Energy-Energy Correlation in QCD and SCET

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with L. Dixon, M.X. Luo, V. Shtabovenko, T.Z. Yang, 1801.03219, EEC in QCD
with I. Moult: 1801.02627, EEC in SCET

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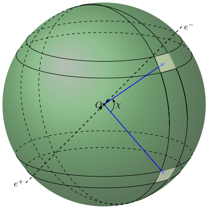
Energy-Energy Correlation (EEC)

- EEC measures the correlations of energy deposition in two detectors with angle χ [Basham, Brown, Ellis, Love, 1978]

$$\frac{1}{\sigma}\frac{d\Sigma_{\mathrm{EEC}}(\chi)}{d\cos\chi} = \frac{1}{\Delta\chi N_{\mathrm{events}}}\sum_{N_{\mathrm{events}}}\sum_{ij}\frac{E_iE_j}{E^2}$$



- Thanks to momentum conservation sum rule for collinear fragmentation function $\sum_h \int_0^1 dx\ x\ f_{q\to h}(x,\mu_F^2) = 1$. Direct calculable in terms of quark/gluon D.O.F. [see also Lorenzo Zoppi's talk]
- Relation to inclusive cross section: $\int \mathrm{d}\chi \mathrm{d}\Sigma/\mathrm{d}\chi = \sigma$
- Moment of EEC related to Energy Correlation Function for jet substructure [Larkoski, Salam, Thaler]

Properties of EEC

- Exhibit rich perturbative structure: collinear single log at $\chi = 0^o$, Sudakov double log at $\chi = 180^o$
- Introduce the scaling variable $z = (1 - \cos\chi)/2$, then the LO QCD prediction is

$$\frac{1}{\sigma_0}\frac{d\Sigma}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi} C_F \frac{3-2z}{4(1-z)z^5}\left[3z(2-3z) + 2(2z^2-6z+3)\log(1-z)\right]$$

  - Collinear limit: $z \to 0$,

$$\mathrm{d}\Sigma/\mathrm{d}\cos\chi \sim \frac{\alpha_s(\mu)}{2\pi} C_F \frac{3}{8z}$$

  - Back-to-back limit: $z \to 1$,

$$\mathrm{d}\Sigma/\mathrm{d}\cos\chi \sim \frac{\alpha_s(\mu)}{2\pi} C_F \frac{-\frac{1}{2}\log(1-z) - \frac{3}{4}}{1-z}$$

Numerical calculation for EEC at NLO

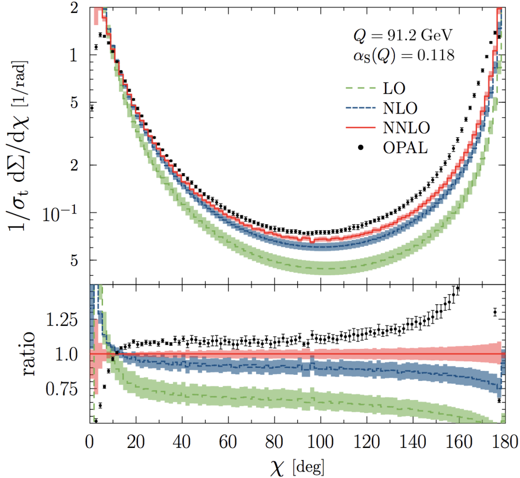
- Numerical calculation for EEC at NLO starts at 1980s, by several different groups
- Sensitive to soft radiation even for z far from 0 or 1
- Several different methods (subtraction, phase space slicing, hybrid), disagreement found for moment $\int \mathrm{d}\Sigma(\chi)\sin^{2+m}\chi\cos^{n}\chi \mathrm{d}\cos\chi$ [Nason et al., hep-ph/9602288]

m	n	N	G	S	C
0	0	50.82 ± 0.05	50.54 ± 0.03	50.72 ± 0.02	46.4 ± 0.2
1	0	35.76 ± 0.04	35.53 ± 0.02	35.64 ± 0.02	32.09 ± 0.06
2	0	28.94 ± 0.03	28.75 ± 0.02	28.82 ± 0.02	25.73 ± 0.04
3	0	24.92 ± 0.03	24.75 ± 0.02	24.80 ± 0.02	22.03 ± 0.04
4	0	22.20 ± 0.03	22.05 ± 0.02	22.09 ± 0.02	19.54 ± 0.04
5	0	20.21 ± 0.03	20.07 ± 0.02	20.10 ± 0.02	17.74 ± 0.03
0	1	-6.468 ± 0.006	-6.50 ± 0.01	-6.455 ± 0.005	-6.0 ± 0.15
1	1	-2.356 ± 0.004	-2.365 ± 0.009	-2.344 ± 0.003	-2.15 ± 0.03
2	1	-1.189 ± 0.003	-1.194 ± 0.008	-1.177 ± 0.003	-1.06 ± 0.02
3	1	-0.714 ± 0.003	-0.718 ± 0.007	-0.702 ± 0.003	-0.62 ± 0.01
4	1	-0.478 ± 0.003	-0.479 ± 0.007	-0.466 ± 0.003	-0.41 ± 0.01
5	1	-0.344 ± 0.003	-0.344 ± 0.006	-0.331 ± 0.003	-0.28 ± 0.01

