RF Basics and TM Cavities

Erk JENSEN, CERN
DC versus RF

DC accelerator

RF accelerator

\[ \Delta W = q \varepsilon d \]
Lorentz force

- A charged particle moving with velocity \( \dot{v} = \frac{\dot{p}}{m \gamma} \) through an electromagnetic field in vacuum experiences the Lorentz force \( \frac{d\dot{p}}{dt} = q(\dot{E} + \dot{v} \times \dot{B}) \).

- The total energy of this particle is \( W = \sqrt{(mc^2)^2 + (pc)^2} = \gamma mc^2 \), the kinetic energy is \( W_{kin} = mc^2(\gamma - 1) \).

- The role of acceleration is to increase \( W \).

- Change of \( W \) (by differentiation):
  \[
  W dW = c^2 \dot{p} \cdot d\dot{p} = q c^2 \dot{p} \cdot (\dot{E} + \dot{v} \times \dot{B}) dt = q c^2 \dot{p} \cdot \dot{E} dt
  
  dW = q \dot{v} \cdot \dot{E} dt
  \]

Note: Only the electric field can change the particle energy!
Maxwell’s equations (in vacuum)

\[
\begin{align*}
\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} &= \mu_0 \vec{J} \quad &\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} &= 0 \quad &\nabla \cdot \vec{E} &= \mu_0 c^2 \rho
\end{align*}
\]

1. Why not DC?

DC \((\frac{\partial}{\partial t} \equiv 0)\): \(\nabla \times \vec{E} = 0\), which is solved by \(\vec{E} = -\nabla \Phi\)

Limit: If you want to gain 1 MeV, you need a potential of 1 MV!

2. Circular machine: DC acceleration impossible since \(\oint \vec{E} \cdot d\vec{s} = 0\)

With time-varying fields:

\[
\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}, \quad \oint \vec{E} \cdot d\vec{s} = -\iiint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}.
\]
Maxwell’s equations in vacuum (continued)

Source-free:
\[
\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0
\]
\[
\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 \quad \nabla \cdot \vec{E} = 0
\]

curl (rot, \(\nabla \times\)) of 3\textsuperscript{rd} equation and \(\frac{\partial}{\partial t}\) of 1\textsuperscript{st} equation:
\[
\nabla \times \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0.
\]

Using the vector identity \(\nabla \times \nabla \times \vec{E} = \nabla \nabla \cdot \vec{E} - \nabla^2 \vec{E}\) and the 4\textsuperscript{th} Maxwell equation, this yields:
\[
\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0,
\]

i.e. the 4-dimensional Laplace equation.
From waveguide to cavity
Homogeneous plane wave

\[ \vec{E} \propto \vec{u}_y \cos(\omega t - \vec{k} \cdot \vec{r}) \]
\[ \vec{B} \propto \vec{u}_x \cos(\omega t - \vec{k} \cdot \vec{r}) \]
\[ \vec{k} \cdot \vec{r} = \frac{\omega}{c} (z \cos \varphi + x \sin \varphi) \]

**Wave vector \( \vec{k} \):**
- the direction of \( \vec{k} \) is the direction of propagation,
- the length of \( \vec{k} \) is the phase shift per unit length.
- \( \vec{k} \) behaves like a vector.

\[ k_\perp = \frac{\omega c}{c} \]
\[ k = \frac{\omega}{c} \]
\[ k_z = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega c}{\omega}\right)^2} \]
Wave length, phase velocity

- The components of $\vec{k}$ are related to the wavelength in the direction of that component as $\lambda_z = \frac{2\pi}{k_z}$ etc., to the phase velocity as $v_{\varphi,z} = \frac{\omega}{k_z} = f\lambda_z$. 

$$k_{\perp} = \frac{\omega_c}{c}$$

$$k_z = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

$$k = \frac{\omega}{c}$$
Superposition of 2 homogeneous plane waves

\[ + = \]

Metallic walls may be inserted where \( E_y \equiv 0 \)
without perturbing the fields.
Note the standing wave in \( x\)-direction!

This way one gets a hollow rectangular waveguide.
Rectangular waveguide

Fundamental (TE$_{10}$ or H$_{10}$) mode in a standard rectangular waveguide.

**Example 1:** “S-band”: 2.6 GHz ... 3.95 GHz,

Waveguide type WR284 (2.84” wide), dimensions: 72.14 mm x 34.04 mm.
cut-off: $f_c = 2.078$ GHz.

**Example 2:** “L-band”: 1.13 GHz ... 1.73 GHz,

Waveguide type WR650 (6.5” wide), dimensions: 165.1 mm x 82.55 mm.
cut-off: $f_c = 0.908$ GHz.

Both these pictures correspond to operation at $1.5 f_c$.

power flow: $\frac{1}{2} \text{Re} \left\{ \iint \vec{E} \times \vec{H}^* \cdot d\vec{A} \right\}$
Waveguide dispersion

What happens with different waveguide dimensions (different width $a$)?

1: $a = 52$ mm, $f/f_c = 1.04$

2: $a = 72.14$ mm, $f/f_c = 1.44$

3: $a = 144.3$ mm, $f/f_c = 2.88$

$$k = \frac{\omega}{c}$$

$$k_z = \frac{2\pi}{\lambda_g} = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}$$

cutoff: $f_c = \frac{c}{2a}$
Phase velocity $v_{\varphi,z}$

The phase velocity $v_{\varphi,z}$ is the speed at which the crest (or zero-crossing) travels in $z$-direction.

