NSPT in ϕ^4 theory

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- φ^4 theory on the lattice

$$S(\varphi) = \sum_x \left(\frac{1}{2}\varphi(x)\Delta\varphi(x) + \frac{g}{4!}\varphi(x)^4 \right) \qquad \Delta = (-\partial_\mu\partial_\mu^* + \frac{1}{2}m^2)$$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int D\varphi\, e^{-S(\varphi)}\mathcal{O}(\varphi)$$

- Generate field configurations $\phi^{(n)}$ with distribution $P[\phi^{(n)}] \propto e^{-S(\phi^{(n)})}$

$$\overline{\mathcal{O}} \equiv \frac{1}{N_{cnfg}} \sum_{n=0}^{N_{cnfg}} \mathcal{O}(\phi^{(n)}) = \langle \mathcal{O} \rangle + O\left(\frac{1}{\sqrt{N_{cnfg}}} \right)$$

- Perturbation theory: generate a field as a power series in g

$$\phi = \sum_r \phi_r g^r \qquad \mathcal{O} = \sum_r \mathcal{O}_r(\phi_0, ..., \phi_r) g^r \qquad \phi^2 = \phi_0^2 + 2\phi_0\phi_1 g + ...$$

such that

$$\overline{\mathcal{O}_r(\phi_0, ..., \phi_r)} = k_r \quad \text{with} \quad \langle \mathcal{O} \rangle = \sum_r k_r g^r$$

- Stochastic error

$$\sigma^2_{\mathcal{O}_r} = N^{-1}_{cnfg} \times \tau_{int}(\mathcal{O}_r) \times \mathrm{Var}(\mathcal{O}_r)$$

$$\mathrm{Var}(\mathcal{O}_r) = \overline{\mathcal{O}^2_r} - \overline{\mathcal{O}}^2_r \quad \neq \left(\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2\right)_{2r} \text{ if } r \neq 0$$

- Different methods to generate ϕ_r
  - ISPT, based on the concept of the trivializing map M. Lüscher (2015)
  - Langevin NSPT, the first to have been introduced, based on Markov process F. Di Renzo, G. Marchesini, P. Marenzoni (1994)
  - GHMC, based on Markov process, it reduces to Langevin for an appropriate tuning of the parameters A. D. Kennedy, B. Pendleton (2000)

ISPT

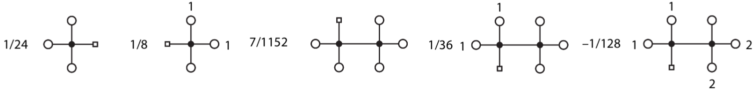
- based on the trivilaizing map. It transform a set of Gaussian distributed noise η_i in a field ϕ such that

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_\eta = \langle \varphi(x_1)\cdots\varphi(x_n)\rangle$$
$$\langle \eta_i(x)\rangle_\eta = 0, \qquad \langle \eta_i(x)\eta_j(y)\rangle_\eta = \delta_{ij}a^{-4}\delta_{xy}$$

- The trivializing map can be expanded order by order in perturbation theory. At the lowest order:

$$\phi_0 = (-\partial_\mu\partial_\mu^* + m)^{1/2}\eta_0$$

A possible choice for the higher orders is to expressed them as a sum of tree graphs e.g. M. Lüscher (2015)

$\phi_1 =$ 

where 1^{st} diagram $= (-\partial_\mu\partial_\mu^* + m)^{-1}\phi_0^3$

ISPT

- Program package for the automatic construction of the trivializing map http://luscher.web.cern.ch/luscher/ISPT/index.html
- At the same link recently available a simulation program for SU(3)

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✓ Uncorrelated configurations
$\sigma^2_{\mathcal{O}_r} = N^{-1}_{cnfg} \times \cancel{\tau_{int}(\mathcal{O}_r)} \times \mathrm{Var}(\mathcal{O}_r)$
✓ Exact algorithm, no systematic errors

× Number and complexity of diagrams increases rapidly

Langevin NSPT

(Di Renzo et. al. '94; Di Renzo, Scorzato '04)