Figure: [N: Nason; G: Glover; S: Seymour; C: Clay]

EEC at NNLO

- Most event shape variables have been computed to NNLO since 2007 [Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Weinzierl], except EEC
- Only recently calculated using the CoLoRfulNNLO subtraction [Del Duca, Duhr, Kardos, Somogyi, Trocsanyi, 2016; Tulipant, Kardos, Somogyi, 2017]



- Sizable corrections from LO to NLO to NNLO

Analytic calculation for event shape observables

- Much less is known analytically for event shape beyond LO
  - Thrust
    $$T = \max_{\boldsymbol{n}} \frac{\sum_i |\boldsymbol{p}_i \cdot \boldsymbol{n}|}{\sum_i |\boldsymbol{p}_i|}$$
    Known analytically at LO only
  - C parameter:
    $$C = \frac{3}{2} \frac{\sum_{i,j} \left[|\boldsymbol{p}_i||\boldsymbol{p}_j| - (\boldsymbol{p}_i \cdot \boldsymbol{p}_j)^2 / |\boldsymbol{p}_i||\boldsymbol{p}_j| \right]}{(\sum_i |\boldsymbol{p}_i|)^2}$$
    Analytically not known even at LO [Ellis, Ross, Terrano, 1981]

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}C} = \frac{\alpha_s}{2\pi C_F}\int_{x_2^-(C)}^{x_2^+(C)} \mathrm{d}x \frac{6x[C(x^3+(x-2)^2)-6(1-x)(1+x^2)]}{C(C+6)^2(x-6/(C+6))\sqrt{(6/(C+6)-x)(x_2^+-x)(x-x_2^-)x}}$$

  - Measurement function for EEC
    $$\sum_{i,j} E_i E_j \delta(\cos\chi - \cos\theta_{ij}) = \sum_{i,j} (E_i E_j)^2 \delta(E_i E_j (1-\cos\chi) - p_i \cdot p_j)$$
    No maximization/minimization. Measurement function only involves a pair of momentum at one time

EEC as a four-point correlation function

- Energy flow operator [Korchemsky, Oderda, Sterman; Bauer, Fleming, Lee, Sterman]

$$\mathcal{E}(\boldsymbol{n}) = \int_0^\infty \mathrm{d}t \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\boldsymbol{n})$$

- Simple field theoretic definition of EEC in terms of 4-point Wightman correlation [Hofman, Maldacena]

$$\langle \mathcal{E}(\boldsymbol{n}_1)\mathcal{E}(\boldsymbol{n}_2)\rangle_Q = \sigma^{-1}\int \mathrm{d}^4x\, e^{ix\cdot Q}\langle 0|O^\dagger(x)\mathcal{E}(\boldsymbol{n}_1)\mathcal{E}(\boldsymbol{n}_2)O(0)|0\rangle_W$$

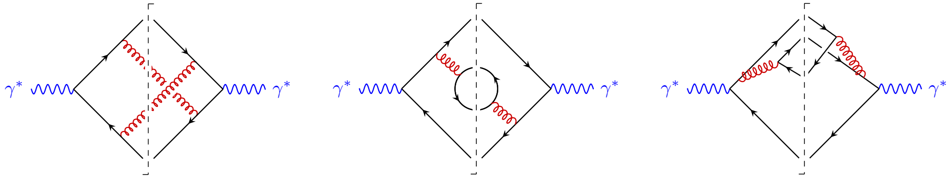
- In $\mathcal{N}=4$ SYM, such 4-point function can be related to scalar 4-point function, which has simple Lorentz structure

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle_E = \frac{1}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}\Phi(u,v;a) \quad u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, v = \frac{x_{23}^2x_{41}^2}{x_{13}^2x_{24}^2}$$

- Analytic continuation from Eucildean correlation to Minkowski, Wightman function via Mellin representation [Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, 2013]
- Full EEC at NLO in $\mathcal{N}=4$ SYM written in terms of ln, Li_2, Li_3 etc

