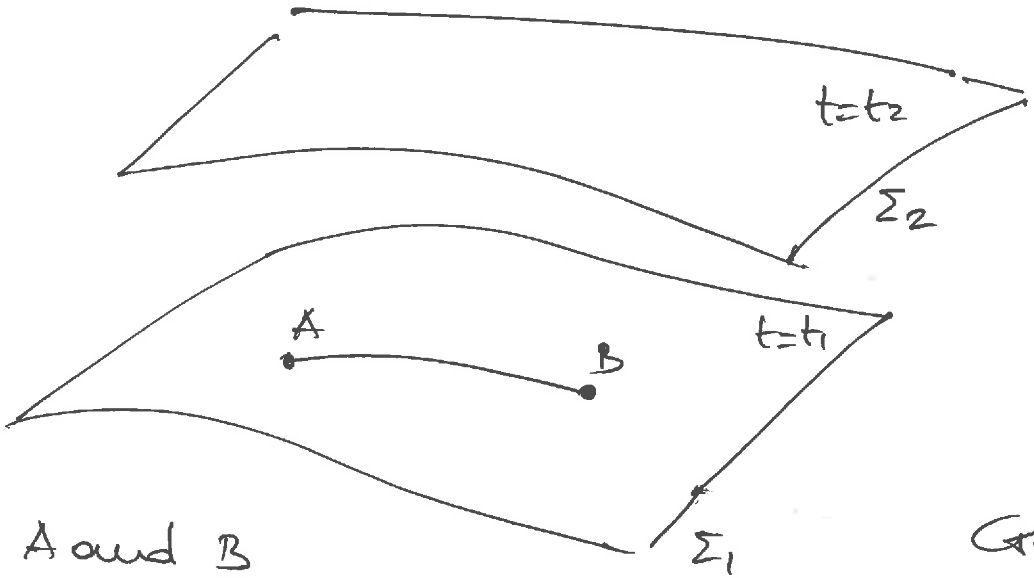
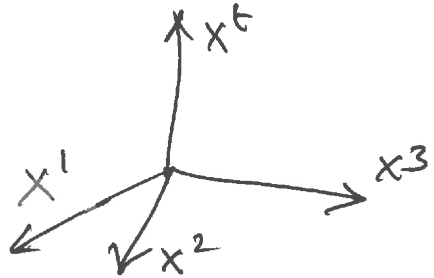
3+1 splitting of spacetime

Let $\mathcal{M}$ be a 4-dimensional manifold endowed with a metric $\underline{g}$; $\mathcal{M}$ can be split in purely spatial hypersurfaces $\Sigma(t)$ ordered by the coordinate t





A and B are spacelike seperated

Given a surface, a first possible operation is the definition of the local normal vector $\underline{n}$

$$\boxed{\Omega_\mu := \nabla_\mu t}$$

$\tilde{\Omega}$ is a one-form expressing the rate at which the time coordinate t is changing

$\underline{n}$: normal unit timelike vector normal to Σ_t

$n_\mu := A\,\Omega_\mu$ $\qquad$ $\tilde{n}$ will be proportional to $\tilde{\Omega}$

$$|\Omega|^2 = \Omega_\mu \Omega^\mu = g^{\mu\nu}\,\Omega_\mu \Omega_\nu =$$

$$= g^{\mu\nu}\,\nabla_\mu t\,\nabla_\nu t$$

$$= g^{tt}\,\nabla_t t\,\nabla_t t = g^{tt}$$

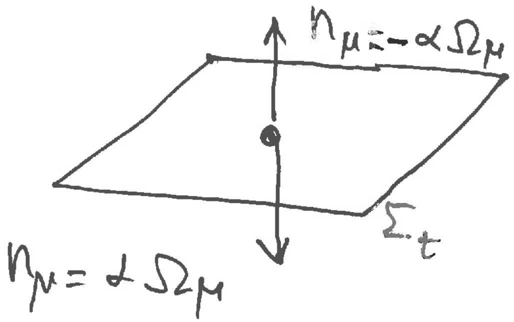
We can now compute the norm of n and fix the proportionality constant between $\tilde{n}$ and $\tilde{\Omega}$

