

Supersymmetric construction of self-consistent condensates in large N GN model: solitons on finite-gap potentials

arXiv:1712.08693

Adrián Arancibia



Abstract

In the present talk the set of stationary solutions of the Gross-Neveu model in t'Hooft limit is extended. Such extension is obtained striving a hidden supersymmetry associated to disconnected sets of stationary solutions.

It is shown how the supersymmetry arises from the Darboux-Miura transformations between Lax pairs of stationary modified Korteweg-de Vries and the stationary Korteweg-de Vries hierarchies, associating the correspondent supercharges to self-consistent condensates for the Gross-Neveu model

The Gross-Neveu model in Large N limit

The Gross-Neveu model corresponds to a quantum field theory for nonlinear interacting fermions without mass. The model presents some interesting properties: dynamical mass generation, is asymptotically free and presents a spontaneous breaking of symmetry..

The GN model, is described by the Lagrangian:

$$\mathcal{L}_{\text{GN}} = \sum_{j=1}^N \bar{\psi}_j i \not{\partial} \psi_j + \frac{g^2}{2} \left(\sum_{j=1}^N \bar{\psi}_j \psi_j \right)^2$$

$\psi_j, j = 1, 2, \dots, N$ are N fermions of different flavors.

For this model a bosonization is allowed, where the bosonic field corresponds to the fermionic condensate

$$\Delta = -g^2 \left(\sum_{j=1}^N \bar{\psi}_j \psi_j \right)$$

And through path integral approach we can obtain an effective action for stationary Δ in the form

$$S_{\text{eff}} = - \int \frac{\Delta^2}{2g^2} dx dt - i N \ln \det [i \not{\partial} - \Delta]$$

At the t'Hooft limit $N \rightarrow \infty$ and $g^2 N \sim 1$, it is possible to use the saddle point method to ensure the convergence of the two-point propagator. The convergence happens for the minimas of effective action. In this direction the variation of such action yields the consistency equations

$$\Delta(x) = -i N g^2 \text{tr}_{D,\mathcal{E}} [\gamma^0 R(x; \mathcal{E})] , \quad R(x; \mathcal{E}) \equiv \langle x | (H^D - \mathcal{E})^{-1} | x \rangle$$

$$H^D = \begin{pmatrix} -i \frac{d}{dx} & \Delta(x) \\ \Delta(x) & i \frac{d}{dx} \end{pmatrix}, \quad H^D \Psi = \mathcal{E} \Psi$$

The Gorkov diagonal resolvent of the Bogoliubov-de Gennes operator or Dirac Hamiltonian in 1+1D

$$R(x; \mathcal{E}) \equiv \langle x | (H^D - \mathcal{E})^{-1} | x \rangle \quad H^D = \begin{pmatrix} -i \frac{d}{dx} & \Delta(x) \\ \Delta(x) & i \frac{d}{dx} \end{pmatrix}$$

satisfies the following algebraic properties

$$R = R^\dagger, \operatorname{tr}_D (R \sigma_3) = 0, \det R = -\frac{1}{4}$$

and also satisfies the Dickey-Eilenberger equation

$$\frac{\partial}{\partial x} R \sigma_3 = i \left[\begin{pmatrix} \mathcal{E} & -\Delta \\ \Delta & -\mathcal{E} \end{pmatrix}, R \sigma_3 \right]$$

The Görkov resolvent power series expansion in energy variable can be truncated in order to find analytic solutions for the condensate $\Delta(x)$. In this case the resolvent takes the form

$$R_n(x; \mathcal{E}) = \mathcal{N}(\mathcal{E}) \sum_{l=0}^n \mathcal{E}^{n-l} \begin{pmatrix} \hat{g}_l(x) & \hat{f}_{l-1}(x) \\ \hat{f}_{l-1}^*(x) & \hat{g}_l(x) \end{pmatrix}$$

$$\hat{f}_l = -\frac{\imath}{2} \hat{f}'_{l-1} + \Delta \hat{g}_l,$$

$$\hat{g}_l = i \int \left(\hat{f}_{l-1} - \Delta \hat{f}_{l-1}^* \right) dx + c_l^D,$$

$$\hat{f}_{-1} = 0, \quad \hat{f}_0 = \Delta(x), \quad \hat{g}_0 = c_0^D = 1,$$

under the truncation condition

$$\hat{f}_n = 0,$$

The truncation condition $\hat{f}_n = 0$ defines $\Delta(x)$ as a solution of the s-mKdVh. The first five equations in the hierarchy correspond to