- Integrate numerically the Langevin equation order by order

$$\partial_t \phi(t,x) = -\frac{\delta S(\phi)}{\delta \phi(t,x)} + \eta(t,x), \qquad \langle \eta(t,x)\eta(t',y)\rangle = 2a^{-4}\delta(t-t')\delta_{xy}$$

$$\partial_t \phi_0(t,x) = -\Delta\phi_0(t,x) + \eta(t,x)$$

$$\partial_t \phi_1(t,x) = -\Delta\phi_1(t,x) + \frac{g}{4!}\phi_0^2(t,x)\phi_0(t,x)$$

$$\ldots$$

- Known scaling of $\mathrm{Var}(\mathcal{O}_r)$ and $\tau_{int}(\mathcal{O}_r)$ to all order with 5-dim field theory analysis

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- Known scaling of $\mathrm{Var}(\mathcal{O}_r)$ and $\tau_{int}(\mathcal{O}_r)$ to all order with 5-dim field theory analysis

✓ Variance $\mathrm{Var}(\mathcal{O}_r) \propto \ln a$

luscher.web.cern.ch/luscher//notes/enspt.pdf

✓ Mild computational cost with the order $\mathcal{C} \propto \frac{r_{max}(r_{max}-1)}{2}$

× Correlated configurations

$\tau_{int}(\mathcal{O}_r) \propto a^{-2}$

Jean Zinn-Justin (1986)

× Inexact algorithm

$\lim_{Ncnfg\to\infty} \overline{\mathcal{O}}_r = k_r + O(\epsilon^p)$

GHMC-based NSPT

- Consist in the following two steps
  1. Molecular dynamics: an approximate integration of Hamilton equations on phase space: area preserving and reversible $(\pi', \phi') = U(\tau)(\pi, \phi)$
  2. Partial momentum refreshment $\pi'' \to e^{-\gamma \delta t}\pi' + \sqrt{1 - e^{-2\gamma\delta t}}\eta$
- Special cases
  1. $\gamma \to \infty$ complete momentum refreshment arbitrary τ HMD
  2. $\tau = \delta t$ single integration step arbitrary γ SMD
  3. $\gamma \to \infty$ and $\tau = \delta t$ Langevin
- Free theory theoretical analysis shows that HMD with $\tau \propto 1/(am)$ KSPT with $\gamma \propto am$ have $\tau_{int}(\mathcal{O}_{r=0}) \propto a^{-1}$

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✓ Include Langevin as special case

✓ Mild computational cost with the order $\mathcal{C} \propto \frac{r_{max}(r_{max}-1)}{2}$

× Unknown high orders $\mathrm{Var}(\mathcal{O}_{r>0})$ and $\tau_{int}(\mathcal{O}_{r>0})$ scaling

× Inexact algorithm $\lim_{Ncnfg\to\infty} \overline{\mathcal{O}}_r = k_r + O(\epsilon^p)$

- The methods presented can be extended to theories with more than one coupling
- Power series in the counterterm

$$S(\varphi) = a^4 \sum_x \left(\frac{1}{2}\varphi(x)\partial_\mu\partial^*_\mu\varphi(x) + \frac{1}{2}(m_R^2 + \delta m^2)\varphi(x)^2 + \frac{g_0}{4!}\varphi(x)^4 \right)$$

- Field expanded as double power series

$$\phi = \sum_{k,l} \phi_{k,l}(\delta m^2)^k g_0^l \quad \longrightarrow \quad \mathcal{O} = \sum_{k,l} \mathcal{O}_{k,l}(\delta m^2)^k g_0^l$$

$$\overline{\mathcal{O}}_{k,l} = K_{k,l} \quad \text{with} \quad \langle \mathcal{O} \rangle = \sum_{k,l} K_{k,l}(\delta m^2)^k g_0^l$$

- Imposing a renormalization condition we can determine $\delta m^2 = \sum_n c_n g_0^n$
- Same situation with Wilson fermions

