The Physics of Gravitational Waves

Neil J. Cornish

eXtreme Gravity Institute

MONTANA STATE UNIVERSITY

Outline

- Nearly Minkowski spacetimes
- Coordinate (gauge) freedom
- Linearized Field Equations
- Revisiting the Newtonian Limit
- Wave equation and solution
- Transverse-Traceless and Fermi normal coordinates
- Generating gravitational waves - the multipole expansion
- Energy carried by gravitational waves

Nearly Minkowski Spacetimes

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

This description is valid for our solar system, galaxy and even out to a few Gpc's. Really good for gravitational waves.

Solar system, galaxy $|h_{\mu\nu}| < 10^{-6}$ Universe out to 1 Gpc $|h_{\mu\nu}| < 10^{-2}$ Gravitational Waves $|h_{\mu\nu}| < 10^{-20}$

Recall Riemann normal coordinates and their extension, Fermi Normal coordinates

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \mathcal{O}(x^3)$$

Coordinate Transformations

There are two kinds of coordinate transformations that preserve the nearly Minkowski form:
Lorentz transformations and infinitesimal coordinate (gauge) transformations

$$x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\mu}\, x^{\mu} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\, x^{\mu}$$

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\mu}{}_{\bar{\mu}} \Lambda^{\nu}{}_{\bar{\nu}}\, g_{\mu\nu}$$

Lorentz transformations: $x^{\bar{\mu}} = L^{\bar{\mu}}{}_{\mu}\, x^{\mu} \quad \Rightarrow \quad h_{\bar{\mu}\bar{\nu}} = L^{\mu}{}_{\bar{\mu}} L^{\nu}{}_{\bar{\nu}}\, h_{\mu\nu}$

Gauge transformations: $x^{\bar{\mu}} = x^{\nu} + \zeta^{\nu}, \qquad |\zeta^{\nu}| \ll 1 \quad \Rightarrow \quad h_{\bar{\mu}\bar{\nu}} = h_{\mu\nu} - \partial_{\mu}\zeta_{\nu} - \partial_{\nu}\zeta_{\mu}$

Riemann tensor components invariant under gauge transformation:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_{\beta}\partial_{\mu}h_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}h_{\beta\mu} - \partial_{\beta}\partial_{\nu}h_{\alpha\mu} - \partial_{\alpha}\partial_{\mu}h_{\beta\nu})$$

Linearized Field Equations

$$R_{\mu\nu} = \frac{1}{2}(-\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial^\alpha\partial_\mu h_{\nu\alpha} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial_\mu\partial_\nu h) \qquad R = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^\mu\partial_\mu h$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,\eta_{\mu\nu} = 8\pi T_{\mu\nu}$$

Define $\Box = \partial^\alpha\partial_\alpha = -\partial_t^2 + \nabla^2$ and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu}$

$$\Rightarrow \quad -\Box\bar{h}_{\mu\nu} + \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} + \partial_\mu\partial^\alpha\bar{h}_{\nu\alpha} - \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} = 16\pi\,T_{\mu\nu}$$

Now, under the Gauge transform $x^{\bar{\nu}} = x^\nu + \zeta^\nu$ we have $\partial^{\bar{\alpha}}\bar{h}_{\bar{\mu}\bar{\alpha}} = \partial^\alpha\bar{h}_{\mu\alpha} - \Box\zeta_\mu$

Setting $\Box\zeta_\mu = \partial^\alpha\bar{h}_{\mu\alpha}$ selects the Lorentz gauge family $\partial^{\bar{\alpha}}\bar{h}_{\bar{\mu}\bar{\alpha}} = 0$

Note that the Lorentz gauge is not fully gauge fixed. Free to shift by a homogeneous term: $\zeta^\nu \to \zeta^\nu + \lambda^\nu, \quad \Box\lambda^\nu = 0$

Linearized Field Equations: Lorentz Gauge

$$\Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu} \qquad\qquad \partial^\nu \bar{h}_{\mu\nu} = 0$$

$$\text{c.f. Maxwell's equations} \quad \Box A^\mu = J^\mu, \quad \partial_\mu A^\mu = 0$$