$$\begin{aligned}\hat{f}_{-1}(x) &= 0, \\ \hat{f}_0(x) &= \Delta(x), \\ \hat{f}_1(x) &= -\frac{i}{2}\Delta', \\ \hat{f}_2(x) &= -\frac{1}{4}(\Delta'' - 2\Delta^3) + c_2^D \Delta, \\ \hat{f}_3(x) &= \frac{i}{8}(\Delta''' - 6\Delta^2\Delta') - \frac{ic_2^D}{2}\Delta'.\end{aligned}$$

The coefficients c_k^D are related with the edges of spectrum of H^D Hamiltonian operator

$$c_k^D = \sum_{\substack{i=j_0, j_1, \dots, j_n=0 \\ j_0+j_1+\dots+j_n=k}}^k 2^{-2k} \prod_{i=0}^{2n+1} \frac{(2j_i)!}{(j_i!)^2(2j_i-1)} (\mathcal{E}_i)^{j_i}. \quad \sigma(H^D) = (-\infty, \mathcal{E}_0] \cup [\mathcal{E}_1, \mathcal{E}_2] \cup \dots \cup [\mathcal{E}_{2j-1}, \mathcal{E}_{2j}] \cup [\mathcal{E}_{2n+1}, \infty),$$

In the search of condensate solutions we introduce the Lax pair formulation of stationary sector of GN model or s-mKdV hierarchy

$$[P_n^D, H^D] = \begin{pmatrix} 0 & 2\hat{f}_n(x) \\ -2\hat{f}_n^*(x) & 0 \end{pmatrix} = \mathbf{0}$$

$$P_n^D = \sum_{\ell=0}^n \begin{pmatrix} \hat{g}_\ell(x) & \hat{f}_{\ell-1}(x) \\ \hat{f}_{\ell-1}^*(x) & \hat{g}_\ell(x) \end{pmatrix} \sigma_3 H^{Dn-\ell}, \quad H^D = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta(x) & i\frac{d}{dx} \end{pmatrix}$$

where P is a 2×2 matrix differential operator of order n and take the role of the Lax-Novikov integral of the Dirac Hamiltonian. An important behavior of the Lax pair operators is a like Burchnall-Chaundy relationship between matrix differential operator, that relate potencies of the Lax pair operator in the following form

$$P_n^{D2} = \prod_{\ell=0}^{2n-1} (H^D - \mathcal{E}_\ell),$$

that relates the eigenvalues z^D of H^D and y^D of P^D over a hyper-elliptic curve

$$(y^D)^2 = \prod_{\ell=0}^{2n-1} (z^D - \mathcal{E}_\ell),$$

this relation is in the basis of algebro-geometric solution method of the s-mKdVh.

s-KdVh s-mKdVh Miura transformation

The Miura transformation is defined by

$$u = v^2 - v_x,$$

if v is any s-mKdVh solution $\hat{f}_{2n+1} = 0$, then u is a s-KdVh solution

$$f_{n,x}(u) = i(2v - \partial_x)\hat{f}_{2n-1}, \quad f_{\ell,x} = -\frac{1}{4}f_{\ell-1,xxx} + uf_{\ell-1,x} + \frac{1}{2}u_x f_{\ell-1},$$

where s-KdVh corresponds to $2f_{\ell,x} = 0$. Note that the inverse affirmation is not correct.

Explicitly, one finds

$$2f_{0,x} = 0,$$

$$2f_{1,x} = u_x$$

$$2f_{2,x} = -\frac{1}{4}(u_{xxx} - 6uu_x - 4c_1u_x)$$

$$2f_{3,x} = \frac{1}{8}(16u_{xxxxx} - 5u_x u_{xx} - 5u_x^2 - 5uu_{xx} + 15u^2u_x - 2c_1(u_{xxx} - 6uu_x) + 8c_2u_x),$$

⋮

How to obtain s-mKdV solutions from s-KdV solutions?

The s-mKdVh is invariant under the change $v \rightarrow -v$, then the Miura Transformation of v allows to define

$$u^+ = v^2 + v_x, \quad \text{and} \quad u^- = v^2 - v_x,$$

where u^\pm are both s-KdVh solutions dependent on v .

