

Relevance of monopoles (topological defects) in QFT/QM

Both Abelian and non-Abelian monopoles (as well as Skyrmion in the low-energy limit of QCD) play a fundamental role in the strong coupling phase of gauge theories

Common features of monopoles, Skyrmions, Instantons (and others defects)

In all the cases in which we want to describe objects with topological charge and at the same time spherical symmetry

$$\pi = + + - - + - + + \dots + \text{admitable}$$

symmetry

Important: these two requirement are not compatible
in the naive sense

Example 1: Skyrmion

$$U = Y_0 \mathbb{I} + Y_k t^k; \quad t^k = i\sigma^k; \quad (Y_0)^2 + \sum_{k=1}^3 (Y_k)^2 = 1 \Rightarrow U \in \text{SU}(2)$$

$$B = \int d^3x \operatorname{tr} \{ (\bar{U}^\dagger \partial_i U) (\bar{U}^\dagger \partial_j U) (\bar{U}^\dagger \partial_k U) \} \epsilon_{ijk}$$

$$\downarrow \\ \{t = \text{const}\}$$

topological
charge

If we want naive spherical symmetry then the all the components of the matrix-valued functions $U = U(x^\mu)$ should actually depend only on the radial coordinate: $U = U(r)$

$$\text{However } U = U(r) \Rightarrow B = 0$$

However $U = U(r) \Rightarrow U =$

thus, the Hedgehog ansatz was introduced

$$\left\{ \begin{array}{l} Y_0 = \cos\alpha(r), \quad Y_i = \sin\alpha(r) n_i \\ n_1 = \sin\theta \cos\varphi, \quad n_2 = \sin\theta \sin\varphi, \quad n_3 = \cos\theta \end{array} \right. \quad \begin{array}{l} \text{It depends} \\ \text{explicitly on} \\ \text{the angles} \end{array}$$

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$\xrightarrow{\text{L}_{\vec{R}_{SO(3)}}} U \neq 0 \text{ but } \xrightarrow{\text{L}_{\vec{R}_{SO(3)}}} T_{00}[U] = 0$$

"Symmetry up to internal rotations"

The solitons is not spherically symmetric
but the lack of spherical symmetry can be
compensated by an internal rotation
so that the energy density is spherically symmetric

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Example 2: non-Abelian monopole

$$A_\mu = K(r) U_0^{-1} \partial_\mu U_0 \quad (\rightarrow \text{equivalent to 't Hooft notation})$$

$$A_i^a \sim K(r) \{ \epsilon_{ijk} \sigma^k \}$$

$$U_0 = n_k t^k;$$

$$n_1 = \sin\theta \cos\varphi; n_2 = \sin\theta \sin\varphi; n_3 = \cos\theta; \quad [\rightarrow A_\mu \text{ depends explicitly on the angles}]$$

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2);$$

$$\text{Also in this case } L_{R_{SO(3)}} A_\mu \neq 0 \text{ but } L_{T_{SO(3)}} [A_\mu] = 0$$

"symmetry up to internal rotation"

the non-Abelian monopole is not spherically symmetric but the lack of spherical symmetry ... is compensated by an internal (gauge) rotation

"J" can be compensated by an internal (gauge) rotation

$$Q_B \sim \int_{S_\infty} d^2 S^i E_{ijk} + \text{tr}\{(\bar{U}_0^\dagger \partial_j U_0)(\bar{U}_0^{-1} \partial_k U_0)\}$$

↳ Non-Abelian Magnetic charge

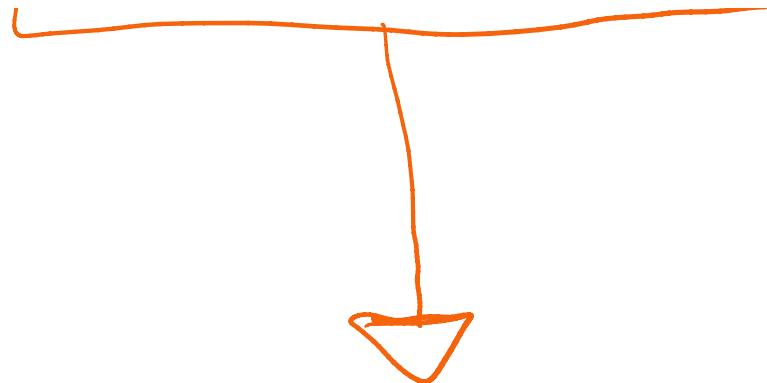
Also in this case, if the non-Abelian gauge potential would be strictly spherically symmetric then $Q_B = 0$. Far from the core ($r \rightarrow +\infty \sim IR$ region) the non-Abelian monopole coincides with the Dirac monopole.

Example 3: Dirac monopole

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$A = \frac{g}{4\pi} (\pm 1 + \cos\theta) d\varphi j$$

$$F = dA = -\frac{g}{4\pi} \sin\theta d\theta \wedge d\varphi$$



this is the volume form on the 2-sphere so that it is spherically symmetric

this is the gauge potential associated to the Dirac monopole and is not spherically symmetric

On the other hand, the lack of spherical symmetry can be compensated by an internal gauge transformation

$$L_{\frac{d}{dt}}_{R_{SO(3)}} A \neq 0, \left\{ L_{\frac{d}{dt}}_{R_{SO(3)}} + G.T. \right\} A = 0$$

thus, once again

"Spherical symmetry up to internal rotation"

To dimensional consequences of this very important

The physical consequences of this very important phenomenon is that in the presence of topological defects the generators of rotations are modified:
extra terms appear ...

I will consider the case of Dirac monopole
since the large- r behavior around Skyrmions
and non-Abelian monopole is the same