The methods for solving the linearized Einstein equations are almost identical to E&M, retarded Green's functions etc

Newtonian Limit Revisited

$|\bar{h}_{\mu\nu}| \ll 1$ $\quad$ $|v| \ll 1$ $\quad$ $|\partial_t^2| \ll |\nabla^2|$ $\quad$ $|T_{tt}| \gg |T_{ti}| \gg |T_{ij}|$

$$\Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu} \quad \Rightarrow \quad \Box \bar{h}_{tt} = -16\pi T_{tt} \quad \Rightarrow \quad \nabla^2 \bar{h}_{tt} = -16\pi\rho$$

Recover Newtonian Gravity by setting $\bar{h}_{tt} = -4\Phi$

$$\Rightarrow \quad ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

$$u^t = \frac{dt}{d\tau} = 1 + \frac{1}{2}v^2 - \Phi \approx 1$$

This is the leading order in the post-Newtonian expansion of Einstein's equations

Solving the wave equation

$$\Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu} \qquad \partial^\nu \bar{h}_{\mu\nu} = 0$$

We can use all the familiar tools: Green's functions, expansion in special functions etc

Start with a plane-wave expansion of the vacuum equations: $\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}\, e^{i\vec{k}\cdot\vec{x}}\right\} \qquad \Box \bar{h}_{\mu\nu} = 0 \qquad \vec{k} \to (\omega, \mathbf{k})$

$$\Rightarrow \quad \vec{k}\cdot\vec{k} = 0, \quad A_{\mu\nu} k^\nu = 0$$

Tells us that gravitational waves travel at the speed of light and are transverse.

The polarization tensor $A_{\mu\nu}$ is symmetric, so has 10 independent components. The transverse condition provides 4 constraints, so 6 dof remain

But, we have the remaining gauge freedom $\zeta^\nu \to \zeta^\nu + \lambda^\nu, \quad \Box \lambda^\nu = 0 \qquad \Rightarrow \qquad \lambda^\nu = i\, C^\nu\, e^{i\vec{k}\cdot\vec{x}}$

Applying this freedom we have $^{(\text{new})}A_{\mu\nu} = {}^{(\text{old})}A_{\mu\nu} + C_\mu k_\nu + C_\nu k_\mu - \eta_{\mu\nu}\, k^\alpha C_\alpha$

Can fully gauge fix by choosing the 4 C_μ to be anything we want. Two degrees of freedom remain.

Finalizing the gauge choice

The residual gauge freedom can be used to finalize our coordinate choice. In the early bar-detector era, Fermi Normal coordinates were a popular choice. Today the transverse-traceless (TT) gauge has risen to prominence. Both have their uses.

TT Gauge:

Use gauge freedom to make traceless: $^{(\mathrm{new})}A^{\mu}_{\mu} = {}^{(\mathrm{old})}A^{\mu}_{\mu} - 4C_{\mu}k^{\mu} = 0$ (1 constraint)

And make orthogonal to observers worldline: $^{(\mathrm{new})}A_{\mu\nu}\,u^{\nu} = 0$ (3 constraints since $^{(\mathrm{old})}A_{\mu\nu}\,u^{\nu}k^{\mu} = 0$)

TT metric for a plane wave propagating in the z direction as seen by inertial observer:

$$\vec{u} \to (1,0,0,0) \qquad \vec{k} \to (\omega,0,0,\omega)$$

$$\Rightarrow \quad A_{\mu t} = 0, \quad A_{\mu z} = 0, \quad A_{yy} = -A_{xx}$$

TT gauge

$$ds^2 = -dt^2 + (1 + h_{xx})dx^2 + (1 - h_{xx})dy^2 + 2h_{xy}\, dxdy + dz^2$$
$$= -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times\, dxdy + dz^2$$

With

$$h_+ = A_+ \cos(\omega(t - z) + \phi_+)$$
$$h_\times = A_\times \cos(\omega(t - z) + \phi_\times)$$

Motion of a test mass? $\dfrac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu}\, u^\mu u^\nu = \dfrac{1}{2}\partial^\alpha h_{\mu\nu}\, u^\mu u^\nu$

If initially stationary $u^\alpha = \delta^\alpha_t \quad \Rightarrow \quad \left.\dfrac{du^\alpha}{d\tau}\right|_0 = \dfrac{1}{2}\partial^\alpha h_{tt} = 0$

Stays fixed at the same ***coordinate*** location!