- As a test observable we chose

$$\mathcal{E} = \frac{t^2}{L^4} \sum_x \langle \phi(x,t)^4 \rangle$$

- $\phi(x,t)$ field at positive flow time

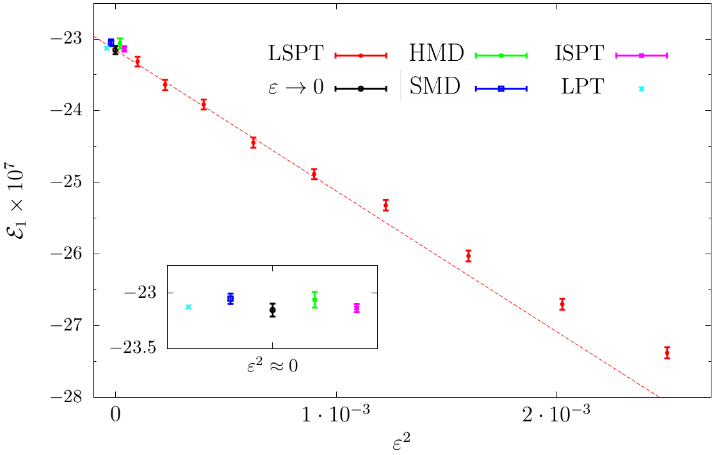
$$\partial_t \phi(x,t) = \partial_\mu \partial_\mu^* \phi(x,t) \implies \phi(p,t) = e^{-p^2 t} \phi(p,t_0)$$
$$\phi(p,t_0) = \phi(p)$$

- The flow smears the field and allows us to define composite operators that are automatically finite after mass, field and coupling renormalization. $Z_{\phi^4} = Z_\phi^4$ at $t > 0$
- Continumm limit

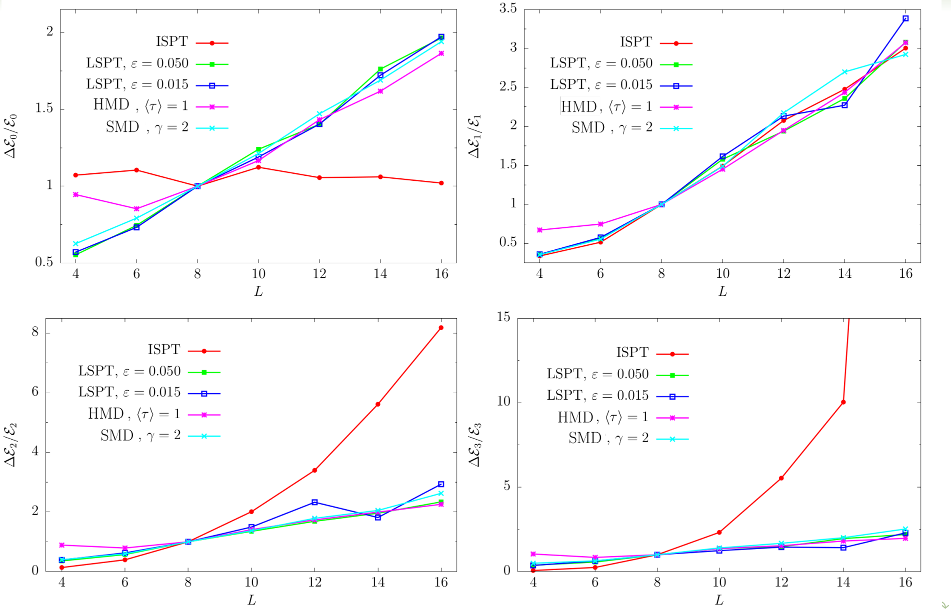
$$m_R L = 4 \qquad \sqrt{8t}/L = 0.2$$

Result

- RK2 to integrate Langevin-based NSPT step size ε
- OMF4 to integrate the MD equation in HMD and SMD-based NSPT with large $\delta t = 0.5$ no sign of sistematic errors