$$n_\mu n^\mu = A^2 \Omega_\mu \Omega^\mu = A^2 g^{tt} = -1$$ (we want unit timelike normal)

$$\Rightarrow \quad A^2 = -\frac{1}{g^{tt}} = +\alpha^2$$

lapse function: $\alpha = \alpha(x^\mu)$ ie function of space and time

so that $A = \begin{cases} +\alpha \\ -\alpha \end{cases}$: choice necessary if we want to consider the future directed normal



$$\boxed{n_\mu = -\alpha \Omega_\mu = -\alpha \nabla_\mu t}$$

$$= (-\alpha, 0, 0, 0)$$

The Contravariant component will be

$$n^\mu = g^{\mu\nu} n_\nu = -g^{\mu\nu} \alpha \nabla_\nu t = -\alpha \nabla^\mu t$$

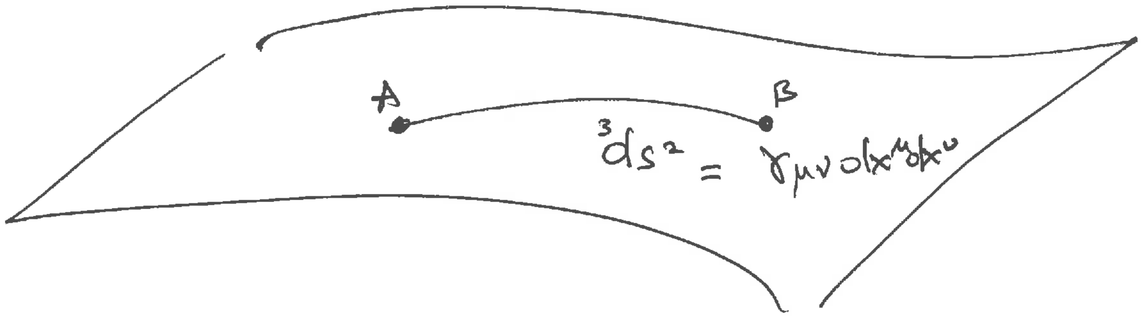
Using now $g_{\mu\nu}$ and n_μ we can build a tensor that is orthogonal to $\underline{n}$: ie the metric on Σ_t

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$$

$$\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}$$: purely spatial tensor

$\underline{\gamma} \cdot \underline{n} = 0$ $\quad$ $\gamma_{\mu\nu} n^\mu = g_{\mu\nu} n^\mu + n_\mu n_\nu n^\mu$

$$= n_\nu - n_\nu = 0$$



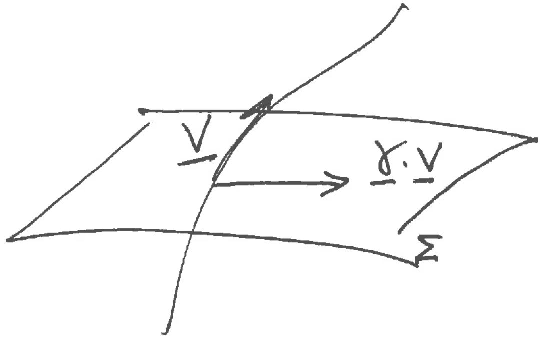
The mixed components of $\underline{\gamma}$ are given by

$$\gamma^\mu{}_\nu = g^\mu{}_\nu + n^\mu n_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$

Hence $\gamma^\mu{}_\nu A_\mu = A_\nu$ if $A_\mu n^\mu = 0$ ie

$\underline{\gamma}$ can be used to raise/lower indices but only of purely spatial tensors. For fully 4D tensor the metric is needed to raise/lower indices.

$\underline{\gamma}$ is therefore a <u>spatial projection tensor</u>, ie a tensor that given a 4D tensor will return the spatial projection of it on Σ.