EEC in QCD using on-shell method

- While the correlation function and Mellin amplitude method is very powerful in $\mathcal{N} = 4$ SYM, extending it to QCD is difficult
    - Two-loop four-point correlation function unknown in QCD
    - Complicated Lorentz structure, challenging Mellin transformation and inverse transformation
- Method based on on-shell amplitudes turn out to be more convenient in this case
- At NLO, the most complicated part is the $2 \to 4$ tree-level subprocess
- There are in total 72 squared amplitudes (including interference terms). Some representative diagrams are



(a) C_F^2, $C_F C_A$ (b) $C_F N_f T_f$ (c) $C_F(C_F - C_A/2)$

Reverse unitarity and Integration By Parts

- By reverse use of unitarity, convert phase space integral to "loop integral" [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]

$$\frac{d^D p}{(2\pi)^D} 2\pi\delta_+(p^2 - m^2) \to \frac{1}{i}\frac{d^D p}{(2\pi)^D}\left(\frac{1}{p^2 - m^2 - i0_+} - \frac{1}{p^2 - m^2 + i0_+}\right) \equiv \frac{1}{i}\frac{d^D p}{(2\pi)^D}\left[\frac{1}{p^2 - m^2}\right]_{\text{cut}}$$

- The EEC measurement function leads to an unusual cut propagator, with quadratic function of dot product in the denominator

$$\delta((1-\cos\chi)p_1\cdot Q p_2\cdot Q - Q^2 p_1\cdot p_2) \to \frac{1}{2\pi i}\left[\frac{1}{(1-\cos\chi)p_1\cdot Q p_2\cdot Q - Q^2 p_1\cdot p_2}\right]_{\text{cut}}$$

- Additional relation among integrals (absent from the usual IBP relation)

$$\int \text{dLIPS}(1-\cos\chi)(p_1\cdot Q p_2\cdot Q)\mathcal{I} = \int \text{dLIPS}(Q^2 p_1\cdot p_2)\mathcal{I}$$

- After IBP reduction with LiteRed and Fire, only 40 master integrals remain

Differential equation

- The master integrals can then be solved by the methods of differential equation [Kotikov; Gehrmann, Remiddi]
- The system of differential equation can be converted to canonical form [Henn]. This procedure is partially automated by FUCHSIA [Gituliar, Magerya].

$$\frac{\mathrm{d}\vec{f}(y,\epsilon)}{\mathrm{d}y} = \epsilon\left(\frac{\boldsymbol{a}}{y} + \frac{\boldsymbol{b}}{1-y} + \frac{\boldsymbol{c}}{1+y}\right)\vec{f}(y,\epsilon)$$

  where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are constant 40×40 matrices, $y = i\sqrt{z}/\sqrt{1-z}$
- Solutions in terms of harmonic polylogarithms
- Fixing the boundary condition (most difficult part)
  - scaling behavior in the collinear limit, $z \to 0$
  - scaling behavior in the non-physical $z \to \infty$ limit
  - sum rule: integrate over z should reproduce inclusive MIs
- Much simpler results after combining real and virtual. E.g., functions like $\mathrm{Li}_2(y) - \mathrm{Li}_2(-y) - \log(-iy)\log\frac{1+y}{1-y}$ (The Bloch-Wigner single valued dilogarithm) that have manifest i dependence in the argument cancel in the sum

EEC in QCD at NLO

- Infrared divergences cancel between real and virtual

$$\frac{1}{\sigma_0}\frac{d\Sigma}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi}A(z) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left(\beta_0 \log\frac{\mu}{Q}A(z) + B(z)\right) + \mathcal{O}(\alpha_s^3)$$

- Color decomposition for the NLO coefficient

$$B = C_F^2 B_{\rm lc} + C_F(C_A - 2C_F)B_{\rm nlc} + C_F N_f T_f B_{N_f}$$

- All color coefficients can be written in the following transcendental function basis (not unique)

$$g_1^{(1)} = \log(1-z)\,, \qquad g_2^{(1)} = \log(z)\,, \qquad g_1^{(2)} = 2(\mathrm{Li}_2(z)+\zeta_2) + \log^2(1-z)\,,$$

$$g_2^{(2)} = \mathrm{Li}_2(1-z) - \mathrm{Li}_2(z)\,, \qquad g_3^{(2)} = -2\,\mathrm{Li}_2\left(-\sqrt{z}\right) + 2\,\mathrm{Li}_2\left(\sqrt{z}\right) + \log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\log(z)\,, \qquad g_4^{(2)} = \zeta_2\,,$$