We have used the original gauge freedom $\Box\lambda^\mu = 0$ to absorb the GW into the coordinate system.

The TT gauge is great for doing calculations (globally defined, test particles stay fixed), but hides the physical nature of the wave

"Ripples of Curvature"

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\alpha\partial_\mu h_{\beta\nu})$$

Non-vanishing components:

$$R_{ytyt} = R_{yzyz} = R_{xtxz} = -R_{xtxt} = -R_{xzxz} = -R_{ytyz} = \frac{1}{2}\ddot{h}_+$$

$$R_{xtyz} = R_{ytxz} = -R_{xzyz} = -R_{xtyt} = -R_{ytxt} = -R_{yzxz} = \frac{1}{2}\ddot{h}_\times$$

Geodesic deviation equation tells us that GWs generate a time varying tidal field

Can be seen more directly in locally inertial Fermi Normal coordinates. Recall that $g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \mathcal{O}(x^3)$

$$\Rightarrow \quad ds^2 \approx -d\bar{t}^2(1+R_{titj}\bar{x}^i\bar{x}^j) - \frac{4}{3}\,d\bar{t}d\bar{x}^i\,(R_{tjik}\bar{x}^j\bar{x}^k) + d\bar{x}^i d\bar{x}^j\,(\delta_{ij} - \frac{1}{3}R_{ikjl}\bar{x}^k\bar{x}^l)$$

$$= -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 + \left(\ddot{h}_\times(t-z)\bar{x}\bar{y} + \frac{1}{2}\ddot{h}_+(t-z)(\bar{x}^2 - \bar{y}^2)\right)(d\bar{t} - d\bar{z})^2$$

Fermi Normal Coordinates

$$ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 + \left(\ddot{h}_\times(t-z)\bar{x}\bar{y} + \frac{1}{2}\ddot{h}_+(t-z)(\bar{x}^2 - \bar{y}^2)\right)(d\bar{t} - d\bar{z})^2$$

Remarkably, while FNC are only valid locally, this particular form for the metric is valid globally.
See [M. Rakhmanov, Class. Quantum Grav. 31 (2014) 085006]

To leading order, the FNC and TT coordinates are related via

$$x = \bar{x} - \frac{1}{2}h_+\,\bar{x} - \frac{1}{2}h_\times\,\bar{y} - \frac{1}{2}\bar{z}(\bar{x}\,\dot{h}_+ + \bar{y}\,\dot{h}_\times)$$

$$y = \bar{y} + \frac{1}{2}h_+\,\bar{y} - \frac{1}{2}h_\times\,\bar{x} + \frac{1}{2}\bar{z}(\bar{y}\,\dot{h}_+ - \bar{x}\,\dot{h}_\times)$$

$$z = \bar{z} + \frac{1}{4}(\bar{x}^2 - \bar{y}^2)\dot{h}_+ + \frac{1}{2}\bar{x}\bar{y}\,\dot{h}_\times$$

$$t = \bar{t} - \frac{1}{4}(\bar{x}^2 - \bar{y}^2)\dot{h}_+ - \frac{1}{2}\bar{x}\bar{y}\,\dot{h}_\times$$

We see that the wave gets put into the TT coordinates

Fermi Normal Coordinates

$$ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 + \left(\ddot{h}_\times(t-z)\bar{x}\bar{y} + \frac{1}{2}\ddot{h}_+(t-z)(\bar{x}^2 - \bar{y}^2)\right)(d\bar{t} - d\bar{z})^2$$