Let's assume now from another perspective that we have two functions u^+ and u^- given in function of $v(x)$, and suppose that both functions u^+ and u^- satisfy the same equation in the s-KdVh. In this case $v(x)$ must satisfy simultaneously

$$f_{n,x}(u) = i(2v - \partial_x) \hat{f}_{2n-1},$$

for v and for $-v$. Adding these two equations we obtain $4v \hat{f}_i = 0$, which implies that v must satisfy the s-mKdV equation

$$\begin{aligned} \hat{f}_{-1}(x) &= 0, \\ \hat{f}_0(x) &= \Delta(x), \\ \hat{f}_1(x) &= -\frac{i}{2} \Delta', \\ \hat{f}_2(x) &= -\frac{1}{4} (\Delta'' - 2\Delta^3) + c_2^D \Delta, \\ \hat{f}_3(x) &= \frac{i}{8} (\Delta''' - 6\Delta^2 \Delta') - \frac{ic_2^D}{2} \Delta'. \end{aligned}$$

KdV's Lax pair formulations

The KdV hierarchy is defined recursively in the form

$$2f_{\ell,x} = 0$$

$$f_{\ell,x} = -\frac{1}{4}f_{\ell-1,xxx} + uf_{\ell-1,x} + \frac{1}{2}u_x f_{\ell-1},$$

The Lax equation for KdV takes the form

$$2f_{\ell+1,x} = i[P_{2\ell+1}, H],$$

where the Lax pair corresponds to a Schrödinger operator and a Lax-Novikov integral

$$H = H(u) = -\frac{d^2}{dx^2} + u, \quad P_{2g+1} = P_{2\ell+1}(u, \partial\sigma(H(u)))$$

$$= -i \sum_{j=1}^{\ell} \left(f_{\ell-j} \frac{d}{dx} - \frac{1}{2} f_{\ell-j,x} \right) H^{\ell},$$

$$2f_{0,x} = 0,$$

$$2f_{1,x} = u_x$$

$$2f_{2,x} = -\frac{1}{4}(u_{xxx} - 6uu_x - 4c_1u_x)$$

$$2f_{3,x} = \frac{1}{8}(16u_{xxxxx} - 5u_x u_{xx} - 5u_x^2 - 5uu_{xx} + 15u^2u_x - 2c_1(u_{xxx} - 6uu_x) + 8c_2u_x),$$

⋮

Inverse Miura transformation and hidden supersymmetry

Using the function defined in Miura transformation we can construct a superalgebra in the form

$$[\mathcal{H}, \mathcal{Q}_a] = 0, \quad \{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}(\mathcal{H} - E_\star)$$

Where the fermionic integrals corresponds to

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{dx} + v \\ -\frac{d}{dx} + v & 0 \end{pmatrix}, \quad \mathcal{Q}_2 = i\sigma_3 \mathcal{Q}_1$$

Note that these corresponds, to a unitary transformation of H^D , $\mathcal{Q}_1 = e^{-i\frac{\pi}{4}\sigma_1} H^D e^{i\frac{\pi}{4}\sigma_1}$ and

$$\begin{aligned} \mathcal{H} - E_\star &= \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} & H^+ &= -\frac{d^2}{dx^2} + u^+ \\ &= -\frac{d^2}{dx^2} + v^2 + \sigma_3 v_x & H^- &= -\frac{d^2}{dx^2} + u^- \end{aligned}$$

it is an extended Schrödinger Hamiltonian. This frame is known as Witten supersymmetric quantum mechanics.

It is natural to ask why we want to change the problem from the search of one solution of the mKdVh to the search of two connected solutions of the KdVh?

In the next we show how starting from an initial Schrödinger potential u^+ construct via Darboux transformation a second one u^- . A characteristic of the Darboux transformation is that maintain the symmetries of u^+ , thus if u^+ have a Lax-Novikov integral then u^- also has its respective Lax-Novikov integral.