$$g_1^{(3)} = -6\left[\mathrm{Li}_3\left(-\frac{z}{1-z}\right) - \zeta_3\right] - \log\left(\frac{z}{1-z}\right)\left(2(\mathrm{Li}_2(z)+\zeta_2) + \log^2(1-z)\right)\,,$$

$$g_2^{(3)} = -12\left[\mathrm{Li}_3(z) + \mathrm{Li}_3\left(-\frac{z}{1-z}\right)\right] + 6\,\mathrm{Li}_2(z)\log(1-z) + \log^3(1-z)\,,$$

$$g_3^{(3)} = 6\log(1-z)\left(\mathrm{Li}_2(z) - \zeta_2\right) - 12\,\mathrm{Li}_3(z) + \log^3(1-z)\,,$$

$$g_4^{(3)} = \mathrm{Li}_3\left(-\frac{z}{1-z}\right) - 3\,\zeta_2\log(z) + 8\,\zeta_3\,,$$

$$g_5^{(3)} = -8\left[\mathrm{Li}_3\left(-\frac{\sqrt{z}}{1-\sqrt{z}}\right) + \mathrm{Li}_3\left(\frac{\sqrt{z}}{1+\sqrt{z}}\right)\right] + 2\mathrm{Li}_3\left(-\frac{z}{1-z}\right) + 4\zeta_2\log(1-z) + \log\left(\frac{1-z}{z}\right)\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\,.$$

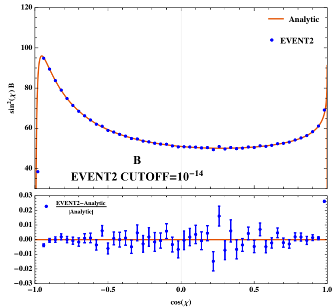
Example of analytical results

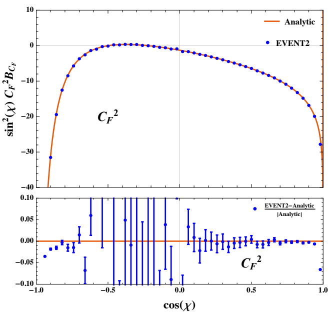
- The leading color coefficient

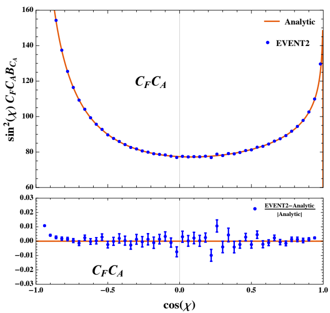
$$
\begin{aligned}
B_{\mathrm{lc}} = &+ \frac{122400z^7 - 244800z^6 + 157060z^5 - 31000z^4 + 2064z^3 + 72305z^2 - 143577z + 63298}{1440(1-z)z^4} \\
&- \frac{-244800z^9 + 673200z^8 - 667280z^7 + 283140z^6 - 48122z^5 + 2716z^4 - 6201z^3 + 11309z^2 - 9329z + 3007}{720(1-z)z^5} g_1^{(1)} \\
&- \frac{244800z^8 - 550800z^7 + 422480z^6 - 126900z^5 + 13052z^4 - 336z^3 + 17261z^2 - 38295z + 19938}{720(1-z)z^4} g_2^{(1)} \\
&+ \frac{4z^7 + 10z^6 - 17z^5 + 25z^4 - 96z^3 + 296z^2 - 211z + 87}{24(1-z)z^5} g_1^{(2)} \\
&+ \frac{-40800z^8 + 61200z^7 - 28480z^6 + 4040z^5 - 320z^4 - 160z^3 + 1126z^2 - 4726z + 3323}{120z^5} g_2^{(2)} \\
&- \frac{1-11z}{48z^{7/2}} g_3^{(2)} - \frac{120z^6 + 60z^5 + 160z^4 - 2246z^3 + 8812z^2 - 10159z + 4193}{120(1-z)z^5} g_4^{(2)} \\
&- 2\left(85z^4 - 170z^3 + 116z^2 - 31z + 3\right) g_1^{(3)} + \frac{-4z^3 + 18z^2 - 21z + 5}{6(1-z)z^5} g_2^{(3)} + \frac{z^2+1}{12(1-z)} g_3^{(3)}
\end{aligned}
$$