Geodesic equation for a test mass in FNC:

$$\frac{d^2\bar{x}}{d\bar{t}^2} \approx \frac{1}{2}\bar{x}\ddot{h}_+ + \frac{1}{2}\bar{y}\ddot{h}_\times$$

$$\frac{d^2\bar{y}}{d\bar{t}^2} \approx \frac{1}{2}\bar{x}\ddot{h}_\times - \frac{1}{2}\bar{y}\ddot{h}_+$$

$$\frac{d^2\bar{z}}{d\bar{t}^2} \approx 0$$

Using the long wavelength limit:

$$|\bar{x}|, |\bar{y}| \ll \lambda = \frac{2\pi}{\omega}$$

For the general expression, valid everywhere, see [M. Rakhmanov, Class. Quantum Grav. 31 (2014) 085006]

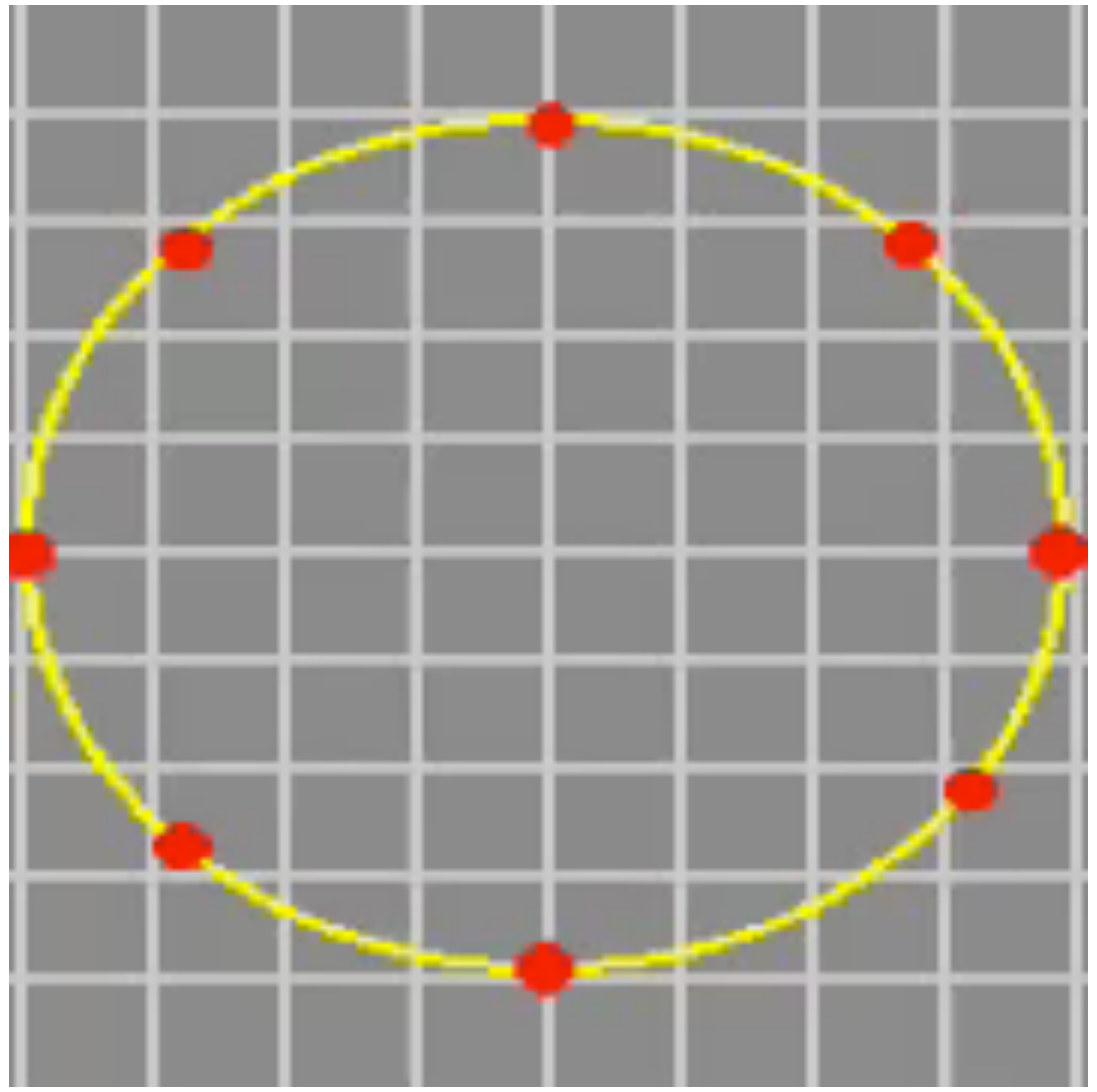
Fermi Normal Coordinates

$$\frac{d^2\bar{x}}{d\bar{t}^2} \approx \frac{1}{2}\bar{x}\ddot{h}_+ + \frac{1}{2}\bar{y}\ddot{h}_\times \qquad \frac{d^2\bar{y}}{d\bar{t}^2} \approx \frac{1}{2}\bar{x}\ddot{h}_\times - \frac{1}{2}\bar{y}\ddot{h}_+$$

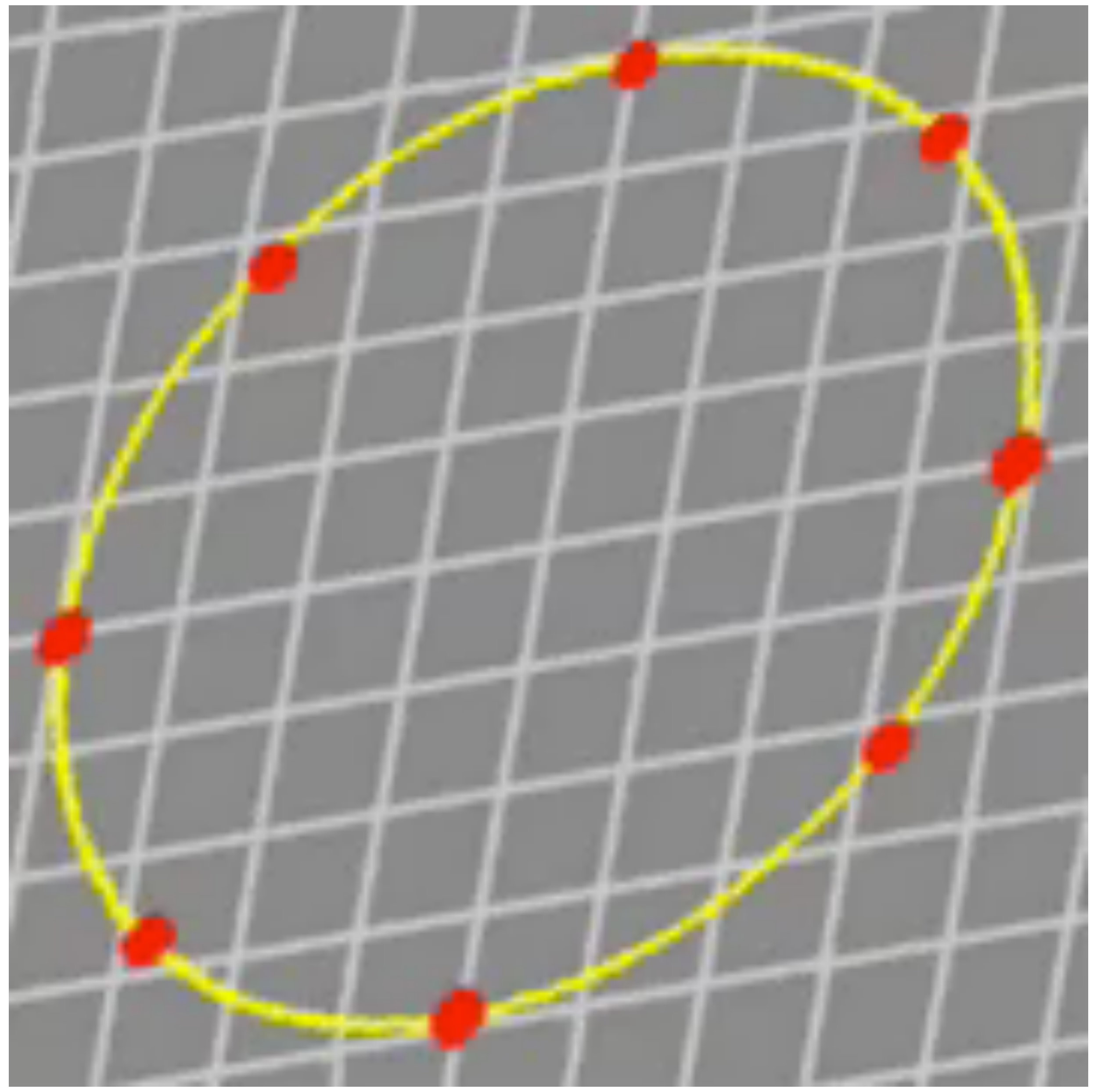
Ring of test particles, initially with $\bar{x} = L\cos\phi, \quad \bar{y} = L\sin\phi, \quad \bar{z} = 0$