Darboux transformation to Schrödinger operators

$$H\varphi = \lambda\varphi,$$

$$A = \varphi \frac{d}{dx} \frac{1}{\varphi},$$

Requirements: we need to know at least one of the eigenstates of the initial Hamiltonian

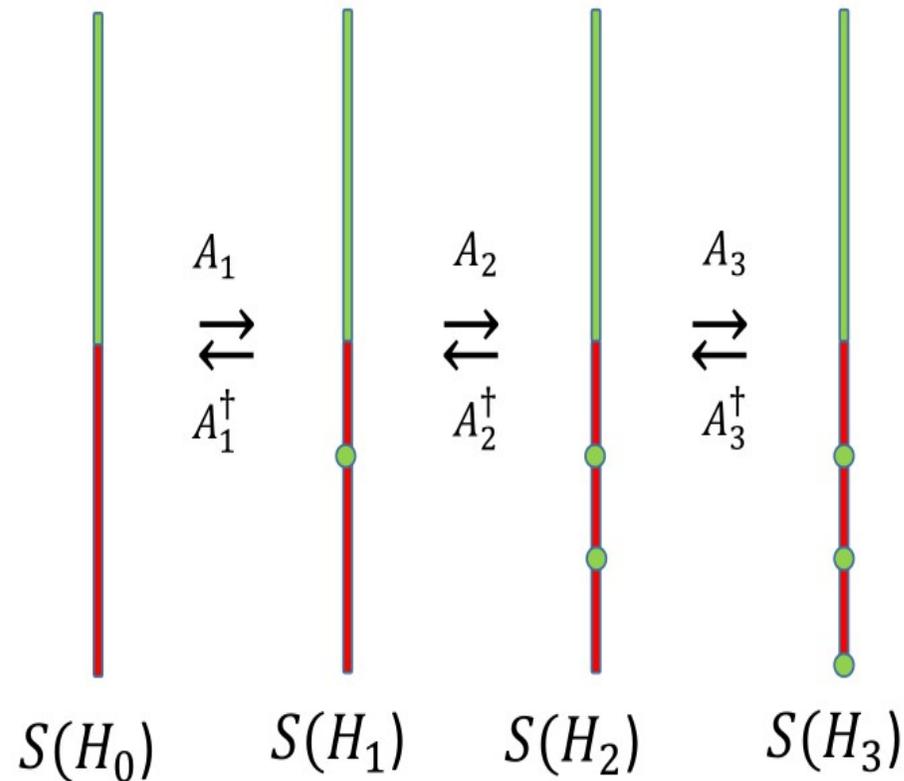
Factorizations: located at the base of the supersymmetric structure of the Darboux transformation

$$A^\dagger A = H - \lambda, \quad AA^\dagger = H' - \lambda,$$

Intertwining relation:
These map the spectrum of a Hamiltonian to the spectrum the other

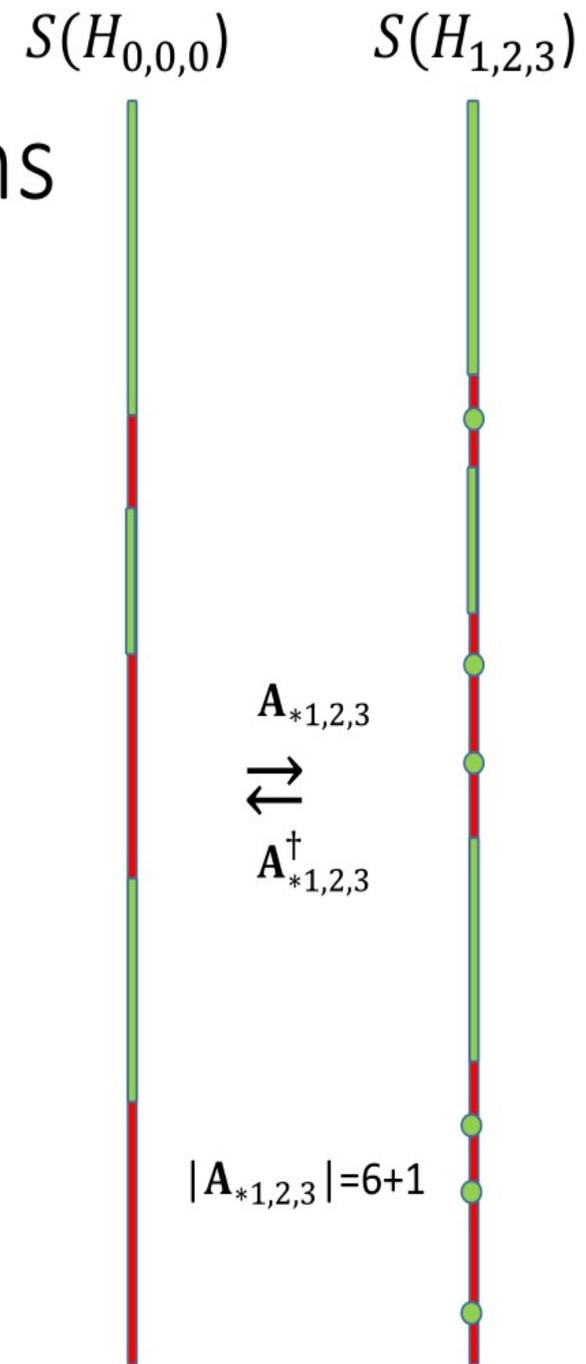
$$AH = H'A, \quad A^\dagger H' = HA^\dagger,$$