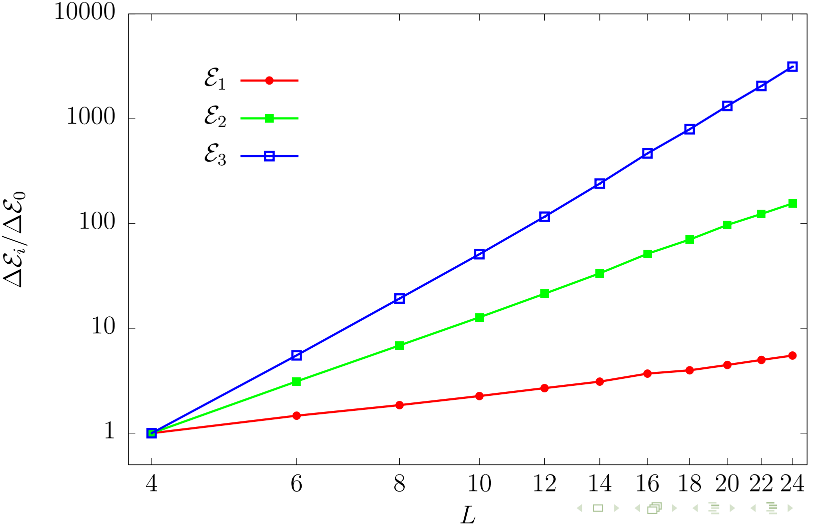


$$\Delta^2\mathcal{E} = \mathrm{Var}(\mathcal{E}_r) \times 2\tau_{int}(\mathcal{E}_r)/N_{cnfg}.$$

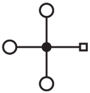
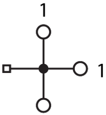


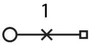
ISPT error

As the continuum limit is approached the statistical variance of perturbative coefficients computed using ISPT grows with an increasing power of L



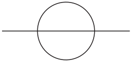
- the divergences in the variance of ISPT Lüscher, Mainz 31.8.-11.9. (2015)

$\phi_1 =$ 1/24  1/8 

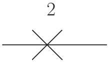
1/2 

$\langle \phi_1 \rangle_\eta = 0$

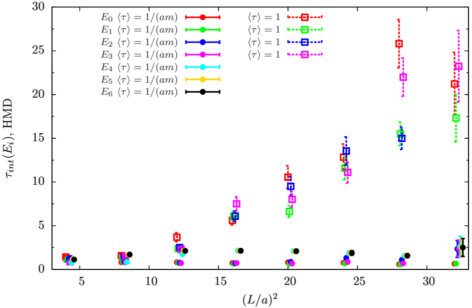
- $\mathrm{Var}(\phi_1) = \langle \phi_1^2 \rangle_\eta - \langle \phi_1 \rangle_\eta^2$

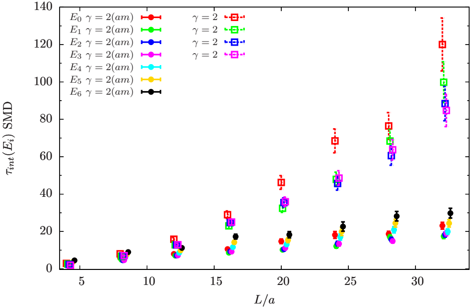
$=$  $+ \ldots,$

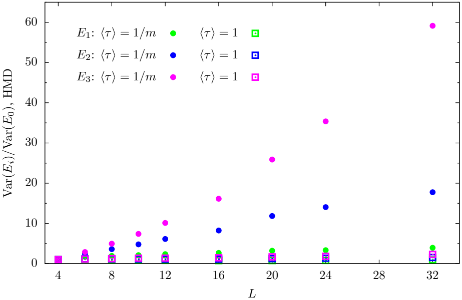
$$\delta m^2 = c_1 g + c_2 g^2 + \ldots$$

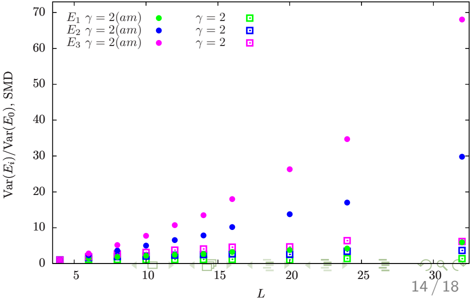
- missing : 