$$\Rightarrow \quad \bar{x} = L\left(\cos\phi + \frac{1}{2}\left(\cos\phi\, h_+ + \sin\phi\, h_\times\right)\right) \qquad \bar{y} = L\left(\sin\phi + \frac{1}{2}\left(\cos\phi\, h_\times - \sin\phi\, h_\times\right)\right)$$

h_+



$h_\times$



Generating Gravitational Waves

$$\Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu}$$

Solution via Green's function: $$\bar{h}_{\mu\nu}(\vec{x}) = -16\pi \int d^4x' G(\vec{x} - \vec{x}') T_{\mu\nu}(\vec{x}')$$

Where $$G(\vec{x} - \vec{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(t_{\rm ret} - t'_{\rm ret})$$ satisfies $$\Box G(\vec{x} - \vec{x}') = \delta^4(\vec{x} - \vec{x}')$$

The retarded time is defined as usual: $$t_{\rm ret} = t - |\mathbf{x} - \mathbf{x}'|$$

$$\Rightarrow \quad \bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int d^3x' \, \frac{T_{\mu\nu}(t_{\rm ret}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Generating waves in the TT gauge

The general solution can be expressed in the TT gauge by using the 4-index projection tensor $\Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$

Which is defined in terms of the 2-index projection tensor $P_{ij} = \delta_{ij} - n_i n_j$

The P projection removes and components along the propagation direction, and the Lambda projection additionally removes the trace

$$h_{ij}^{TT} = \Lambda_{ijkl}\,\bar{h}^{kl}$$

Denoting the extent of the source by d, and the distance to a distance observer by r, we can approximate:

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}\left(\frac{d^2}{r}\right)$$

$$\Rightarrow \quad h_{ij}^{TT} = \frac{4}{r}\Lambda_{ij}^{\ \ kl} \int d^3x'\, T_{kl}(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}},\, \mathbf{x}')$$

Post-Newtonian Expansion

$$h_{ij}^{TT} = \frac{4}{r}\Lambda_{ij}{}^{kl} \int d^3x' \, T_{kl}(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}, \, \mathbf{x}')$$

In the limit that the material is moving at low velocities we can expand:

$$T_{kl}(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}, \, \mathbf{x}') \approx T_{kl}(t-r) + x'_i n^i \, \partial_t T_{kl} + \frac{1}{2} x'_i x'_j n^i n^j \, \partial_t^2 T_{kl} + \dots$$

$$\Rightarrow \quad h_{ij}^{TT}(t, \mathbf{x}) = \frac{4}{r}\Lambda_{ijkl} \left[S^{kl}(t-r) + n_m \dot{S}^{klm}(t-r) + \frac{1}{2} n_m n_p \ddot{S}^{klmp}(t-r) + \dots \right]$$

Here we have introduced the multipole moments

$$S^{ij}(t) = \int d^3x' \, T^{ij}(t, \mathbf{x}') \qquad S^{ijk}(t) = \int d^3x' \, T^{ij}(t, \mathbf{x}') {x'}^k \qquad S^{ijkp}(t) = \int d^3x' \, T^{ij}(t, \mathbf{x}') {x'}^k {x'}^p$$

