\phi) = \Delta\phi_r + V_r(\phi_0,\ldots,\phi_{r-1})]$$

Observables

$$\mathcal{O}(\phi) = \mathcal{O}_0(\phi_0) + g\,\mathcal{O}_1(\phi_0,\phi_1) + \ldots$$ **Ex.:** $\phi^2 = \phi_0^2 + g\,2\phi_0\phi_1 + \ldots$

$$\lim_{t\to\infty}\langle\mathcal{O}_r(\phi_0,\ldots,\phi_r)\rangle_\eta = k_r^{\mathcal{O}}$$ where $\langle\mathcal{O}\rangle = k_0^{\mathcal{O}} + g\,k_1^{\mathcal{O}} + \ldots$

Numerical stochastic perturbation theory

Go numerical! (Di Renzo et. al. '94; Di Renzo, Scorzato '04)

Computerize

$$t \to t_n = n\varepsilon, \quad n \in \mathbb{N} \quad \text{and} \quad r = 0, 1, \ldots, N$$

Discrete dynamics

Ex.:

$$\phi_0(t_{n+1}) = -\varepsilon \Delta \phi_0(t_n) + \sqrt{\varepsilon}\, \eta(t_n)$$

$$\phi_1(t_{n+1}) = -\varepsilon \Delta \phi_1(t_n) - \frac{\varepsilon}{3!} \phi_0(t_n)^3$$

$$\ldots$$

$$\phi_r(t_{n+1}) = -\varepsilon F_r(\phi_0(t_n), \ldots, \phi_r(t_n)) + \delta_{r0} \sqrt{\varepsilon}\, \eta(t_n)$$

Order-by-order ops.

$$\phi = \{\phi_0, \ldots, \phi_N\} \quad \text{Ex.:} \quad \phi + \chi \to \phi_r + \chi_r, \quad \phi^2 \to \phi_r^2 = \sum_{s=0}^{r} \phi_{r-s} \phi_s$$

Stochastic estimates

$$\overline{\mathcal{O}}_r = \frac{1}{N_{\text{cnfg}}} \sum_{n=0}^{N_{\text{cnfg}}} \mathcal{O}_r(\phi_0(t_n), \ldots, \phi_r(t_n)) \quad \text{Ex.:} \quad \phi^2 = \left\{ \phi_0^2,\ 2\phi_0\phi_1,\ \ldots \right\}$$

Langevin based NSPT

Some known (and proven) facts

[Statistical errors]

$$\sigma(\overline{\mathcal{O}}_r)^2 = N_{\rm cnfg}^{-1} \times \tau_{\rm int}(\mathcal{O}_r) \times {\rm var}(\mathcal{O}_r) \qquad [\mathcal{O} \equiv \text{multiplicatively ren.}]$$

- **Autocorrelations** (Zinn-Justin '86; Zinn-Justin, Zwanziger '88)

  $\tau_{\rm int}(\mathcal{O}_r) \overset{a\to 0}{\propto} a^{-2}$

- **Variances** (Lüscher '15)

  ${\rm var}(\mathcal{O}_r) = \overline{\mathcal{O}_r^2} - (\overline{\mathcal{O}_r})^2 \overset{a\to 0}{\propto} \ln(a)$

  IMPORTANT: ${\rm var}(\mathcal{O}_r) \neq 2r$-order of $\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2 \quad [r \neq 0]$

  $\Rightarrow$ they are **NOT** given by the theory **ALONE**!

[Systematic errors] (Batrouni et. al. '85; Kronfeld '93)

- $\lim_{N_{\rm cnfg}\to\infty} \overline{\mathcal{O}}_r = k_r^{\mathcal{O}} + {\rm O}(\varepsilon^p) \qquad [p \equiv \text{order integration scheme}]$

  IMPORTANT: NO accept-reject step: **NOT** analytic in g!