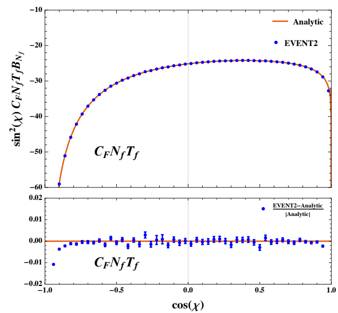
- The results are sum of $\frac{P(z)}{Q(z)} \times g_i^{(j)}$. $P(z)$ and $Q(z)$ are polynomial of z with largest power 9 and 6, respectively. Intriguing behavior as $z \to 0$ and $z \to \infty$

Cross check against EVENT2









Expansion in collinear limit

- The collinear limit is $\chi \to 0$, or $z \to 0$. With the full NLO results, we can expand to any order in z

$$B(z) = C_F \left\{ \frac{1}{z} \left[\log(z) \left(-\frac{107 C_A}{120} + \frac{25 C_F}{32} + \frac{53 N_f T_f}{240} \right) \right.\right.$$
$$+ C_A \left(-\frac{25\zeta_2}{12} + \frac{\zeta_3}{2} + \frac{17683}{2700} \right)$$
$$\left.\left. + C_F \left(\frac{43\zeta_2}{12} - \zeta_3 - \frac{8263}{1728} \right) - \frac{4913 N_f T_f}{3600} \right] \right\} + \mathcal{O}(z)\,,$$

- The leading logarithms $\alpha_s^n (\log^{n-1} z)/z$ can be predicted using jet calculus [Konishi, Ukawa, Veneziano, 1979; Richards, Stirling, Ellis, 1983]. Full agreement with analytic NLO results
- Note that the individual term in the NLO results can diverge as $1/z^5$. Require cancellation between different terms to reduce to $1/z$ divergence

Large z limit

- For real, on-shell external momentum, $0 < z < 1$
- With the analytic NLO results, one can view $\Sigma_{\mathrm{EEC}}(z)$ as a function of z and analytically continue into the complex plane
- Individual terms can diverge as most as z^3 in the large z limit. However, the actual asymptotically scaling is

$$B_{\mathrm{lc}}(z) = \frac{1}{z^3}\left[\left(4\,\zeta_2 + \frac{4699}{288}\right)\log(-z) - 8\,\zeta_3 + \frac{991}{84}\,\zeta_2 - \frac{85595}{1728}\right] + \frac{i\,\mathrm{sign}(\mathrm{Im}(z))}{z^3}\left[\frac{11}{8}\,\zeta_2\sqrt{-z} + \pi\left(-\frac{1459}{140}\log(-z) + \frac{466259}{19600}\right)\right] + \mathcal{O}(1/z^{7/2})$$

- In $\mathcal{N}=4$ SYM, the surprising scaling behavior can be readily understood from the master formula

$$\mathrm{EEC} = \frac{a}{4z^2(1-z)}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\mathrm{d}j_1\mathrm{d}j_2}{(2\pi i)^2}M(j_1,j_2;a)K(j_1,j_2)\left(\frac{1-z}{z}\right)^{j_1+j_2}$$

The back-to-back limit

- Back-to-back limit: $\chi \to \pi$ or $z \to 1$

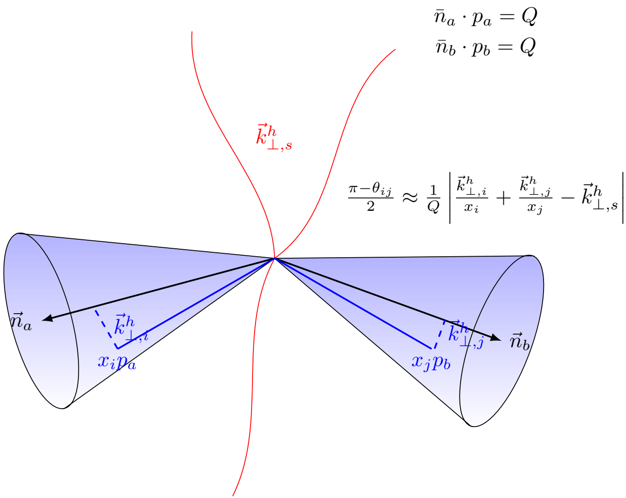
$$B(z) = C_F \left\{ \frac{1}{1-z} \left[+ \frac{1}{2} C_F \log^3(1-z) + \log^2(1-z) \left(\frac{11 C_A}{12} + \frac{9 C_F}{4} - \frac{N_f T_f}{3} \right) \right. \right.$$
$$+ \log(1-z) \left(C_A \left(\frac{\zeta_2}{2} - \frac{35}{72} \right) + C_F \left(\zeta_2 + \frac{17}{4} \right) + \frac{N_f T_f}{18} \right) + C_A \left(\frac{11\zeta_2}{4} + \frac{3\zeta_3}{2} - \frac{35}{16} \right)$$
$$\left. \left. + C_F \left(3\zeta_2 - \zeta_3 + \frac{45}{16} \right) + N_f T_f \left(\frac{3}{4} - \zeta_2 \right) \right] \right\}$$