In a similar way we want an operator that projects "out" of Σ, ie in the time direction

$$N^\mu{}_\nu = - n^\mu n_\nu$$

where of course

$$\underline{N} \cdot \underline{\gamma} = 0 \qquad N^\mu{}_\nu \gamma^\nu{}_\mu = -(n^\mu n_\nu)(\delta^\mu{}_\nu + n^\nu n_\mu)$$

$$= -n^\mu n_\mu - n^\mu n_\mu n_\nu n^\nu$$

$$= +1 - 1 = 0$$

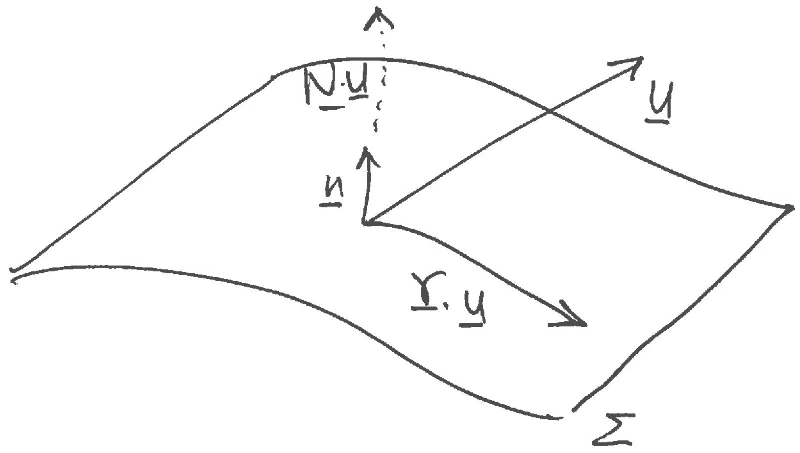
In this way we have the mathematical tools not only to split the manifold $\mathcal{M}$ in 3+1 but also any tensor in it.

Eg Take 4-vector $\underline{u}$

$$\underline{u} = \underline{\gamma}\cdot\underline{u} + \underline{N}\cdot\underline{u}$$

: ie split $\underline{u}$ in a purely time-like part and in a purely space-like part

$$= (\underline{\gamma} + \underline{N})\cdot\underline{u}$$



$$u^\mu{}_\nu = \underbrace{\gamma^\mu{}_\nu u^\nu}_{\text{purely spatial}} + \underbrace{N^\mu{}_\nu u^\nu}_{\text{purely timelike}}$$

$$= \underline{A} + \underline{B}$$

$$A^\mu = (0, A^1, A^2, A^3)$$

$$B^\mu = (B^0, 0, 0, 0)$$

Note: $\underline{n}$ is the unit normal to any point in Σ

$\underline{n}$ does not represent the direction along which the time coordinate varies. To check this, let's compute

$$\underline{n}\cdot\tilde{\Omega} = n^{\mu}\Omega_{\mu} \underset{n_\mu = A\,\Omega_\mu}{=} \frac{1}{A}\, n^{\mu} n_{\mu} \underset{A=-\alpha}{=} \frac{1}{\alpha} \neq 1 \qquad \text{since } \alpha = \alpha(x^i)$$

In other words, $\boxed{\underline{n}\cdot\tilde{\Omega} = \alpha^{-1}}$ expresses the fact that the gradient of the time coordinate ($\tilde{\Omega}$) along a given direction ($\underline{n}$) is a function!

We need a new timelike 4-vector $\underline{t}$ such that

$\boxed{\underline{t}\cdot\tilde{\Omega} = 1}$ and that acts as the basis vector of the time coordinate

$$\underline{t} := \underline{e}_t = \underbrace{\alpha\,\underline{n}}_{\text{part proportional to } \underline{n}} + \overset{\text{purely spatial part}}{\underline{\beta}} \qquad : \text{most generic definition}$$

α: lapse function

$\underline{\beta}$: shift vector $\qquad \underline{\beta}\cdot\underline{n} = 0$ (purely spatial)