  $\Rightarrow$ inexact algorithm and step-size errors

Stochastic molecular dynamics algorithm

The Horowitz's way (Horowitz '85, '87, '91; Jansen, Liu '95; Lüscher, Schaefer '11)

Stochastic molecular dynamics (SMD)

$$\partial_t \phi(t,x) = \pi(t,x)$$

$$\partial_t \pi(t,x) = -\frac{\delta S(\phi)}{\delta \phi(t,x)} - \gamma \pi(t,x) + \eta(t,x)$$

$$\langle \eta(t,x)\, \eta(s,y) \rangle_\eta = 2\gamma \delta_{xy} \delta(t-s)$$

- Adjustable "friction" parameter $\gamma > 0$
- Coincides with Langevin equation for $\gamma \to \infty$
- Once perturbatively expanded, if $\gamma = \text{const.}$: (MDB, Lüscher '17; MDB, Garofalo, Kennedy '17)
  - **Autocorrelations:** $\tau_{\text{int}}(\mathcal{O}_r) \overset{a\to 0}{\approx} c_{\mathcal{O}_r}(\gamma)\, a^{-2}$
  - **Variances:** $\text{var}(\mathcal{O}_r) \overset{a\to 0}{\approx} c'_{\mathcal{O}_r}(\gamma) \ln(a)$

Phase-space stochastic quantization

$$\lim_{t\to\infty} \langle \phi(t,x_1) \ldots \phi(t,x_n) \rangle_\eta = \langle \varphi(x_1) \ldots \varphi(x_n) \rangle$$

$$P_{\text{eq}}(\pi,\phi) \propto e^{-H(\pi,\phi)}, \quad H(\pi,\phi) = \frac{1}{2} \sum_x \pi(x)^2 + S(\phi)$$

SMD based NSPT

The algorithm in a nutshell (MDB, Garofalo, Kennedy '15, '17; MDB, Lüscher '16, '17)

Start:

$$t \to t_n = n\epsilon,\ n \in \mathbb{N}, \quad \phi = \sum_{r=0}^{N} g^r \phi_r, \quad \pi = \sum_{r=0}^{N} g^r \pi_r$$

Step 1: Momentum rotation

$$\pi \to c_1\, \pi + c_2\, v$$

$$c_1 = e^{-\gamma\epsilon}, \quad c_2 = (1 - c_1^2)^{1/2}, \quad \langle v(x) v(y) \rangle_v = \delta_{xy} \quad (v = v_0)$$

Step 2: Molecular dynamics step

$$\left.\begin{array}{l} I_{\pi,h} : \pi \to \pi - hF(\phi) \\ I_{\phi,h} : \phi \to \phi + h\,\pi \end{array}\right\}$$

Symplectic reversible int.

$$t_n \to t_{n+1}$$

Ex. LPF: $I_\epsilon = I_{\pi,\epsilon/2}\, I_{\phi,\epsilon}\, I_{\pi,\epsilon/2}$

Step 1 $\to$ Step 2 $\to$ Step 1 $\to \ldots \to$ Step 2

Step 3: Measure $$\overline{\mathcal{O}}_r = \frac{1}{N_{\text{cnfg}}} \sum_{n=0}^{N_{\text{cnfg}}} \mathcal{O}_r(\phi_0(t_n), \ldots, \phi_r(t_n))$$

SMD based NSPT

Convergence to a stationary distribution

(MDB, Lüscher '17)

The discrete stochastic processes $\{\phi_r(t_n), \pi_r(t_n)\}$, $r = 0, 1, \ldots$ all converge for $t_n \to \infty$ to a **unique** and **stationary** distribution **iff**:

1. $\Delta > 0$
2. $\epsilon^2 ||\Delta|| < \kappa$

$$S_0 = \frac{1}{2} \sum_x \phi_0(x) \Delta \phi_0(x)$$

$$||\Delta|| \equiv \lambda_{\max} \approx 16$$

Leading-order distros $[r = 0]$

$$P_{\rm eq}(\pi_0, \phi_0) \propto e^{-\hat{H}_0(\pi_0, \phi_0)}$$

$$\hat{H}_0(\pi_0, \phi_0) = \frac{1}{2} \sum_x \pi_0(x)^2 + \hat{S}_0(\phi_0), \qquad \hat{S}_0 = \frac{1}{2} \sum_x \phi_0(x) \hat{\Delta} \phi_0(x)$$

- **LPF:** $\hat{\Delta} = \Delta(1 - \frac{1}{4}\epsilon^2 \Delta)$ $\qquad \hat{\Delta} > 0 \quad \Rightarrow \quad \kappa = 4$
- **OMF2:** $\hat{\Delta} = \Delta(1 + a_1\, \epsilon^2 \Delta + \mathrm{O}(\epsilon^4))$ $\qquad a_1 \approx -2.46 \times 10^{-3}, \quad \kappa = 6.51$
- **OMF4:** $\hat{\Delta} = \Delta(1 + a_2\, \epsilon^4 \Delta^2 + \mathrm{O}(\epsilon^6))$ $\qquad a_2 \approx -2.58 \times 10^{-5}, \quad \kappa = 9.87$