Note on the 3 animations on the right that, at constant $f$, $v_{\varphi,z} \propto \lambda_g$. Note also that at $f = f_c$, $v_{\varphi,z} = \infty$!

With $v \to \infty$, $v_{\varphi,z} \to c$!

1: $a = 52$ mm, $f/f_c = 1.04$

2: $a = 72.14$ mm, $f/f_c = 1.44$

3: $a = 144.3$ mm, $f/f_c = 2.88$

**cutoff:** $f_c = \frac{c}{2a}$

$k = \frac{\omega}{c}$

$k_z = \frac{2\pi}{\lambda_g} = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = \frac{\omega}{v_{\varphi,z}}$
Summary waveguide dispersion and phase velocity:

In a **general** cylindrical waveguide:

\[
k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_{\perp}^2} = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}
\]

Propagation in \(z\)-direction: \(\propto e^{j(\omega t - k_z z)}\)

\[
Z_0 = \frac{\omega \mu}{k_z} \text{ for TE, } Z_0 = \frac{k_z}{\omega \varepsilon} \text{ for TE}
\]

\[
k_z = \frac{2\pi}{\lambda_g}
\]

Example: TE10-mode in a rectangular waveguide of width \(a\):

\[
k_{\perp} = \frac{\pi}{a}
\]

\[
\gamma = j \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2}
\]

\[
Z_0 = \frac{\omega \mu}{k_z}
\]

\[
\lambda_{\text{cutoff}} = 2a.
\]

In a hollow waveguide: phase velocity \(v_\varphi > c\), group velocity \(v_{gr} < c\), \(v_{gr} \cdot v_\varphi = c^2\).
Rectangular waveguide modes

plotted: $E$-field
Some more standard rectangular Waveguides

<table>
<thead>
<tr>
<th>Waveguide name</th>
<th>Recommended frequency band of operation (GHz)</th>
<th>Cutoff frequency of lowest order mode (GHz)</th>
<th>Cutoff frequency of next mode (GHz)</th>
<th>Inner dimensions of waveguide opening (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WR2300</td>
<td>0.32 — 0.45</td>
<td>0.257</td>
<td>0.513</td>
<td>23.0 × 11.5</td>
</tr>
<tr>
<td>WR1150</td>
<td>0.63 — 0.97</td>
<td>0.513</td>
<td>1.026</td>
<td>11.50 × 5.75</td>
</tr>
<tr>
<td>WR340</td>
<td>2.2 — 3.3</td>
<td>1.736</td>
<td>3.471</td>
<td>3.40 × 1.70</td>
</tr>
<tr>
<td>WR75</td>
<td>10 — 15</td>
<td>7.869</td>
<td>15.737</td>
<td>0.75 × 0.375</td>
</tr>
<tr>
<td>WR10</td>
<td>75 — 110</td>
<td>59.015</td>
<td>118.03</td>
<td>0.10 × 0.05</td>
</tr>
<tr>
<td>WR3</td>
<td>220 — 330</td>
<td>173.571</td>
<td>347.143</td>
<td>0.034 × 0.017</td>
</tr>
</tbody>
</table>

![Waveguide Diagram](source: Eric Montesinos/CERN)

**Peak Power vs Frequency**

- WR2300
- WR2100
- WR1800
- WR1500
- WR1150
- WR975

courtesy: Eric Montesinos/CERN
Radial waves

Also radial waves may be interpreted as superposition of plane waves. The superposition of an outward and an inward radial wave can result in the field of a round hollow waveguide.

\[
E_z \propto H_n^{(2)}(k_r \rho) \cos(n\varphi) \quad E_z \propto H_n^{(1)}(k_r \rho) \cos(n\varphi) \quad E_z \propto J_n(k_r \rho) \cos(n\varphi)
\]
Round waveguide

$\frac{f_c}{a}$ GHz $= \begin{cases} 87.9 & \text{TE}_{11} \text{ fundamental} \\ 114.8 & \text{TM}_{01} \text{ axial field} \\ 182.9 & \text{TE}_{01} \text{ low loss} \end{cases}$

$f/f_c = 1.44$
Circular waveguide modes

- TE_{11}
- TE_{21}
- TE_{31}
- TE_{01}
- TM_{01}
- TM_{11}

Plotted: $E$-field
**General waveguide equations:**

Transverse wave equation (membrane equation): \( \Delta T + \left( \frac{\omega_c}{c} \right)^2 T = 0. \)

<table>
<thead>
<tr>
<th></th>
<th><strong>TE (or H-)</strong> modes</th>
<th><strong>TM (or E-)</strong> modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary condition:</td>
<td>( \vec{n} \cdot \nabla T = 0 )</td>
<td>( T = 0 )</td>
</tr>
<tr>
<td>Longitudinal wave equations (transmission line equations):</td>
<td>( \frac{dU(z)}{dz} + jk_z Z_0 I(z) = 0 )</td>
<td>( \frac{dI(z)}{dz} + jk_z U(z) = 0 )</td>
</tr>
<tr>
<td>Propagation constant:</td>
<td>( k_z = \frac{\omega}{c} \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2} )</td>
<td></td>
</tr>
<tr>
<td>Characteristic impedance:</td>
<td>( Z_0 = \frac{\omega \mu}{k_z} )</td>
<td>( Z_0 = \frac{k_z}{\omega \varepsilon} )</td>
</tr>
<tr>
<td>Ortho-normal eigenvectors:</td>
<td>( \vec{e} = \vec{u}_z \times \nabla T )</td>
<td>( \vec{e} = -\nabla T )</td>
</tr>
<tr>
<td>Transverse fields:</td>
<td>( \vec{E} = U(z) \vec{e} )</td>
<td>( \vec{H} = I(z) \vec{u}_z \times \vec{e} )</td>
</tr>
<tr>
<td>Longitudinal fields:</td>
<td>( H_z = \left( \frac{\omega_c}{\omega} \right)^2 \frac{T}{I(z)} \frac{U(z)}{j\omega \mu} )</td>
<td>( E_z = \left( \frac{\omega_c}{\omega} \right)^2 \frac{T}{I(z)} \frac{I(z)}{j\omega \varepsilon} )</td>
</tr>
</tbody>
</table>
### Special cases: rectangular and round waveguide