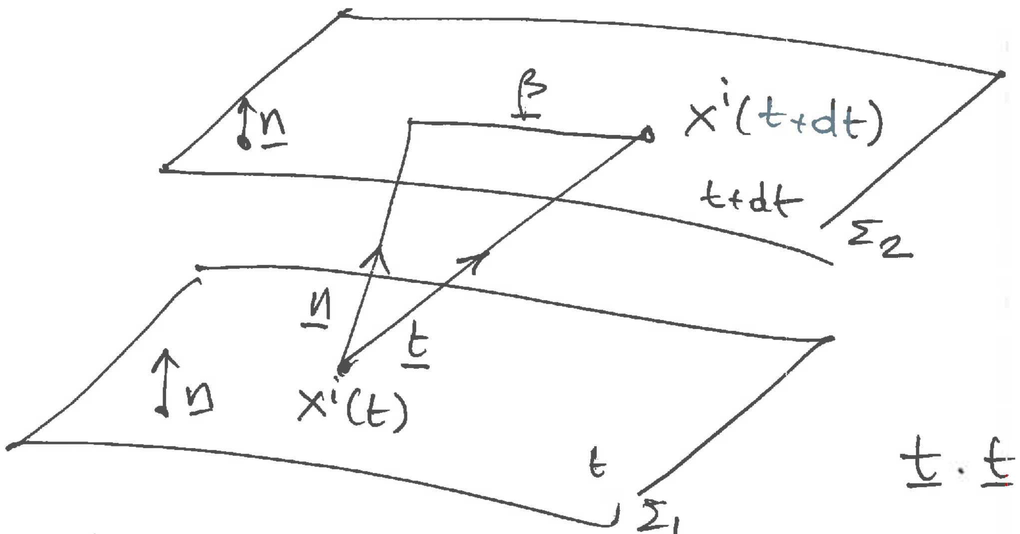
$$\underline{t}\cdot\tilde{\Omega} = (\alpha\underline{n} + \underline{\beta})(\tilde{\Omega}) \iff$$

$$\left(\alpha n^\mu + \beta^\mu\right)\left(-\frac{n_\mu}{\alpha}\right) = 1 + 0 = 1$$

$\underline{t}$ and $\underline{\Omega}$ are // always

$\underline{t}$ and $\underline{n}$ are // if $\alpha = 1$, $\underline{\beta} = 0$

(flat spacetime)



$$n_\mu = -\alpha\nabla_\mu t$$

$$x^i(t+dt) = x^i(t) - \beta^i(t, x^i)\,dt$$

$$\underline{t}\cdot\underline{t} = g_{tt} = -\alpha^2 + \beta^\mu\beta_\mu$$

$$= -\alpha^2 + \beta^i\beta_i$$

$\neq -1$: $\underline{t}$ is not a unit vector

Using the lapse and shift we can write the tt and ti parts of the metric as

$$g_{tt} = \underline{t} \cdot \underline{t} = -\alpha^2 + \beta^i \beta_i \qquad \left[(\alpha n^\mu + \beta^\mu)(\alpha n_\mu + \beta_\mu) = \alpha^2 n^\mu n_\mu + 2\alpha \cancel{n^\mu \beta_\mu} + \beta^\mu \beta_\mu \right]$$

$$g_{ti} = \underline{t} \cdot \underline{\gamma} = t^\mu \gamma_{\mu i} = t^\mu (g_{\mu i} + n_\mu \cancel{n_i}) = t_i =$$

$$\underset{\underline{t}\cdot\underline{n}=0}{=} (\alpha n^\mu + \beta^\mu) g_{\mu i} = \alpha n_i + \beta^\mu g_{\mu i}$$

$$= \alpha \cdot 0 + \beta^j \gamma_{ij}$$

$$= + \beta_i$$

so that the line element reads

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -(\alpha^2 - \beta^i \beta_i) & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} ; \quad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