(Omelyan, Mryglod, Folk '03)

N.B.: $||\hat{\Delta}_{\rm LPF}/\Delta|| \approx 25 ||\hat{\Delta}_{\rm OMF2}/\Delta|| \approx 300 ||\hat{\Delta}_{\rm OMF4}/\Delta||$ [@ fixed cost per MDU]

NSPT in lattice QCD

SU(3) Yang-Mills theory

(MDB, Lüscher '17)

SMD equations

$$\partial_t U_t(x,\mu) = g_0 \pi_t(x,\mu) U_t(x,\mu)$$

$$\partial_t \pi_t(x,\mu) = -g_0 (\partial^a_{x,\mu} S_G)(U_t) T^a - \gamma \pi_t(x,\mu) + \eta_t(x,\mu)$$

$$\langle \eta^a_t(x,\mu)\, \eta^b_s(y,\nu) \rangle = 2\gamma \delta^{ab} \delta_{\mu\nu} \delta_{xy} \delta(t-s)$$

Perturbative fields

$$U(x,\mu) = e^{g_0 A_\mu(x)} = 1 + \sum_{r=0}^{N} g_0^{r+1} U_r(x,\mu), \quad \pi(x,\mu) = \sum_{r=0}^{N} g_0^r \pi_r(x,\mu)$$

Step 1. Momentum rotation (very much as for ϕ^4)

Step 2. Molecular dynamics step

$$\left.\begin{aligned} I_{\pi,h} &: \pi \to \pi - h g_0 \partial S_G(U) \\ I_{U,h} &: U \to e^{h g_0 \pi} U \end{aligned}\right\}$$

Symplectic reversible int. $t_n \to t_{n+1}$

WARNING: Convergence to a unique and stationary distro. requires special care due to gauge symmetry, but it can be guaranteed! [@ BACKUP]

The gradient flow coupling

A highly non-trivial case study

Gradient flow

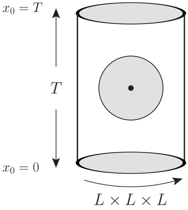
$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) \qquad [t \equiv \text{flow time}]$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad B_\mu(0,x) = A_\mu(x)$$

Schrödinger functional (Lüscher et. al. '92)

$$A_k(x)|_{x_0=0} = A_k(x)|_{x_0=T} = 0$$

$$A_\mu(x + \hat{k}L) = A_\mu(x)$$



Basic quantity (Lüscher '10; Fodor et. al. '12; Fritzsch, Ramos '13)

$$t^2\langle E(t,x)\rangle, \quad E(t,x) = \tfrac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)$$

$$T = L, \quad x_0 = T/2, \quad \sqrt{8t} = 0.3 \times L$$

NSPT observables

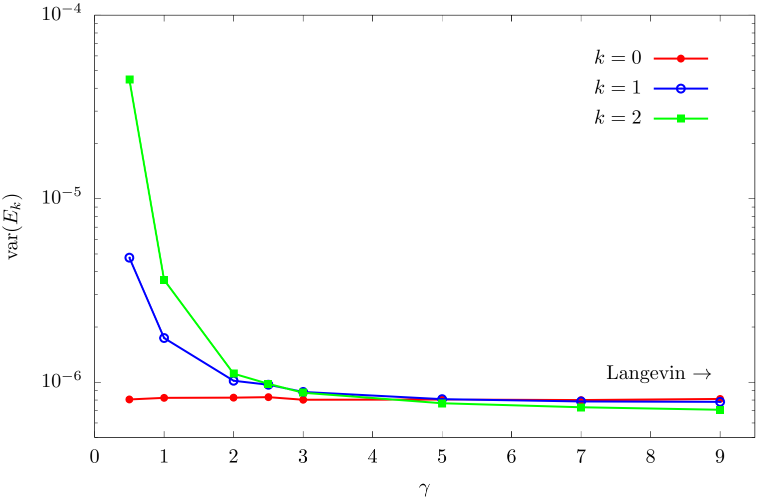
[using $\alpha_{\overline{\mathrm{MS}}} \leftrightarrow g_0$ (Lüscher, Weisz '95)]