#### Rectangular waveguide: transverse eigenfunctions

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE (H-) modes:</td>
<td>[ T_{mn}^{(H)} = \frac{1}{\pi} \sqrt{\frac{a b \varepsilon_m \varepsilon_n}{(mb)^2 + (na)^2}} \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) ]</td>
</tr>
<tr>
<td>TM (E-) modes:</td>
<td>[ T_{mn}^{(E)} = \frac{2}{\pi} \sqrt{\frac{a b}{(mb)^2 + (na)^2}} \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) ]</td>
</tr>
</tbody>
</table>

#### Round waveguide: transverse eigenfunctions

<table>
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<tr>
<th>Mode Type</th>
<th>Expression</th>
</tr>
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<tbody>
<tr>
<td>TE (H-) modes:</td>
<td>[ T_{mn}^{(H)} = \frac{\varepsilon_m}{\sqrt{\pi (\chi_{mn}^2 - m^2)}} \frac{J_m \left( \chi_{mn}^r \frac{\rho}{a} \right)}{J_m \left( \chi_{mn}^l \right)} \left{ \cos(m\varphi) \right} \left{ \sin(m\varphi) \right} ]</td>
</tr>
<tr>
<td>TM (E-) modes:</td>
<td>[ T_{mn}^{(E)} = \frac{\varepsilon_m}{\sqrt{\pi}} \frac{J_m \left( \chi_{mn}^r \frac{\rho}{a} \right)}{J_{m-1} \left( \chi_{mn} \right)} \left{ \sin(m\varphi) \right} \left{ \cos(m\varphi) \right} ]</td>
</tr>
</tbody>
</table>

where in both cases \( \varepsilon_i = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i \neq 0 \end{cases} \)
Waveguide perturbed by notches

Reflections from notches lead to a superimposed standing wave pattern. “Trapped mode”
Short-circuited waveguide

$TM_{010}$ (no axial dependence)

$TM_{011}$

$TM_{012}$
Single WG mode between two shorts

Eigenvalue equation for field amplitude $a$:

$$a = e^{-j k z \ell} a$$

Non-vanishing solutions exist for $2 k_z \ell = 2 \pi m$:

With $k_z = \frac{\omega}{c} \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2}$, this becomes $f_0^2 = f_c^2 + \left( c \frac{m}{2 \ell} \right)^2$. 
Simple pillbox (only 1/2 shown)

*TM*$_{010}$-mode

- Electric field (purely axial)
- Magnetic field (purely azimuthal)
Pillbox cavity field (w/o beam tube)

$$T(\rho, \varphi) = \frac{1}{\sqrt{\pi}} \frac{J_0 \left( \frac{\chi_0 \rho}{a} \right)}{\chi_0 J_1 \left( \frac{\chi_0}{a} \right)}$$

with $\chi_0 = 2.40483 \ldots$

The only non-vanishing field components:

$$E_z = \frac{1}{j \omega \varepsilon} \frac{\chi_0}{a} \sqrt{\frac{1}{\pi}} \frac{J_0 \left( \frac{\chi_0 \rho}{a} \right)}{a J_1 \left( \frac{\chi_0}{a} \right)}$$

$$B_\varphi = \mu_0 \sqrt{\frac{1}{\pi}} \frac{J_1 \left( \frac{\chi_0 \rho}{a} \right)}{a J_1 \left( \frac{\chi_0}{a} \right)}$$

$$\omega_0 |_{\text{pillbox}} = \frac{\chi_0 c}{a}, \quad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \ \Omega$$

$$\frac{Q}{\Omega} |_{\text{pillbox}} = \frac{\sqrt{2a \eta \sigma \chi_0}}{2 \left( 1 + \frac{a}{h} \right)}$$

$$\frac{R}{\frac{Q}{\Omega}} |_{\text{pillbox}} = \frac{4 \eta \sin^2 \left( \frac{\chi_0 h}{2a} \right)}{\chi_0^3 \pi J_1^2 (\chi_0)} h/a$$
Pillbox with beam pipe

One needs a hole for the beam passage – circular waveguide below cutoff

**TM**$_{010}$-mode (only 1/4 shown)

- Electric field
- Magnetic field
A more practical pillbox cavity

Rounding of sharp edges (to reduce field enhancement!)

TM_{010}\text{-mode} \quad \text{(only 1/4 shown)}

electric field

magnetic field
A (real) elliptical cavity

$\text{TM}_{010}$-mode  (only 1/4 shown)

Electric field  Magnetic field
Choice of frequency

• **Size:**
  - Linear dimensions scale as $f^{-1}$, volume as $f^{-3}$.
  - amount of material, mass, stiffness, tolerances, ...
  - Outer radius of elliptical cavity $\sim 0.45 \lambda$.