Examples



Crum-Darboux transformations

Crum-Darboux transformations are chains of Darboux transformations that allow generally make more of a spectral deformation to the initial Hamiltonian operator.



A order n Crum-Darboux transformation to the Schrödinger operator $H_0 = -\frac{d^2}{dx^2} + V_0(x)$, result in the operator

$$H_n = -\frac{d^2}{dx^2} + V_n(x),$$
$$V_n = V_0 - 2\frac{d^2}{dx^2} \log \mathbb{W}_n,$$

where

$$\mathbb{W}_n = \mathbb{W}(\psi_1, \dots, \psi_n) = \det \mathcal{A},$$
$$\mathcal{A}_{ij} = \frac{d^{i-1}}{dx^{i-1}} \psi_j, \quad i, j = 1, \dots, n$$

is the wronskian of n states of H_0 .

We can introduce the first-order differential intertwining operators

$$A_n = \frac{d}{dx} + \mathcal{W}_n,$$
$$\mathcal{W}_n = -\frac{d}{dx} \log \mathbb{W}_n + \frac{d}{dx} \log \mathbb{W}_{n-1}$$

These operators and their conjugated factorize H_{n-1} and H_n in the form

$$A_n^\dagger A_n = H_{n-1} - E_n, \quad A_n A_n^\dagger = H_n - E_n$$

and intertwine them as follow

$$A_n H_{n-1} = H_n A_n, \quad A_n^\dagger H_n = H_{n-1} A_n^\dagger.$$

Which allow us to construct the linear SUSY

$$[\mathcal{H}, \mathcal{Q}_a] = 0, \quad \{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}(\mathcal{H} - E_\star)$$

With the intention of obtain condensates of gros Neveu model we ned demand that V_{n-1} and V_n be solutions of the same equation in KdV hierarchy, it is have a Lax- Novikov integral of motion.

Enogh that V_{n-1} be a solution of a equation of the KdV the hierarchy to fullfil this condition.

The Its-Matveev formula for potentials with $(g - 1)$ -gaps potentials, solutions of s.KdVh, is given for