$$t^2\langle E(t,x)\rangle = g_0^2 E_0 + g_0^4 E_1 + g_0^6 E_2 + \ldots$$

$$\overset{a/L\to 0}{=} k_0 \left\{\alpha_{\overline{\mathrm{MS}}}(q) + k_1\, \alpha_{\overline{\mathrm{MS}}}(q)^2 + k_2\, \alpha_{\overline{\mathrm{MS}}}(q)^3 + \ldots\right\}, \quad q^{-1} = \sqrt{8t}$$

Algorithm dependence of the variances

Variances $\mathrm{var}(E_k)$ **as a function of** γ **for** $L/a = 16$

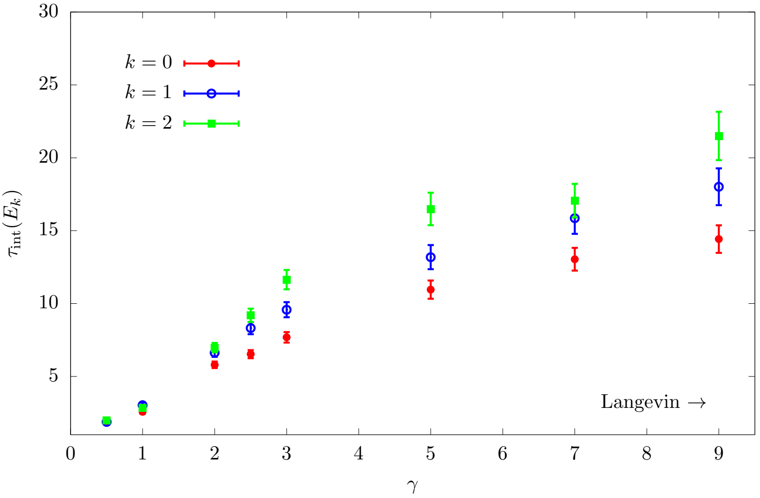


IMPORTANT:

$\mathrm{var}(E_k) \neq [\langle E^2\rangle - \langle E\rangle^2]|_{g_0^{4(k+1)}}$ if $k \neq 0 \Rightarrow \mathrm{var}(E_{1,2})$ are **algorithm dependent**!

Autocorrelations

Integrated autocorrelations $\tau_{\text{int}}(E_k)$ (in MDU) as a function of γ for $L/a = 16$

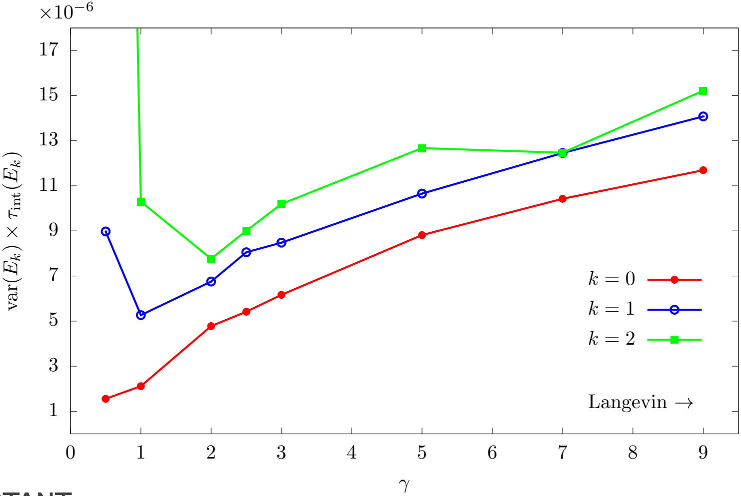


REMARK:

The autocorrelations of the different orders have a similar γ-dependence

Statistical errors

Products $\mathrm{var}(E_k) \times \tau_{\mathrm{int}}(E_k)$ **as a function of** γ **for** $L/a = 16$