• **Beam interaction:**
  - $r/Q$ increases with $f$ – but also for HOMs!
  - short bunches are easier with higher $f$.

• **Technology:**
  - superconducting: BCS resistance $\propto f^2$.
  - Power sources available?
  - Max. accelerating voltage?
Characterizing a cavity
Acceleration voltage and $R/Q$

- I define
  \[ V_{acc} = \int_{-\infty}^{\infty} E_z e^{j \frac{\omega}{\beta c} z} \, dz. \]

- The exponential factor accounts for the variation of the field while particles with velocity $\beta c$ are traversing the cavity gap.

- With this definition, $V_{acc}$ is generally complex – this becomes important with more than one gap (cell).

- For the time being we are only interested in $|V_{acc}|$.

- The square of the acceleration voltage $|V_{acc}|^2$ is proportional to the stored energy $W$; the proportionality constant defines the quantity called “$R$-upon-$Q$”:
  \[ \frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}. \]

› Attention – different definitions are used in literature!
Transit time factor

- The transit time factor is the ratio of the acceleration voltage to the (non-physical) voltage a particle with infinite velocity would see:

\[
TT = \left| \frac{V_{acc}}{\int E_z \, dz} \right| = \left| \frac{\int E_z e^{i\beta c z} \, dz}{\int E_z \, dz} \right|
\]

- The transit time factor of an ideal pillbox cavity (no axial field dependence) of height (gap length) \( h \) is:

\[
TT = \frac{\sin \left( \frac{\chi_{01} h}{2a} \right)}{\frac{\chi_{01} h}{2a}}
\]

(remember: \( \omega_0 = \frac{2\pi c}{\lambda} = \frac{\chi_{01} c}{a} \))

Field rotates by 360° during particle passage.
Stored energy

- The energy stored in the electric field is
  \[ W_E = \iiint_{\text{cavity}} \frac{\varepsilon}{2} |\vec{E}|^2 \, dV. \]

- The energy stored in the magnetic field is
  \[ W_M = \iiint_{\text{cavity}} \frac{\mu}{2} |\vec{H}|^2 \, dV. \]

- Since \( \vec{E} \) and \( \vec{H} \) are 90° out of phase, the stored energy continuously swaps from electric energy to magnetic energy.

- On average, electric and magnetic energy must be equal.

- In steady state, the Poynting vector describes this energy flux.

- In steady state, the total energy stored (constant) is
  \[ W = \iiint_{\text{cavity}} \left( \frac{\varepsilon}{2} |\vec{E}|^2 + \frac{\mu}{2} |\vec{H}|^2 \right) \, dV. \]
Stored energy and Poynting vector

- Electric field energy
- Poynting vector
- Magnetic field energy
Wall losses & $Q_0$

- The losses $P_{\text{loss}}$ are proportional to the stored energy $W$.

- The tangential $\vec{H}$ on the surface is linked to a surface current $\vec{j}_A = \vec{n} \times \vec{H}$ (flowing in the skin depth $\delta = \sqrt{2/ (\omega \mu \sigma)}$).

- This surface current $\vec{j}_A$ sees a surface resistance $R_s$, resulting in a local power density $R_s |H_t|^2$ flowing into the wall.

- $R_s$ is related to skin depth $\delta$ as $\delta \sigma R_s = 1$.
  - Cu at 300 K has $\sigma \approx 5.8 \cdot 10^7$ S/m, leading to $R_s \approx 8$ mΩ at 1 GHz, scaling with $\sqrt{\omega}$.
  - Nb at 2 K has a typical $R_s \approx 10$ nΩ at 1 GHz, scaling with $\omega^2$.

- The total wall losses result from $P_{\text{loss}} = \iint_{\text{wall}} R_s |H_t|^2 \, dA$.

- The cavity $Q_0$ (caused by wall losses) is defined as $Q_0 = \frac{\omega_0 W}{P_{\text{loss}}}$.

- Typical $Q_0$ values:
  - Cu at 300 K (normal-conducting): $\mathcal{O}(10^3 \ldots 10^5)$, should improve at cryogenic $T$ by a factor $10^3 \ldots 10^5$.
  - Nb at 2 K (superconducting): $\mathcal{O}(10^9 \ldots 10^{11})$
Shunt impedance

- Also the power loss $P_{\text{loss}}$ is also proportional to the square of the acceleration voltage $|V_{\text{acc}}|^2$; the proportionality constant defines the “shunt impedance”

$$R = \frac{|V_{\text{acc}}|^2}{2 P_{\text{loss}}}.
$$

› **Attention, also here different definitions are used!**

- Traditionally, the shunt impedance is the quantity to optimize in order to minimize the power required for a given gap voltage.
- Now the previously introduced term “$R$-upon-$Q$” makes sense:

$$\left(\frac{R}{Q}\right) = R/Q$$
Geometric factor

- With

\[ Q_0 = \frac{\omega_0 W}{\iint_{\text{wall}} R_s |H_t|^2 \, dA}, \]

and assuming an average surface resistance \( R_s \), one can introduce the “geometric factor” \( G \) as

\[ G = Q_0 \cdot R_s = \frac{\omega_0 W}{\iint_{\text{wall}} |H_t|^2 \, dA}. \]

- \( G \) has dimension Ohm, depends only on the cavity geometry (as the name suggests) and typically is \( \mathcal{O}(100 \, \Omega) \).