As a result

$$n_\mu = (-\alpha, 0, 0, 0) \; ; \qquad n^\mu = \frac{1}{\alpha}(1, -\beta^i)$$

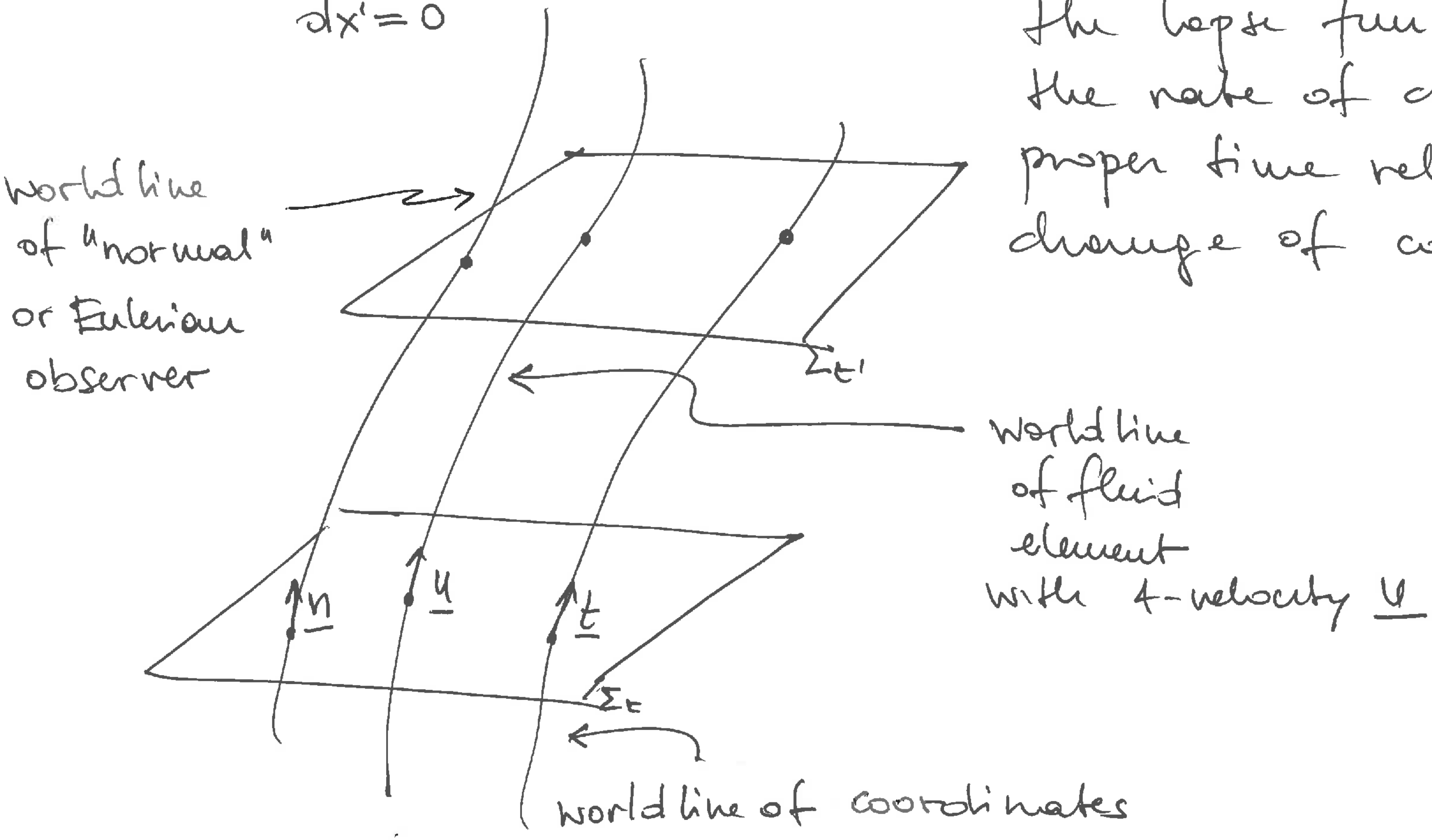
$$n^i = g^{\mu i} n_\mu = g^{0i} n_0 = -\alpha\left(\frac{\beta^i}{\alpha^2}\right) = -\beta^i/\alpha$$