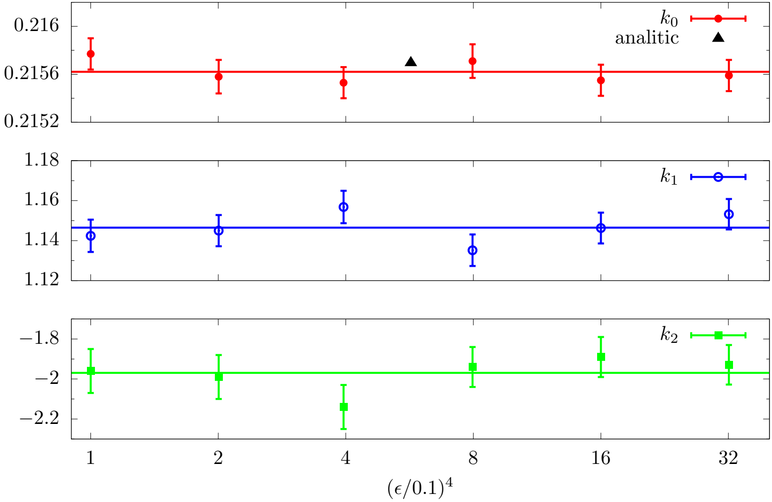


IMPORTANT:

Algorithms tuned for **small** autocorrelations tend to have **large** variances and viceversa: a **compromise** needs to be found!

Systematic errors

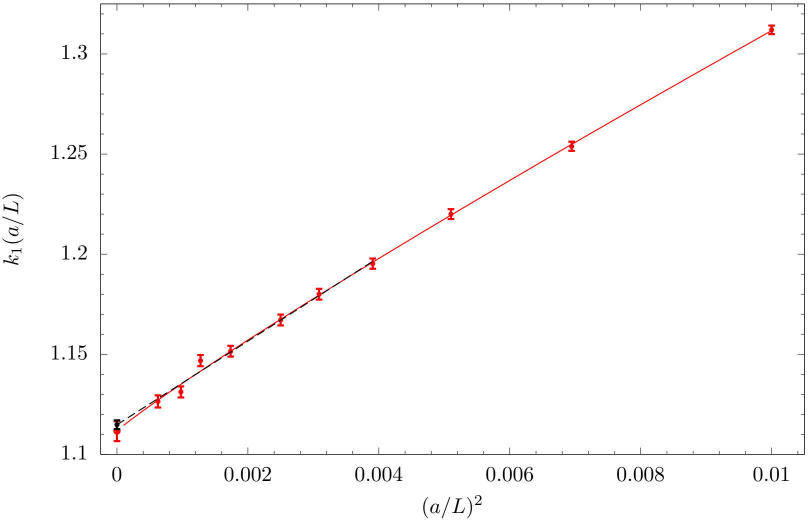
Coefficients $k_i(a/L)$, $i = 0, 1, 2$, for $L/a = 24$, 10^4 measurements per ϵ, OMF4 integrator



NOTE:

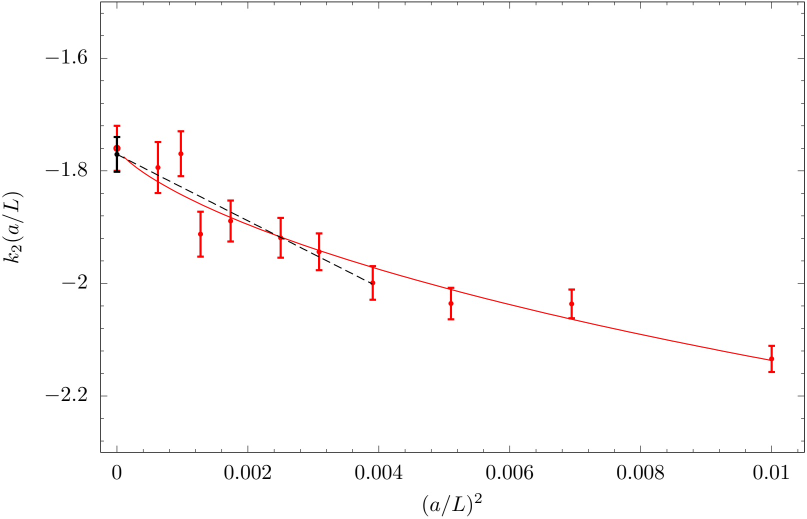
The energy violations per MD step are $\Delta H_i \propto \epsilon^5$, where $H = H_0 + g_0\, H_1 + \ldots$