- Note that \( R_s \cdot R = G \cdot (R/Q) \) (dimension \( \Omega^2 \), purely geometric)

- \( G \) is only used for SC cavities.
Cavity resonator – equivalent circuit

Simplification: single mode

\[ R / \beta \quad C \quad L \quad R \]

\[ \beta: \text{coupling factor} \]
\[ R: \text{shunt impedance} \]
\[ \sqrt{L / C} = \frac{R}{Q}: R \text{-upon-} Q \]
Power coupling - Loaded $Q$

- Note that the generator inner impedance also loads the cavity – for very large $Q_0$ more than the cavity wall losses.

- To calculate the loaded $Q$ ($Q_L$), losses have to be added:
  \[
  \frac{1}{Q_L} = \frac{P_{\text{loss}} + P_{\text{ext}} + \cdots}{\omega_0 W} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} + \frac{1}{\cdots}.
  \]

- The coupling factor $\beta$ is the ratio $P_{\text{ext}}/P_{\text{loss}}$.

- With $\beta$, the loaded $Q$ can be written
  \[
  Q_L = \frac{Q_0}{1 + \beta}.
  \]

- For NC cavities, often $\beta = 1$ is chosen (power amplifier matched to empty cavity); for SC cavities, $\beta = \mathcal{O}(10^4 \ldots 10^6)$. 

While a high \( Q_0 \) results in small wall losses, so less power is needed for the same voltage. 

On the other hand the bandwidth becomes very narrow.

Note: a 1 GHz cavity with a \( Q_0 \) of \( 10^{10} \) has a natural bandwidth of 0.1 Hz!

... to make this manageable, \( Q_{\text{ext}} \) is chosen much smaller!
Loss factor

\[ k_{\text{loss}} = \frac{\omega_0}{2} \left( \frac{R}{Q} \right) = \frac{|V_{\text{acc}}|^2}{4W} = \frac{1}{2C} \]

Energy deposited by a single charge \( q \): \( k_{\text{loss}}q^2 \)

Voltage induced by a single charge \( q \): \( 2k_{\text{loss}}q \)

Voltage induced by a single charge \( q \):

\[ \frac{V_{\text{acc}}}{2k_{\text{loss}}q} \]

Impedance seen by the beam

\[ L = R/(Q_0\omega_0) \]

\[ C = Q_0/(R\omega_0) \]

\[ f_0t \]
Summary: relations $V_{acc}$, $W$ and $P_{loss}$

Attention – different definitions are used in literature!

$V_{acc}$
Accelerating voltage

$$ R \frac{Q}{2} = \frac{|V_{acc}|^2}{2\omega_0 W} $$

$k_{loss} = \frac{\omega_0 R}{2Q} = \frac{|V_{acc}|^2}{4W}$

$$ R = \frac{|V_{acc}|^2}{2P_{loss}} = \frac{R}{Q} Q_0 $$

$W$
Energy stored

$P_{loss}$
wall losses

$$ Q_0 = \frac{\omega_0 W}{P_{loss}} $$
Beam loading

• The beam current “loads” the cavity, in the equivalent circuit this appears as an impedance in parallel to the shunt impedance.

• If the generator is matched to the unloaded cavity ($\beta = 1$), beam loading will (normally) cause the accelerating voltage to decrease.

• The power absorbed by the beam is $-\frac{1}{2} \Re\{V_{acc}I_B^*\}$.

• For high power transfer efficiency RF $\rightarrow$ beam, beam loading must be high!

• For SC cavities (very large $\beta$), the generator is typically matched to the beam impedance!

• Variation in the beam current leads to transient beam loading, which requires special care!

• Often the “impedance” the beam presents is strongly reactive – this leads to a detuning of the cavity.
The words “multipactor”, “to multipact” and “multipacting” are artificially composed of “multiple” “impact”.

Multipactor describes a resonant RF phenomenon in vacuum:
- Consider a free electron in a simple cavity – it gets accelerated by the electric field towards the wall
- when it impacts the wall, secondary electrons will be emitted, described by the secondary emission yield (SEY)
- in certain impact energy ranges, more than one electron is emitted for one electron impacting! So the number of electrons can increase
- When the time for an electron from emission to impact takes exactly ½ of the RF period, resonance occurs – with the SEY>1, this leads to an avalanche increase of electrons, effectively taking all RF power at this field level, depleting the stored energy and limiting the field!

For this simple “2-point MP”, this resonance condition is reached at \( \frac{1}{4\pi} \frac{e}{m} V = (fd)^2 \) or \( \frac{V}{112} = \left( \frac{f}{MHz} \frac{d}{m} \right)^2 \). There exist other resonant bands.

courtesy: Sarah Aull/CERN
Multipactor (contd.)

• Unfortunately, good metallic conductors (Cu, Ag, Nb) all have SEY>1!

• 1-point MP occurs when the electron impact where they were emitted

• Electron trajectories can be complex since both $\vec{E}$ and $\vec{B}$ influence them; computer simulations allow to determine the MP bands (barriers)

• To reduce or suppress MP, a combination of the following may be considered:
  - Use materials with low SEY
  - Optimize the shape of your cavity (→ elliptical cavity)
  - Conditioning (surface altered by exposure to RF fields)
  - Coating (Ti, TiN, NEG, amorphous C …)
  - Clearing electrode (for a superimposed DC electric field)
  - Rough surfaces
Many gaps
What do you gain with many gaps?

- The $R/Q$ of a single gap cavity is limited to some 100 Ω.

Now consider to distribute the available power to $n$ identical cavities: each will receive $P/n$, thus produce an accelerating voltage of $\sqrt{2RP/n}$. (Attention: phase important!)