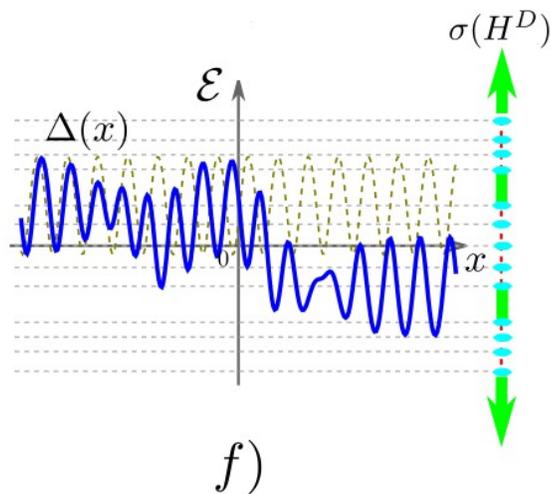
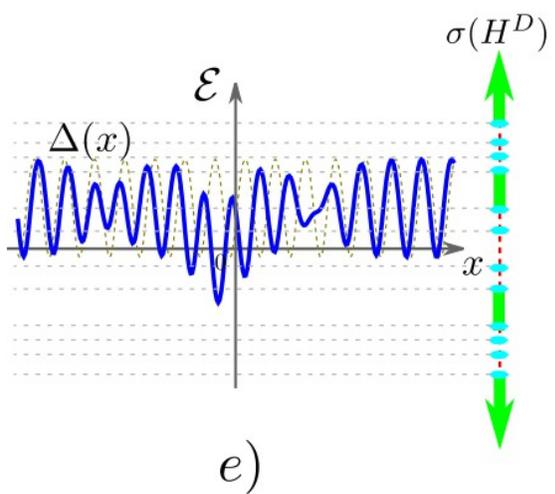
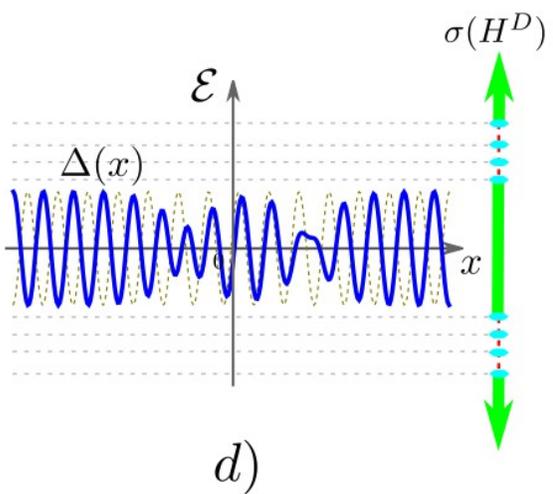
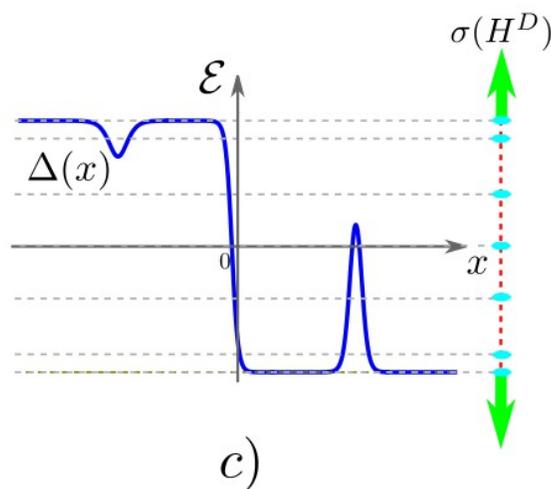
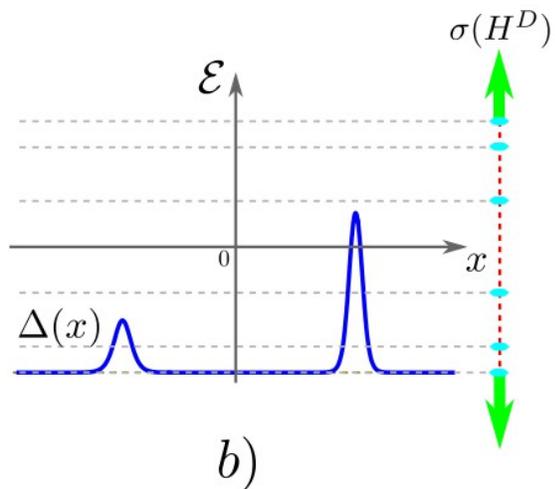
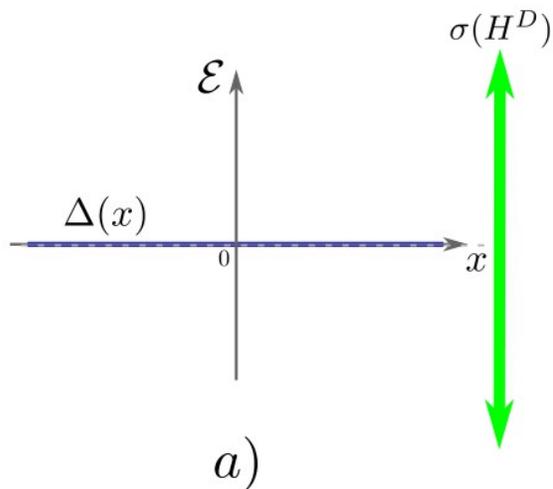
$$u_{g,0}(x) = -2 \frac{d^2}{dx^2} \ln(\theta(x\mathbf{v} + \boldsymbol{\phi}, \tau)) + \Lambda_0,$$

The solutions in Its-Matveev form correspond to finite-gap potentials, while the Crum-Darboux transformations to solutions in Its-Matveev form correspond to finite-gap systems with bound states in initial potential forbidden bands. The genus g of the Riemann theta function corresponds to the number of band gaps in the spectrum of asociated Schrödinger operator.