Continuum limit of k_1



Symanzik theory: $k_1 \overset{a \to 0}{=} a_0 + \{a_1 + b_1 \ln(L/a)\}(a/L)^2 + \mathrm{O}(a^3)$

Continuum limit of k_2



Symanzik theory: $k_2 \stackrel{a\to 0}{=} a_0 + \{a_1 + b_1 \ln(L/a) + c_1[\ln(L/a)]^2\}(a/L)^2 + \mathrm{O}(a^3)$

Outlook & conclusions

Conclusions:

- NSPT is a **powerful** tool for automatizing LPT calculations
- Complicated lattice set-ups and observables give rise to hardly any difficulty
- The recent algorithmic advances opened the way for **precise** and **accurate** results
- A full implementation for the SU(3) gauge theory can be downloaded at:

  luscher.web.cern.ch/luscher/NSPT

Outlook:

- The inclusion of fermions is in principle straightforward and does not pose any technical difficultly (MDB, Lüscher '17)
- Fermions are not expected to slow down the simulations by a big factor (Di Renzo, Scorzato '04)


Thanks

BACKUP

NSPT in lattice QCD

Convergence to a stationary distribution

Leading order dynamics

$$S_G = S_0 + g_0 S_1 + g_0^2 S_2 + \ldots$$

$$S_0 = \frac{1}{2} \sum_{x,y} A_\mu(x) \Delta_{\mu\nu}(x,y) A_\nu(y), \quad A_\mu(x) = U_0(x,\mu)$$

PROBLEM: $\Delta \geq 0$

1. Gauge modes $A_\mu(x) \sim 0$
2. In a finite volume w/ periodic bc. $\exists\ A_\mu(x) \not\sim 0$ s.t. $\Delta A = 0$

(Gonzalez-Arroyo, Jurkiewicz, Korthals-Altes '83)

N.B.: Issues of the perturbative expansion not (N)SPT!

SOLUTION:

1. Gauge damping a.k.a. stochastic gauge fixing (Zwanziger '81)
2. Choose proper boundary conditions for the fields
   e.g. $A_k(x)|_{x_0=0} = A_k(x)|_{x_0=T} = 0$

RESULT: Separate convergence criteria for the gauge modes (MDB, Lüscher '17)

Gauge damped SMD equations

Time-dependent gauge transf.

$$\pi_t(x,\mu) \to \Lambda_t(x)\pi_t(x,\mu)\Lambda_t(x)^{-1}$$

$$U_t(x,\mu) \to \Lambda_t(x)U_t(x,\mu)\Lambda_t(x+\hat{\mu})^{-1}$$

Modified SMD equations

$$\partial_t U_t(x,\mu) = g_0\big\{\pi_t(x,\mu) - \nabla_\mu \omega_t(x)\big\}U_t(x,\mu)$$

$$\partial_t \pi_t(x,\mu) = -g_0(\partial^a_{x,\mu} S_G)(U_t)T^a - \gamma\pi_t(x,\mu) + \eta_t(x,\mu) + g_0[\omega_t(x), \pi_t(x,\mu)]$$

$$\langle \eta^a_t(x,\mu)\, \eta^b_s(y,\nu)\rangle = 2\gamma\delta^{ab}\delta_{\mu\nu}\delta_{xy}\delta(t-s)$$

with

$$\omega_t(x) = g_0^{-1}\partial_t\Lambda_t(x)\Lambda_t(x)^{-1}$$

$$\nabla_\mu\omega_t(x) = U_t(x,\mu)\omega_t(x+\hat{\mu})U_t(x,\mu)^{-1} - \omega_t(x)$$

Gauge damping

$$\partial_t\omega_t(x) = -\gamma\omega_t(x) + \lambda(d^* C_t)(x), \qquad \lambda > 0$$

$$C_t(x,\mu) = \frac{1}{2g_0}\Big\{U_t(x,\mu) - U_t(x,\mu)^{-1} - \frac{1}{3}\mathrm{tr}\Big[U_t(x,\mu) - U_t(x,\mu)^{-1}\Big]\Big\}$$

NSPT in lattice QCD

Gauge damping

Gauge transformation

$$\pi(x,\mu) \to \Lambda(x)\pi(x,\mu)\Lambda(x)^{-1}, \quad U(x,\mu) \to \Lambda(x)U(x,\mu)\Lambda(x+\hat{\mu})^{-1}$$