The total accelerating voltage thus increased, equivalent to a total equivalent shunt impedance of $nR$.

\[
|V_{acc}| = n \sqrt{2RP/n} = \sqrt{2(nR)P}
\]
Standing wave multi-cell cavity

- Instead of distributing the power from the amplifier, one might as well couple the cavities, such that the power automatically distributes, or have a cavity with many gaps (e.g. drift tube linac).

- Coupled cavity accelerating structure (side coupled)

- The phase relation between gaps is important!
Brillouin diagram; Travelling wave structure

\[ \omega L / c \]

speed of light line, \( \omega = \beta / c \)

synchronous

\[ \beta \]

\[ L \]
The elliptical cavity

- The elliptical shape was found as optimum compromise between
  - maximum gradient \( \frac{E_{\text{acc}}}{E_{\text{surface}}} \)
  - suppression of multipactor
  - mode purity
  - machinability

- A multi-cell elliptical cavity is typically operated in \( \pi \)-mode, i.e. cell length is exactly \( \beta \lambda / 2 \).

- It has become de facto standard, used for ions and leptons! E.g.:
  - ILC/X-FEL: 1.3 GHz, 9-cell cavity
  - SNS (805 MHz)
  - SPL/ESS (704 MHz)
  - LHC (400 MHz)

\[ R_0 = 103.3 \text{ mm}, \quad 2L = 115.3 \text{ mm}. \]

\[ D. \text{ Proch, 1993} \]

*): \( \text{http://accelconf.web.cern.ch/accelConf/SRF93/papers/srf93g01.pdf} \)
Elliptical cavity – the *de facto* standard for SRF

- **FERMI 3.9 GHz**
- **CEBAF 1.5 GHz**
- **HEPL 1.3 GHz**
- **S-DALINAC 3 GHz**
- **CESR 0.5 GHz**
- **TESLA/ILC 1.3 GHz**
- **SNS $\beta = 0.61, 0.81, 0.805$ GHz**
- **KEK-B 0.5 GHz**
- **HERA 0.5 GHz**
- **TRISTAN 0.5 GHz**
- **LEP 0.352 GHz**

*cells*
Practical RF parameters 1

- Average accelerating gradient: \( E_{acc} = \frac{\sqrt{\omega W (R/Q)}}{l_{active}} \)

The ratio shows sensitivity of the shape to the field emission of electrons.

The ratio shows limit in \( E_{acc} \) due to the breakdown of superconductivity (quench, Nb: \( \approx 190 \text{ mT} \)).

courtesy: Jacek Sekutovicz/DESY
Practical RF parameters 2

\[ G \cdot (R/Q) \]

- Both \( G \) and \( R/Q \) are purely geometric parameters.
- Like the shunt impedance \( R \), the product \( G \cdot (R/Q) \) is a measure of the power loss for given acceleration voltage \( V_{acc} \) and surface resistance \( R_s \).

\[
P_{loss} = \frac{|V_{acc}|^2 R_s}{2 \cdot G \cdot (R/Q)}
\]

Minimize \( R_s \):
- operation at lower \( T \),
- better surface cleanliness,
- lower residual resistance

Optimize geometry maximizing \( G \cdot (R/Q) \).

courtesy: Jacek Sekutovicz/DESY
Single-cell versus multi-cell cavities

- Advantages of single-cell cavities:
  - It is easier to manage HOM damping
  - There is no field flatness problem.
  - Input coupler transfers less power
  - They are easy for cleaning and preparation

- Advantages of multi-cell cavities:
  - much more acceleration per meter!
  - better use of the power \((R \rightarrow n \ R)\)
  - more cost-effective for most applications

courtesy: Jacek Sekutovicz/DESY
Practical RF parameters 3

- **Cell-to-cell coupling** $k_{cc}$ will determine the width of the passbands in multi-cell cavities.

$$k_{cc} = 2 \frac{\omega_\pi - \omega_0}{\omega_\pi + \omega_0}$$

Brillouin diagram

0-mode

$\omega_0 L/c$

$\pi/2$

$\omega_L/c$

$\pi$

$2\pi$

$\omega_\pi L/c$

$\pi$

$k_z L$

courtesy: Jacek Sekutovicz/DESY
Field flatness

- Field amplitude variation from cell to cell in a multi-cell structure
- Should be small for maximum acceleration.

Field flatness sensitivity factor $a_{ff}$ for a structure made of $N$ cells:

$$\frac{\Delta A_i}{A_i} = a_{ff} \frac{\Delta f_i}{f_i}$$

$a_{ff}$ is related to the cell-to-cell coupling as $a_{ff} = \frac{N^2}{k_{cc}}$ and describes the sensitivity of the field flatness on the errors in individual cells. 

courtesy: Jacek Sekutovicz/DESY
Criteria for Cavity Design (1)

• Here: Inner cells of multi-cell structures

• Parameters for optimization:
  • Fundamental mode: \( \frac{R}{Q'} G, \frac{E_{\text{peak}}}{E_{\text{acc}}}, \frac{B_{\text{peak}}}{E_{\text{acc}}}, k_{cc}. \)
  • Higher order modes: \( k_\perp, k_z. \)

• The elliptical cavity design has distinct advantages:
  • easy to clean (rinse)
  • little susceptible to MP – can be conditioned ...