$$n^0 = g^{00} n_0 = -\frac{1}{\alpha^2}\cdot(-\alpha) = \frac{1}{\alpha}$$

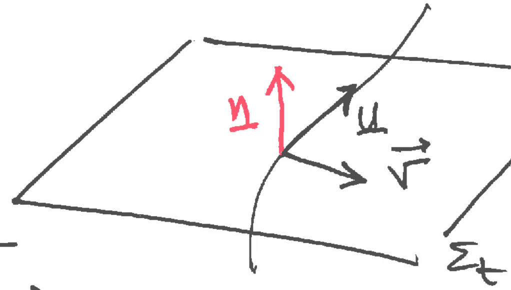
$$d\tau^2 = -ds^2 \underset{dx^i=0}{=} +\alpha^2 dt^2$$

$$\underbrace{d\tau = \pm \alpha dt}$$

the lapse function expresses the rate of change of proper time relative to the change of coordinate time.



An observer with tangent vector $\underline{n}$ is said to be a "normal" observer or "Eulerian" observer. This is the standard observer in a 3+1 split and it is relative to this observer that 3+1 quantities are measured. This is the case also for the fluid four-velocity $\underline{u}$



$$\vec{v}: \left(\begin{array}{c}\text{spatial part}\\ \text{of } \underline{u}\end{array}\right) = \frac{(\text{projection of } \underline{u} \text{ on } \Sigma_t)}{(\text{projection of } \underline{u} \text{ along } \underline{n})}$$

$$= \frac{(\text{space})}{(\text{time})} = \frac{\gamma^i{}_\mu u^\mu}{-u_\mu n^\mu}$$

$W := -n_\mu u^\mu = \alpha u^t$: Lorentz factor

$W = (1 - v^i v_i)^{-1/2}$ as in special relativity (Exercise)

$W \to \infty$ for $v \to 1$

Prove $W = (1 - v^i v_i)^{-1/2}$

Proof

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right) \; ; \; v_i = \gamma_{ij} v^j = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^0} + \beta^j\right)$$

$$v^i v_i = \frac{1}{\alpha^2}\left[\left(\frac{u^i}{u^0} + \beta^i\right)\gamma_{ij}\left(\frac{u^j}{u^0} + \beta^j\right)\right] = \frac{1}{\alpha^2}\left[\gamma_{ij}\frac{u^i u^j}{(u^0)^2} + 2\beta^i \frac{u_i}{u^0} + \beta^i \beta_i\right] =$$

$$-1 = u^\mu u_\mu =$$

$$= g_{00}(u^0)^2_i + 2 g_{0i} u^0 u^i + u^i u_i$$

$$= -(\alpha^2 - \beta^i \beta_i)(u^0)^2_i + 2\beta_i u^i u^0 + u^i u_i \Rightarrow$$

$$= -1 + (\alpha u^0)^2$$

$$= \frac{1}{(\alpha u^0)^2}\left[u^i u_i + 2\beta^i u_i u^0 + (u^0)^2 \ldots\right]$$

$$= \frac{1}{(\alpha u^0)^2}\left(-1 + (\alpha u^0)^2\right) = \frac{-1 + W^2}{W^2} \rightarrow W^2(1 - v^i v_i) = 1 \Rightarrow$$

$$W = (1 - v^i v_i)^{-1/2} \checkmark$$

In component form:

$$v^t = 0 \quad ; \quad v^i = \frac{\gamma^i{}_\mu u^\mu}{\alpha u^t} = \frac{(\delta^i_\mu + n^i n_\mu) u^\mu}{\alpha u^t} = \frac{u^i}{\alpha u^t} + \left(-\frac{\beta^i}{\alpha}\right) \frac{\overbrace{n_\mu u^\mu}^{-\alpha u^t}}{\alpha u^t}$$

$$= \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

$$v_t = 0 \quad ; \quad v_i = \frac{\gamma_{i\mu} u^\mu}{\alpha u^t} = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \frac{\gamma_{i0} u^0}{u^0}\right) = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \beta^j\right);$$

In other words, using $W = \alpha u^t$

$$\boxed{v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}} \qquad \boxed{v_i = \frac{u_i}{W} = \gamma_{ij}\left(\frac{u^j}{W} + \frac{\beta^j}{\alpha}\right)}$$

Note that $\underline{\gamma}$ does not lower indices of $\underline{u}$ (4D object)