- Double logarithmic series, typical Sudakov logarithms
- NLL resummation by [Collins, Soper, 1984] using dihadron production in back-to-back limit
- NNLL resummation by [de Florian, Grazzini, 2004]: direct calculation of NNLL large logarithm and match to CSS formula for pT resummation

$$\frac{1}{\sigma} \frac{d\Sigma^{(\text{res.})}}{d\cos\chi} = \frac{Q^2}{8} H(\alpha_s(Q^2)) \int_0^\infty db\, b J_0(b q_T) \exp\left\{ - \int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[A(\alpha_s(q^2)) \ln \frac{Q^2}{q^2} + B(\alpha_s(q^2)) \right] \right\}$$

- No manifest factorization into operator definition of matrix element. Relation of A and B to universal anomalous obscure
- Do not give prediction to $\delta(1-z)$

Kinematics



- Dominated by soft/collinear radiations $\Rightarrow$ Soft-Collinear Effective Theory (SCET_{II})

Factorization of triple differential distribution

Start from a triple differential distribution

$$\frac{d^3\sigma}{dx_i dx_j dz} = \frac{1}{2\pi}\int d^2\vec{k}_\perp \frac{d^3\sigma}{dx_i dx_j d^2\vec{k}_\perp}\delta\left(1 - z - \frac{\vec{k}_\perp^2}{Q^2}\right)$$

EEC is simply the moment of it

$$\frac{d\sigma}{dz} = \frac{1}{2}\sum_{ij}\int dx_i dx_j x_i x_j \frac{d^3\sigma}{dx_i dx_j dz}$$

The triple differential distribution readily factorized in SCET

$$\frac{d^3\sigma}{dx_i dx_j d^2\vec{k}_\perp} = \frac{1}{2}H(Q,\mu)\int d^2\vec{k}_{\perp,i}\int d^2\vec{k}_{\perp,j}\int d^2\vec{k}_{\perp,s}\ \delta^{(2)}\left(\vec{k}_\perp - \left(\frac{\vec{k}^h_{\perp,i}}{x_i} + \frac{\vec{k}^h_{\perp,j}}{x_j} - \vec{k}_{\perp,s}\right)\right)$$
$$\cdot F_{q\to i}(\vec{k}_{\perp,i}, x_i, \mu, \nu) F_{q\to j}(\vec{k}_{\perp,j}, x_j, \mu, \nu) S_{\mathrm{EEC}}(\vec{k}_{\perp,s}, \mu, \nu)\,.$$

TMD fragmentation function and soft function

$$\frac{d^3\sigma}{dx_i dx_j d^2\vec{k}_\perp} = \frac{1}{2} H(Q,\mu) \int d^2\vec{k}_{\perp,i} \int d^2\vec{k}_{\perp,j} \int d^2\vec{k}_{\perp,s}\ \delta^{(2)}\left(\vec{k}_\perp - \left(\frac{\vec{k}^h_{\perp,i}}{x_i} + \frac{\vec{k}^h_{\perp,j}}{x_j} - \vec{k}_{\perp,s}\right)\right)$$
$$\cdot F_{q\to i}(\vec{k}_{\perp,i}, x_i, \mu, \nu) F_{q\to j}(\vec{k}_{\perp,j}, x_j, \mu, \nu) S_{\mathrm{EEC}}(\vec{k}_{\perp,s}, \mu, \nu)\,.$$

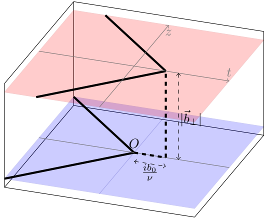
TMD fragmentation function [Collins; Echevarria, Scimemi, Vladimirov]

$$F_{q\to h}(\vec{b}_\perp, z_h) = \frac{1}{4 z_h N_c} \sum_X \int \frac{d\xi^+}{2\pi} e^{-i p_h^- \xi^+ / z_h} \langle 0 | \bar{\chi}_n(\xi) | X, h \rangle \not{\bar{n}} \langle X, h | \chi(0)(0) | 0 \rangle$$
$$\xi = (\xi^+, i b_0/\nu, \vec{b}_\perp)$$