The eigenstates for such Hamiltonian are given in the form

$$\psi(r, x) = \frac{\theta(x\mathbf{v} + \boldsymbol{\phi} + \boldsymbol{\alpha}(r), \tau)}{\theta(x\mathbf{v} + \boldsymbol{\phi}, \tau)} \exp(-ix\xi(r))$$

$$\Delta(x) = \mathcal{W}_n = -\frac{d}{dx} \log \mathbb{W}_n + \frac{d}{dx} \log \mathbb{W}_{n-1}$$



$$[\tilde{\mathcal{H}}, \tilde{\mathcal{Q}}_a] = 0, \quad \{\tilde{\mathcal{Q}}_a, \tilde{\mathcal{Q}}_b\} = 2\delta_{ab}(\tilde{\mathcal{H}} - z_*)$$

$$[\tilde{\mathcal{P}}_1, \tilde{\mathcal{H}}] = 0, \quad [\tilde{\mathcal{P}}_1, \mathcal{Q}_a] = 0,$$

Consistency equations

$$\Delta(x) = -iNg^2 \text{tr}_{D,\mathcal{E}} [\gamma^0 R(x; \mathcal{E})] , \quad \text{tr}_{\mathcal{E}} \equiv \frac{1}{N} \sum_{r=1}^N \int_{C_r} \frac{d\mathcal{E}}{2\pi},$$

$$i \frac{\Delta}{Ng^2} = \text{tr}_{\mathcal{E}} \left[\frac{\mathcal{E}^{n+2l} \Delta + \Sigma}{\sqrt{-\Pi} \prod_{k=1}^l (\mathcal{E}^2 - \mathcal{E}_{b,k}^2)} \right]$$

$$\Sigma = \sum_{j=1}^{\lfloor n/2 \rfloor + l} \mathcal{E}^{n+2l-2j} \hat{f}_{2j},$$

$$\sqrt{-\Pi} = \sqrt{-\prod_{j=0}^{2n+1} (\mathcal{E} - \mathcal{E}_{e,j})},$$

1-gap consistency condition

$$|\mathcal{E}_{b,j}| = M \sin \left(\frac{\pi \nu_j}{2} \right),$$

ν_l Occupation fraction

Decoupled in:

$$\frac{i}{Ng^2} = \text{tr}_{\mathcal{E}} \left[\frac{\mathcal{E}^n}{\sqrt{-\Pi}} \right]$$

UV cut-off

$$0 = \text{tr}_{\mathcal{E}} \left[\frac{\mathcal{E}^{n-j}}{\sqrt{-\Pi}} \right]$$

Occupation of bands

$$j = 1, \dots, 2 \lfloor n/2 \rfloor$$

$$0 = \text{tr}_{\mathcal{E}} \left[\frac{\mathcal{E}^{n-2 \lfloor n/2 \rfloor}}{\sqrt{-\Pi} (\mathcal{E}^2 - \mathcal{E}_{b,j}^2)} \right]$$

Occupation of bound states

$$j = 1, \dots, l$$

2-gap consistency condition

$$|\mathcal{E}_{b,l}| = M \cos \left(\frac{\pi \nu_1}{2} \right) \sqrt{k'^2 + \tan^2 \left(\frac{\pi \nu_1}{2} \right)},$$

k' modular parameter

Conclusion

Using an exotic supersymmetry between finite-gap systems with defects, we have constructed the set of stationary and analytical solutions for the GN model, observing the existence of inhomogeneous and non-periodic condensates, but with bands structures and a finite number of bound states.

The Darboux transformation has allowed us recursively to construct infinite families of exactly solvable Schrödinger systems from a finite-gap potential in the Its-Matveev form.

The process of constructing solitary defects on finite-gap Schrödinger operators has generated Dirac or Bogoliubov-de Gennes operators with scalar potential with solitary defects on finite-gap background, these Dirac systems exhibit an integral of motion corresponding to a Lax operator of the s-mKdVh.

The Darboux dressing of the Lax operator of finite-gap systems have allowed to found the self-consistency equations for all the system, having as main characteristic the independence of the consistence of each defect, depending only in the data of finite-gap background.

Muchas gracias!!!