• Geometric parameters for optimization:
  • iris ellipse half axes: \( a, b: \)
  • iris aperture radius: \( r_i, \)
  • equator ellipse half axes: \( A, B \)

• Problem: 7 parameters to optimize, only 5 to play with – some compromise has to be found!

courtesy: Jacek Sekutovicz/DESY
We see here that $r_i$ is a very “powerful” variable to trim the RF-parameters of a cavity. Of course it has to fit the aperture required for the beam!
Effect of $r_i$

- Smaller $r_i$ allows to concentrate $E_z$ where it is needed for acceleration

$E_z(z)$ for small and big iris radius

courtesy: Jacek Sekutovicz/DESY
Example: cell optimization at 1.5 GHz

A. Mosnier, E. Haebel, SRF Workshop 1991
Equator shape optimization

- $B_{\text{peak}}/E_{\text{acc}}$ (and $G$) change when changing the equator shape.

courtesy: Jacek Sekutovicz/DESY
Iris shape optimization

- $E_{\text{peak}} / E_{\text{acc}}$ changes with the iris shape

Both cells have the same: $f_0$, $R/Q$, and $r_i$. 

courtesy: Jacek Sekutovicz/DESY
Minimizing HOM excitation

HOMs loss factors \( (k_{\text{loss,\perp}}, k_{\text{loss}}) \)

\[
\begin{align*}
R/Q &= 152 \ \Omega \\
B_{\text{peak}}/E_{\text{acc}} &= 3.5 \text{ mT}/(\text{MV/m}) \\
E_{\text{peak}}/E_{\text{acc}} &= 1.9
\end{align*}
\]

\[
\begin{align*}
R/Q &= 86 \ \Omega \\
B_{\text{peak}}/E_{\text{acc}} &= 4.6 \text{ mT}/(\text{MV/m}) \\
E_{\text{peak}}/E_{\text{acc}} &= 3.2
\end{align*}
\]

courtesy: Jacek Sekutovicz/DESY

\( r_i = 20 \text{ mm} \)

\( r_i = 40 \text{ mm} \)
Cell-to-cell coupling $k_{cc}$

$R/Q = 152 \, \Omega$
$B_{\text{peak}}/E_{\text{acc}} = 3.5 \, \text{mT}/(\text{MV/m})$
$E_{\text{peak}}/E_{\text{acc}} = 1.9$

$R/Q = 86 \, \Omega$
$B_{\text{peak}}/E_{\text{acc}} = 4.6 \, \text{mT}/(\text{MV/m})$
$E_{\text{peak}}/E_{\text{acc}} = 3.2$

courtesy: Jacek Sekutovicz/DESY
Scaling the frequency

\[ \times 2 = \]

\[
\begin{array}{|c|c|}
\hline
f_\pi & [\text{MHz}] \\
\hline
2600 & \\
\hline
R/Q & [\Omega] \\
\hline
57 & \\
\hline
r/Q & [\Omega/\text{m}] \\
\hline
2000 & \\
\hline
G & [\Omega] \\
\hline
271 & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
f_\pi & [\text{MHz}] \\
\hline
1300 & \\
\hline
R/Q & [\Omega] \\
\hline
57 & \\
\hline
r/Q & [\Omega/\text{m}] \\
\hline
1000 & \\
\hline
G & [\Omega] \\
\hline
271 & \\
\hline
\end{array}
\]

\[ r/Q = (R/Q)/l \propto f \]

(or \( R/Q / \lambda = \text{const} \))

courtesy: Jacek Sekutovicz/DESY
Operating temperature

At the XFEL gradient, $Q_0$ is higher by 80% at 1.8 K

courtesy: Jacek Sekutovicz/DESY
Historic evolution of inner cell geometry

Example: ILC

<table>
<thead>
<tr>
<th>$r_i$</th>
<th>[mm]</th>
<th>35</th>
<th>30</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{cc}$</td>
<td>[%]</td>
<td>1.9</td>
<td>1.56</td>
<td>1.52</td>
</tr>
<tr>
<td>$E_{\text{peak}}/E_{\text{acc}}$</td>
<td>-</td>
<td>1.98</td>
<td>2.30</td>
<td>2.36</td>
</tr>
<tr>
<td>$B_{\text{peak}}/E_{\text{acc}}$</td>
<td>[mT/(MV/m)]</td>
<td>4.15</td>
<td>3.57</td>
<td>3.61</td>
</tr>
<tr>
<td>$R/Q$</td>
<td>[Ω]</td>
<td>113.8</td>
<td>135</td>
<td>133.7</td>
</tr>
<tr>
<td>$G$</td>
<td>[Ω]</td>
<td>271</td>
<td>284.3</td>
<td>284</td>
</tr>
<tr>
<td>$R/Q \cdot G$</td>
<td>[Ω*Ω]</td>
<td>30840</td>
<td>38380</td>
<td>37970</td>
</tr>
<tr>
<td>$k_{\text{loss}} \perp (\sigma_z = 1 \text{ mm})$</td>
<td>[V/pC/cm²]</td>
<td>0.23</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>$k_{\text{loss}} (\sigma_z = 1 \text{ mm})$</td>
<td>[V/pC]</td>
<td>1.46</td>
<td>1.75</td>
<td>1.72</td>
</tr>
</tbody>
</table>

courtesy: Jacek Sekutovicz/DESY
Cavity optimization example

courtesy: Frank Marhauser/JLAB
Higher order modes

external dampers

\[ R_1, Q_1, \omega_1 \]
\[ R_2, Q_2, \omega_2 \]
\[ R_3, Q_3, \omega_3 \]

\[ n_1 \]
\[ n_2 \]
\[ n_3 \]

\[ I_B \]
Pillbox: $1^{st}$ dipole mode

$\text{TM}_{110}$-mode  (only 1/4 shown)

electric field

magnetic field
Panofsky-Wenzel theorem

For particles moving virtually at \( v = c \), the integrated transverse force (kick) can be determined from the transverse variation of the integrated longitudinal force!

\[
j \frac{\omega}{c} \vec{F}_\perp = \nabla_\perp F_z
\]

Pure TE modes: No net transverse force!