The EEC soft function

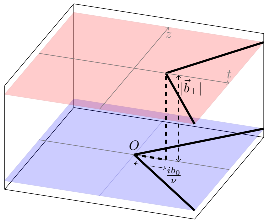
$$S_{\mathrm{EEC}}(\vec{b}_\perp, \mu, \nu) = \lim_{\nu\to+\infty} \frac{1}{N_c} \mathrm{tr} \langle 0 | T\left[S^\dagger_{\bar{n}+}(0) S_{n-}(0)\right] \bar{T}\left[S^\dagger_{n+}\left(y_\nu(\vec{b}_\perp)\right) S_{\bar{n}-}\left(y_\nu(\vec{b}_\perp)\right)\right] | 0 \rangle$$
$$y_\nu(\vec{b}_\perp) = (i b_0/\nu, i b_0/\nu, \vec{b}_\perp)$$

The $ib_0/\nu = i2e^{-\gamma_E}/\nu$ comes from exponential regularization for rapidity divergence [Y. Li, Neill, HXZ]



$$S_{\rm EEC}(\vec{b}_\perp,\mu,\nu) = \lim_{\nu\to+\infty}\frac{\rm tr}{N_c}\langle 0|T\left[S^\dagger_{\bar{n}+}(0)S_{n-}(0)\right]\bar{T}\left[S^\dagger_{n+}\left(y_\nu(\vec{b}_\perp)\right)S_{\bar{n}-}\left(y_\nu(\vec{b}_\perp)\right)\right]|0\rangle$$

Upon time reversal, translation, and parity in $\perp$ space, $\Rightarrow$ the TMD soft function



$$S_\perp(\vec{b}_\perp,\mu,\nu) = \lim_{\nu\to+\infty}\frac{\rm tr}{N_c}\langle 0|T\left[S^\dagger_{\bar{n}-}(0)S_{n+}(0)\right]\bar{T}\left[S^\dagger_{n-}\left(y_\nu(\vec{b}_\perp)\right)S_{\bar{n}+}\left(y_\nu(\vec{b}_\perp)\right)\right]|0\rangle$$

$$\boxed{S_{\rm EEC} = S_\perp}$$

TMD soft function known to 3 loops [Y. Li, HXZ]. For discussion of crossing relation for SCET$_{\rm I}$ soft function, see [Kang, Labun, Lee]

EEC now written as multi-dimensional convolution integral

$$\frac{d\sigma}{dz} = \frac{1}{2}\sum_{ij}\int dx_i dx_j x_i x_j \int d^2\vec{k}_\perp \delta\left(1 - z - \frac{\vec{k}_\perp^2}{Q^2}\right)$$
$$\cdot H(Q,\mu)\int d^2\vec{k}_{\perp,i}\int d^2\vec{k}_{\perp,j}\int d^2\vec{k}_{\perp,s}\delta^{(2)}\left(\vec{k}_\perp - \left(\frac{\vec{k}^h_{\perp,i}}{x_i} + \frac{\vec{k}^h_{\perp,j}}{x_j} - \vec{k}_{\perp,s}\right)\right)$$
$$\cdot F_{q\to i}(\vec{k}_{\perp,i}, x_i, \mu, \nu) F_{q\to j}(\vec{k}_{\perp,j}, x_j, \mu, \nu) S_{\mathrm{EEC}}(\vec{k}_{\perp,s}, \mu, \nu) \tag{1}$$

$F_{q\to i}$ are non-perturbative objects. To use this formula in perturbative regime, we need to perform operator production expansion

$$F_{i\to h}(\vec{k}_\perp, z_h) = \sum_j \int \frac{dz}{z}\mathcal{I}_{ij}\left(\vec{k}_\perp, \frac{z_h}{z}\right) f_{j\to h}(z,\mu)\left[1 + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{(\vec{k}^h_\perp)^2}\right)\right] \tag{2}$$

Substitute (2) into (1), and apply the momentum conservation sum rule

$$\sum_h \int dz\ z\ f_{j\to h}(z) = 1$$

Factorization formula for EEC in the back-to-back limit

$$
\frac{d\sigma}{dz} = \frac{1}{2}\sum_{ij}\int d\tau_i d\tau_j\ \tau_i\tau_j \int d^2\vec{k}_\perp \delta\left(1-z-\frac{\vec{k}_\perp^2}{Q^2}\right)
$$
$$
\cdot H(Q)\int d^2\vec{k}_{\perp,i}\int d^2\vec{k}_{\perp,j}\int d^2\vec{k}_{\perp,s}\delta^{(2)}\left(\vec{k}_\perp - \left(\frac{\vec{k}_{\perp,i}}{\tau_i}+\frac{\vec{k}_{\perp,j}}{\tau_j}-\vec{k}_{\perp,s}\right)\right)
$$
$$
\cdot \mathcal{I}_{qi}(\tau_i,\vec{k}_{\perp,i})\cdot\mathcal{I}_{qj}(\tau_j,\vec{k}_{\perp,j})S_{\mathrm{EEC}}(\vec{k}_{\perp,s})
$$