HMD and SMD-based NSPT

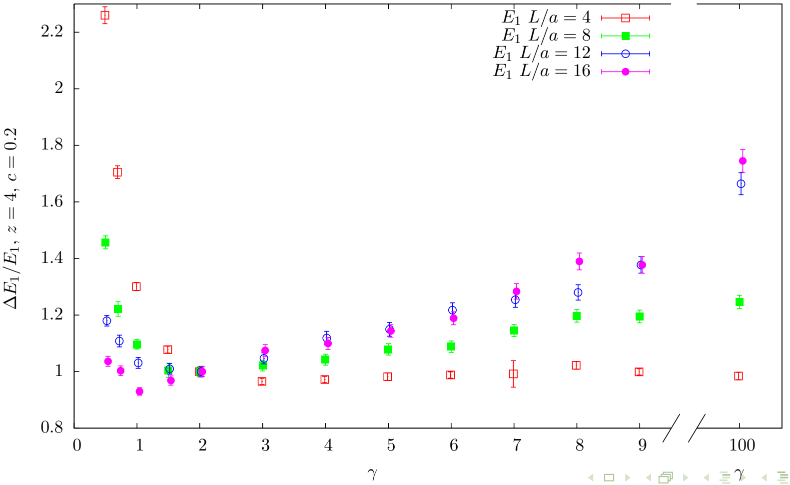




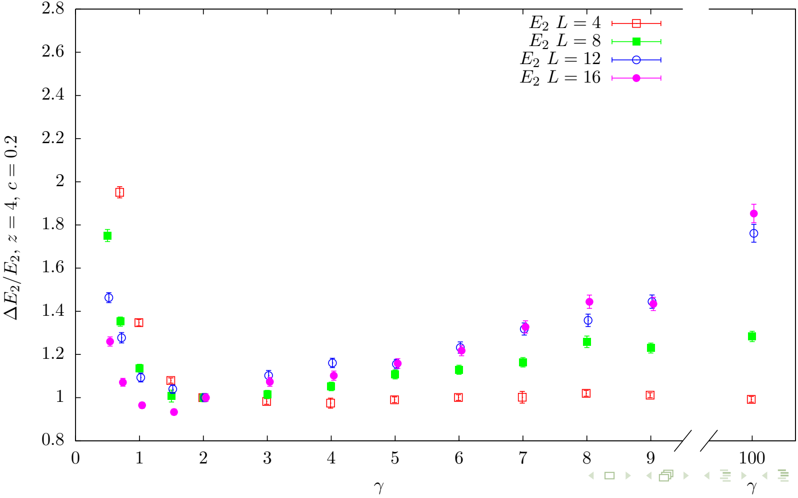




- HMD and SMD-based NSPT show a similar behaviour when the parameter is tuned: Var$(E_i) \uparrow$ and $\tau_{int}(E_i) \downarrow$.
- We concentrate on SMD where we scan in γ seeking for the optimal value



- Similar situation for the second order
- $\gamma^1_{min} \sim \gamma^2_{min}$
- $\gamma \sim 2$ is a good choice for all L
- $\gamma \to \infty$ (Langevin limit) worst than $\gamma \sim 2$



Conclusion

- Utility of ISPT limited to a few low perturbative orders
- HMD and SMD-based NSPT
  - Efficient high-order symplectic integrators as the OMF4 for the MD equation reduce drastically the systematic errors in HMD and SMD-based NSPT
  - Lowest order prediction for scaling of τ_{int} is observed also for higher orders
  - The algorithmic dependence of the $\mathrm{Var}(E_i)$ makes it difficult to find the optimal value of $\langle\tau\rangle$ and γ.
  - HMD similar to SMD with the identification $\langle\tau\rangle \sim 1/\gamma$
- Non constant variance makes the tuning of NSPT algorithms different than the more conventional non-perturbative simulations

Thank you for your attention