Supplemental Assignments for Collider Physics

Tao Han, Jan. 15–19, 2018, The 7^{th} UTFSM summer school, Valparaiso, Chile

A partial list of assignments accompanying the lectures.

Chapter I: Introduction to collider physics

`Exercise 1.1:` Show that the phase space element $d\vec{p}/2p^0$ is Lorentz invariant.

`Exercise 1.2:` A particle of mass M decays to two particles isotropically in its rest frame. What does the momentum distribution look like in a frame in which the particle is moving with a speed β_z? Compare the result with your expectation for the shape change for a basket ball.

`Exercise 1.3:` In the "Standard Model" of elementary particle physics, the amplitude for the scattering of the weak gauge bosons (the force mediator for the nuclear β decay) $W^+W^+ \to W^+W^+$ is calculated to be

$$f(k,\ \theta) = \frac{1}{16\pi k}\left(\frac{-M_H^2}{v^2}\right)\left(\frac{t}{t-M_H^2} + \frac{u}{u-M_H^2}\right)$$

where k is the W momentum in the Center-of-Momentum frame, M_H is the mass of the Higgs boson, and $v \approx 250$ GeV is the Higgs vacuum expectation value. The angular-dependent kinematical variables are

$$t = -2k^2(1-\cos\theta) \quad \text{and} \quad u = -2k^2(1+\cos\theta).$$

Note that the amplitude is give in the "natural units" where $c = \hbar = 1$, and everything is expressed in terms of the energy units electron-volts: 1 GeV = 10^9 eV.

(a). Take the high-energy limit $2k \gg M_H$, compute the partial wave amplitude f_ℓ. Note that for final state identical particles W^+W^+, the angular integration should be $1/2\int_{-1}^{1} d\cos\theta$.

(b). Impose the partial wave unitarity condition on kf_ℓ for w-wave, determine the bound on the mass of the Higgs boson M_H (in units of GeV).

(c). If the Higgs boson did not exist in Nature, then the amplitude for the weak gauge boson scattering for $W^+W^+ \to W^+W^+$ would be expressed by taking the limit $2k \ll M_H \to \infty$. Using the same procedure above, determine at what energy scale $2k$ the Standard Model theory would break down to violate the partial wave unitarity.

(Remark: The "Large Hadron Collider" (LHC) at CERN, Geneva, provides proton-proton collisions at a c.m. energy of 13,000 GeV, which was designed based on the above physics argument. Consequently, we have witnessed the historical discovery of the Higgs boson!)

Chapter II: High-energy colliders

`Exercise 2.1:` A 120 GeV Higgs boson will have a production cross section of 20 pb at the LHC. How many events per year do you expect to produce for the Higgs boson with a designed LHC luminosity $10^{33}/\mathrm{cm}^2/\mathrm{s}$? Do you expect it to be easy to observe and why?

`Exercise 2.2:` **(challenging problem)** For a resonant production $e^+e^- \to V^*$ with a mass M_V and total width Γ_V, derive the Breit-Wigner formula

$$\sigma(e^+e^- \to V^* \to X) = \frac{4\pi(2j+1)\Gamma(V \to e^+e^-)\Gamma(V \to X)}{(s-M_V^2)^2 + \Gamma_V^2 M_V^2}\ \frac{s}{M_V^2},$$

Consider a beam energy spread Δ in Gaussian distribution

$$\frac{dL}{d\sqrt{\hat{s}}} = \frac{1}{\sqrt{2\pi}\ \Delta}\exp[\frac{-(\sqrt{\hat{s}}-\sqrt{s})^2}{2\Delta^2}],$$

obtain the appropriate cross section formulas for (a) $\Delta \ll \Gamma_V$ (resonance line-shape) and (b) $\Delta \gg \Gamma_V$ (narrow-width approximation).

`Exercise 2.3` **(challenging problem)**`:` Derive the Weizsäcker-Williams spectrum for a photon with an energy xE off an electron with an energy E

$$P_{\gamma/e}(x) \approx \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\ln\frac{E^2}{m_e^2}$$

`Exercise 2.4:` For a four-momentum $p \equiv p^\mu = (E, \vec{p})$, define

$$E_T = \sqrt{p_T^2 + m^2}, \quad p_T^2 = p_x^2 + p_y^2, \quad y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z},$$

$$\text{then show } p^\mu = (E_T\cosh y,\ p_T\cos\phi,\ p_T\sin\phi,\ E_T\sinh y),$$

$$\text{and, } \frac{d^3\vec{p}}{E} = p_T dp_T d\phi\ dy = E_T dE_T d\phi\ dy.$$

Due to the random boost between the Lab-frame (O) and the c.m. frame (O') for every event,

$$y' = \frac{1}{2}\ln\frac{E'+p'_z}{E'-p'_z} = \frac{1}{2}\ln\frac{(1-\beta_{cm})(E+p_z)}{(1+\beta_{cm})(E-p_z)} = y - y_{cm},$$

where β_{cm} and y_{cm} are the speed and rapidity of the c.m. frame w.r.t. the lab frame.

In the massless limit, the rapidity y defines the pseudo-rapidity:

$$y \to \eta = \frac{1}{2} \ln \frac{1+\cos\theta}{1-\cos\theta} = \ln \cot \frac{\theta}{2}.$$

`Exercise 2.5:` For a π^0, μ^-, or a τ^- respectively, calculate its decay length if the particle has an energy $E = 10$ GeV.

Chapter III: From kinematics to dynamics

`Exercise 3.1:` Assume that $m_a = m_1$ and $m_b = m_2$. Show that

$$\begin{aligned} t &= -2p_{cm}^2(1-\cos\theta_{a1}^*), \\ u &= -2p_{cm}^2(1+\cos\theta_{a1}^*) + \frac{(m_1^2 - m_2^2)^2}{s}, \end{aligned}$$

$p_{cm} = \lambda^{1/2}(s, m_1^2, m_2^2)/2\sqrt{s}$ is the momentum magnitude in the c.m. frame.
Note: t is negative dfinite; $t \to 0$ in the collinear limit.

`Exercise 3.2:` (**challenging problem** A particle of mass M decays to 3 particles $M \to abc$. Show that the phase space element can be expressed as

$$dPS_3 = \frac{1}{2^7\pi^3}\ M^2 dx_a dx_b.$$
$$x_i = \frac{2E_i}{M},\ (i = a, b, c,\ \sum_i x_i = 2).$$

where the integration limits for $m_a = m_b = m_c = 0$ are

$$0 \le x_a \le 1,\quad 1 - x_a \le x_b \le 1.$$

`Exercise 3.3:` An event was identified to have a μ^+ and a μ^- along with some missing energy. What can you say about the kinematics of the system of the missing particles? Consider for both an e^+e^- and a hadron collider.

`Exercise 3.4:` For a two-body massless final state with an invariant mass squared s, show that

$$\frac{d\hat\sigma}{dp_T} = \frac{4p_T}{s\sqrt{1-4p_T^2/s}}\ \frac{d\hat\sigma}{d\cos\theta^*}.$$

where $p_T = p\sin\theta^*$ is the transverse momentum and θ^* is the polar angle in the c.m. frame. Comment on the apparent singularity at $p_T^2 = s/4$.