Now the non-perturbative fragmentation function disappear. At a perturbative scale, all ingredients in this factorization formula can be calculated perturbatively Further factorize the convolution by Fourier transformation

$$
\delta^{(2)}\left(\vec{k}_\perp - \left(\frac{\vec{k}_{\perp,i}}{\tau_i}+\frac{\vec{k}_{\perp,j}}{\tau_j}-\vec{k}_{\perp,s}\right)\right) = \int \frac{d^2\vec{b}_\perp}{(2\pi)^2}\ \exp\left[-i\vec{b}_\perp\cdot\vec{k}_\perp + i\vec{b}_\perp\cdot\left(\frac{\vec{k}_{\perp,i}}{\tau_i}+\frac{\vec{k}_{\perp,j}}{\tau_j}-\vec{k}_{\perp,s}\right)\right]
$$

The final factorization formula

$$
\boxed{\frac{d\sigma}{dz} = \frac{1}{2}\int d^2\vec{k}_\perp\int\frac{d^2\vec{b}_\perp}{(2\pi)^2}e^{-i\vec{b}_\perp\cdot\vec{k}_\perp}H(Q,\mu)J^q_{\mathrm{EEC}}(\vec{b}_\perp,\mu,\nu)J^{\bar{q}}_{\mathrm{EEC}}(\vec{b}_\perp,\mu,\nu)S_{\mathrm{EEC}}(\vec{b}_\perp,\mu,\nu)\delta\left(1-z-\frac{\vec{k}_\perp^2}{Q^2}\right)}
$$

$$
J^q_{\mathrm{EEC}}(\vec{b}_\perp) = \sum_i\int_0^1 dx\ x\ \mathcal{I}_{qi}\left(\frac{\vec{b}_\perp}{x},x\right)
$$

Resummation for EEC in the back-to-back limit

- RG evolution in virtuality and rapidity [Chiu, Jain, Neill, Rothstein]

$$\frac{d\sigma}{dz} = \frac{1}{4}\int_0^\infty db\, b J_0(bQ\sqrt{1-z}) H(Q,\mu_h) j^q_{\mathrm{EEC}}(b, b_0/b, Q) j^{\bar{q}}_{\mathrm{EEC}}(b, b_0/b, Q) S_{\mathrm{EEC}}(b,\mu_s,\nu_s)$$
$$\cdot \left(\frac{Q^2}{\nu_s^2}\right)^{\gamma^r_{\mathrm{EEC}}(\alpha_s(b_0/b))} \exp\left[\int_{\mu_s^2}^{\mu_h^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma_{\mathrm{cusp}}(\alpha_s(\bar{\mu}))\ln\frac{b^2\bar{\mu}^2}{b_0^2}\right.$$
$$\left. + \int_{\mu_h^2}^{b_0^2/b^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\Gamma_{\mathrm{cusp}}(\alpha_s(\bar{\mu}))\ln\frac{b^2Q^2}{b_0^2} + \gamma^H(\alpha_s(\bar{\mu}))\right) - \int_{\mu_s^2}^{b_0^2/b^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\gamma^s_{\mathrm{EEC}}(\alpha_s(\bar{\mu}))\right]$$

- TMD physics in $e^+e^- \to$ jets [See also Neill, Scimemi, Waalewijn]
- All ingredients known to high enough orders for N^{3}LL resummation
- Rapidity anomalous dimension control Q evolution. Potential enhanced non-perturbative effects [Becher, Bell]

Summary and outlook

- Analytical results for EEC at NLO
  - Provide data for perturbative power corrections to $\mathrm{SCET_{II}}$ observable
  - Mathematical structure for IRC safe observable
- Resummation in the back-to-back region can be done at $\mathrm{N^3LL}$
  - TMD physics in e^+e^- to jets
  - Precision α_s extraction
- Future directions
  - Is there a space of function that make the behavior of EEC in $z \to 0$ and $z \to \infty$ manifest?
  - Calculation of EEC at NNLO. Improvement in IBP reduction and better knowledge of boundary condition needed
  - Resummation in collinear limit
  - Three point energy correlation, at least in the collinear limit
  - Insight for non-perturbative corrections from operator definition