Transverse modes are characterized by
- the transverse impedance in \( \omega \)-domain
- the transverse loss factor (kick factor) in \( t \)-domain!

CERN/PS 80 MHz cavity (for LHC)

inductive (loop) coupling, low self-inductance
Higher order modes

Example shown:
80 MHz cavity PS for LHC.
Color-coded:

\[ \vec{E} \]
Higher order modes (measured spectrum)

Without dampers

With dampers
7-cell 1.3 GHz structure for **bERLInPro**

Band diagram (top) and Q-factors (bottom)

Reminder:

- 0-mode
- π-mode

Galek et al.: IPAC2013
HOMs: Example 5-cell cavity

First passbands of HOMs typically dangerous

courtesy: Rama Calaga/CERN
HOM dampers

• Ferrite absorbers: broadband damper, room temperature
• Waveguides: better suited for higher frequencies (size!)
• Notch filters: narrow-band; target specific mode

- ferrite absorber
- waveguides
- notch filter
- bandpass filter
- double notch

• Multi-cell cavities require broadband dampers

• More tomorrow by Eric Montesinos!
Tuners
Small boundary perturbation

- Perturbation calculation is used to understand the basics for cavity tuning – it is used to analyse the sensitivity to (small) surface geometry perturbations.
  - This is relevant to understand the effect of fabrication tolerances.
  - Intentional surface deformation or introduced obstacles can be used to tune the cavity.

- The basic idea of the perturbation theory is use a known solution (in this case the unperturbed cavity) and assume that the deviation from it is only small. We just used this to calculate the losses (assuming $H_t$ would be that without losses).

- The result of this calculation leads to a convenient expression for the (de)tuning:

\[
\frac{\omega - \omega_0}{\omega} = \frac{\iiint_{\Delta V} (\mu_0 |H_0|^2 - \varepsilon |E_0|^2) \, dV}{\iiint_{V} (\mu_0 |H_0|^2 + \varepsilon |E_0|^2) \, dV}
\]

unperturbed: $\omega_0$

perturbed: $\omega$

**Slater’s Theorem**

John C. Slater

1900 – 1976
Lorentz force detuning ("LFD")

- The presence of electromagnetic fields inside the cavity lead to a mechanical pressure on the cavity.
- Radiation pressure: \( P = \frac{1}{4} (\mu_0 |H|^2 - \varepsilon_0 |E|^2) \)
- Deformation of the cavity shape:

\[ \Delta f = K L |E_{\text{acc}}|^2; \text{typical: } K L \approx - (1 \ldots 10) \text{Hz/} \left( \frac{\text{MV}}{\text{m}} \right)^2 \]

- This requires good stiffness – and the possibility to tune rapidly!
Tuner principle

• Slow tuners:
  • compensate for mechanical tolerances,
  • realized with stepper motor drives

• Fast tuners:
  • compensate Lorentz-force detuning and reactive beam loading
  • realized with piezo crystal (lead zirconate titanate – PZT)

• Tuning of SC cavities is often realized by deforming the cavity:

![Diagram of RF system with RF Power Source, Circulator, RF Transmission Line, Load, Cavity, Beam, and Tuner.](Image)

courtesy: Eiji Kako/KEK
Blade tuner

- Developed by INFN Milano
- Azimuthal motion transferred to longitudinal strain
- Zero backlash
- CuBe threaded shaft used for a screw nut system
- Stepping motor and gear combination driver
- Two piezo actuators for fast action
- All components in cold location

courtesy: Eiji Kako/KEK
“Saclay” lever-arm tuner

- Developed by DESY based on the Saclay design
- Double lever system (leverage 1.25)
- Cold stepping motor and gear combination
- Screw nut system
- Two piezo actuators for fast action in a preloaded frame
- All components in cold location

courtesy: Eiji Kako/KEK
Slide-jack tuner

- Developed by KEK for STF cryomodule
- Slide-jack mechanism
- Single high voltage piezo actuator for fast action
- Warm stepping motor for easy maintenance
- Access port for replacing piezo

courtesy: Eiji Kako/KEK
End of RF Basics and TM Cavities

Thank you very much!
FPC
(Fundamental Power Coupler)
Magnetic (loop) coupling

- The magnetic field of the cavity main mode is intercepted by a coupling loop.
- The coupling can be adjusted by changing the size or the orientation of the loop.
- Coupling: $\propto \iiint \vec{H} \cdot \vec{J}_m \, dV$

courtesy: David Alesini/INFN
Electric (antenna) coupling

- The inner conductor of the coaxial feeder line ends in an antenna penetrating into the electric field of the cavity.
- The coupling can be adjusted by varying the penetration.
- Coupling $\propto \int \int \int \mathbf{E} \cdot \mathbf{j} \, dV$

antenna
courtesy: David Alesini/INFN
The **Fundamental Power Coupler** is the connecting part between the RF transmission line and the RF cavity.

- It is a specific piece of transmission line that also has to provide the cavity vacuum barrier.
- FPCs are amongst the most critical parts of the RF cavity system in an accelerator!
- A good RF design, a good mechanical design and a high quality fabrication are essential for an efficient and reliable operation.

courtesy: Eric Montesinos/CERN