Gauge damping field

$$\Lambda(x) = e^{\epsilon g_0 \omega(x)}, \quad \omega(x) \in \mathfrak{su}(3), \quad \omega(x) = \sum_{r=0}^{N} g_0^r \omega_r(x)$$

Gauge-damping dynamics (leading-order)

1. $\omega_0(x) \to c_1 \omega_0(x)$ $\qquad c_1 = e^{-\epsilon\gamma}$
2. $\omega_0(x) \to \omega_0(x) + \epsilon\lambda(d^*A)(x)$ $\qquad (d^*A)(x) = -\partial_\mu^* A_\mu(x)$
3. $A_\mu(x) \to A_\mu(x) - \epsilon(d\omega_0)_\mu(x)$ $\qquad (d\omega_0)_\mu(x) = \partial_\mu \omega_0(x)$

N.B.: $\omega(x)$ assures a restoring force on the longitudinal modes!

Convergence criteria

1. $\Delta|_{A^T} > 0$
2. $\epsilon^2 ||\Delta|| < \kappa$
3. $\epsilon^2 \lambda ||dd^*|| < 2(1+c_1)$ (**N.B.:** $||dd^*|| \leq 16$)

The Yang-Mills gradient flow in perturbation theory

In PTh, GF obs. are quite non-trivial objects,

$$B_\mu = \sum_{k=1}^{\infty} g_0^k B_{\mu,k} \quad \xrightarrow{\text{GF eq.}} \quad \partial_t B_{\mu,k} - \partial_\nu \partial_\nu B_{\mu,k} = R_{\mu,k}$$

$$B_{\mu,k}|_{t=0} = \delta_{k1} A_\mu$$

where e.g.

$$R_{\mu,1} = 0,$$
$$R_{\mu,2} = 2[B_{\nu,1}, \partial_\nu B_{\mu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,1}],$$
$$R_{\mu,3} = \text{complicated and not very illuminating expression almost as long as this whole sentence} \ldots$$

Simple at lowest order

$$B_{\mu,1}(t,x) = \int \mathrm{d}^4 y \, K_t(x-y) A_\mu(y), \quad K_t(z) = \frac{e^{-z^2/4t}}{(4\pi t)^2}$$

The Yang-Mills gradient flow in perturbation theory

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$$R_{\mu,3} = \text{complicated and not very illuminating expression almost as long as this whole sentence} \ldots$$

Simple at lowest order ... but quickly becomes **involved**,

$$B_{\mu,k}(t,x) = \int_0^t \mathrm{d}s \int \mathrm{d}^4 y \, K_{t-s}(x-y) R_{\mu,k}(s,y), \quad k = 2, 3, \ldots$$

The gradient flow coupling

Lattice formulation

Gradient flow $\to$ Wilson flow

$$\partial_t V_t(x,\mu) V_t(x,\mu)^{-1} = -g_0^2 (\partial^a_{x,\mu} S_G)(V_t) T^a, \quad V_{t=0}(x,\mu) = U(x,\mu)$$

N.B.: Solved using a Runge-Kutta scheme

Schrödinger functional

$$U(x,k)|_{x_0=0} = 1 = U(x,k)|_{x_0=T} \quad \Rightarrow \quad U_r(x,k)|_{x_0=0,T} = 0$$

Perturbative solution (MDB, Hesse '13)

$$V_t(x,\mu) = e^{g_0 B_\mu(t,x)} = 1 + \sum_{r=0}^{N} g_0^{r+1} V_{t,r}(x,\mu), \quad V_{t,r}(x,\mu) = U_r(x,\mu)$$

Basic quantity

$$E(t,x) = g_0^2 E_0 + g_0^4 E_1 + g_0^6 E_2 + \ldots$$

Coupling [using $\alpha_{\overline{\mathrm{MS}}} \leftrightarrow g_0$ (Lüscher, Weisz '95)]

$$t^2 \langle E(t,x) \rangle = k_0(a/L) \{ \alpha_{\overline{\mathrm{MS}}}(q) + k_1(a/L)\, \alpha_{\overline{\mathrm{MS}}}(q)^2 + k_2(a/L)\, \alpha_{\overline{\mathrm{MS}}}(q)^3 + \ldots \}$$

$$\lim_{a/L \to 0} k_i(a/L) = k_i, \quad i = 1,2,3$$