Post-Newtonian Expansion

The lowest order multipole moment dominates for slow moving sources. It is related to mass quadrupole moment

$$Q^{ij}(t) = \int d^3x\, T_{tt}(t,\mathbf{x})\, x^i x^j \approx \int d^3x\, \rho(t,\mathbf{x})\, x^i x^j$$

$$\dot{Q}^{ij}(t) = \int d^3x\, \partial^t T_{tt}(t,\mathbf{x})\, x^i x^j = -\int d^3x\, \partial^k T_{tk}(t,\mathbf{x})\, x^i x^j = \int d^3x \left(T^i_t x^j + T^j_t x^i \right) - \oint T^{tk} x^i x^j n_k d^2S$$

$$\ddot{Q}^{ij}(t) = -\int d^3x\, (\partial_k T^{ki}(t,\mathbf{x})\, x^j + \partial_k T^{kj}(t,\mathbf{x})\, x^i) = 2\int d^3x\, T^{ij}(t,\mathbf{x}) - \oint (T^{ki} x^j + T^{kj} x^i) n_k d^2S$$

$$\Rightarrow \quad S^{ij}(t) = \frac{1}{2}\ddot{Q}^{ij}(t)$$

Post-Newtonian Expansion

Lowest order, Quadrupole approximation: $$h_{ij}^{TT}(t, \mathbf{x}) = \frac{2}{r} \Lambda_{ijkl} \, \ddot{Q}^{kl}(t-r) = \frac{2}{r} \, \ddot{Q}_{ij}^{TT}(t-r)$$

Applying the projections for a wave traveling in the z direction

$$h_+(t) = \frac{1}{r} \left(\ddot{Q}_{xx}(t-r) - \ddot{Q}_{yy}(t-r) \right)$$

$$h_\times(t) = \frac{2}{r} \ddot{Q}_{xy}(t-r)$$

In Lecture 5 we will apply this formalism to binary systems.

Energy carried by gravitational waves

Expand Einstein equations to next order $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + f_{\mu\nu}$ $|f_{\mu\nu}| \sim |h_{\mu\nu}|^2$

$$G^{(1)}_{\mu\nu}(h) = 8\pi T_{\mu\nu} \qquad G^{(2)}_{\mu\nu}(f) = 8\pi \tau_{\mu\nu}(h^2)$$

Energy Momentum Tensor for GWs $\tau^{TT}_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h^{TT}_{jk} \partial_\nu h^{TT\,jk} \rangle$ (GW energy can't be localized)

Traceless $\tau^\mu_\mu = 0$ Conserved $\partial^\nu \tau_{\mu\nu} = 0$

In spherical coordinates $\tau_{tt} = \tau_{rr} = -\tau_{tr} = \frac{1}{32\pi}\langle |\dot{h}^{TT}_{ij}|^2 \rangle = \frac{1}{8\pi r^2}\langle |\dddot{Q}^{TT}_{ij}|^2 \rangle$

Energy radiated $\frac{dE}{dt} = \oint \tau_{tr}\, r^2 \sin^2\theta d\phi$ $\Rightarrow$ $\frac{dE}{dt} = \frac{1}{5}\langle |\dddot{Q}_{ij}(t-r)|^2 \rangle$

Energy, Momentum and Angular Momentum carried by gravitational waves

Energy radiated

$$\frac{dE}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \dot{h}^{ij}_{TT} \rangle$$

Linear momentum radiated

$$\frac{dP^k}{dt} = -\frac{r^2}{32\pi} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \partial^k h^{ij}_{TT} \rangle$$

(responsible for BH kicks)

Angular momentum radiated

$$\frac{dJ^i}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle 2\epsilon^{ikl} \dot{h}_{al}^{TT} \dot{h}_{ak}^{TT} - \epsilon^{ikl} \dot{h}_{ab}^{TT} x_k \partial_l h^{ab}_{TT} \rangle$$