I recall that in special relativity

$$v^i = \frac{u^i}{u^t} = \frac{dx^i/dt}{dt/d\tau} = \frac{dx^i}{d\tau}$$

Recap $\quad v^i = \dfrac{u^i}{W} + \dfrac{\beta^i}{\alpha} = \dfrac{u^i}{\alpha u^t} + \dfrac{\beta^i}{\alpha}$

compare with special-relativistic expression

$$v^i_{SR} = \frac{u^i}{u^t} = \frac{u^i}{\gamma}$$

Hence, the three-velocity gains a dependence on α and $\underline{\beta}$

$v^i = v^i_{SR}$ if $\alpha = 1$; $\underline{\beta} = 0$

It's easy to show that

$$\underline{u} = W(\underbrace{\underline{n}}_{\text{purely time part}} + \underbrace{\underline{v}}_{\text{purely spatial part}})$$

$\Big(W = -\underline{n}\cdot\underline{u} \Rightarrow \underline{n} W = \underline{u}$

$u^i = W v^i - \dfrac{\beta^i}{\alpha} W = W n^i + W v^i$

$u^0 = W n^0 = +\dfrac{1}{\alpha} W \Big)$

Next, we will derive the most famous 3+1 formulation of the Einstein eqs, the ADM formulation (from Arnowitt, Deser, Misner 1962).

Notes

- the ADM formulation was not derived for numerical solutions but for Hamiltonian formulation of the Einstein eqs
- the ADM formulation is seldom used in practice and I'll explain why
- much of the formulation I present comes from the formalism introduced by York (1979)

An important first step in the derivation of the ADM eqs is the definition of the <u>spatial covariant derivative</u>

$$D_\nu := \gamma^\mu{}_\nu \nabla_\mu = (\delta^\mu{}_\nu + n^\mu n_\nu) \nabla_\mu$$

Just like the 4D cov. derivative is compatible with the 4-metric ie

$$\nabla_\mu g^{\mu\nu} = 0$$

so is the 3D cov. derivative with the 3-metric

$$D_\mu \gamma^{\mu\nu} = 0 .$$

In practice, the spatial covariant derivative is just the result of the projection of the 4D one, eg

$$\boxed{D_\alpha T^\gamma{}_\beta = \gamma^\mu{}_\alpha \gamma_\beta{}^\rho \gamma_\nu{}^\gamma \nabla_\mu T^\nu{}_\rho}$$

what is important is that the spatial covariant derivative is based on the 3D Christoffel symbols.

I recall that the Chistoffel symbols or connections are first derivatives of the metric

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma} \right)$$

so that the corresponding 3D (spatial) objects are

$$ {}^{(3)}\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\delta} \left(\partial_\beta \gamma_{\gamma\delta} + \partial_\gamma \gamma_{\delta\beta} - \partial_\delta \gamma_{\beta\gamma} \right)$$

As for the 4D Chistoffels, also the 3D ones follow the same properties: symmetric on lower indices and not proper tensors (do not transform as tensors)

If we want to express a 3+1 decomposition of the Einstein eqs. we have to follow the same route in 4D spacetimes:

$$g_{\mu\nu} \longrightarrow \Gamma^{\alpha}_{\mu\nu} \longrightarrow R^{\alpha}{}_{\beta\mu\nu} \longrightarrow R_{\mu\nu} \longrightarrow G_{\mu\nu}$$

(metric) (Riemann tensor) (Ricci tensor) (Einstein tensor)

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$$

I recall that the 4D curvature tensor is

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta} \Gamma^{\lambda}_{\nu\alpha}$$

$$= f(\partial^2 g, (\partial g)^2) \quad ; \quad [R^{\mu}{}_{\nu\alpha\beta}] = L^{-2}$$

The corresponding spatial 3D tensor is

$$^{(3)}R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha {}^{(3)}\Gamma^{\mu}_{\nu\beta} - \partial_\beta {}^{(3)}\Gamma^{\mu}_{\nu\alpha} + {}^{(3)}\Gamma^{\mu}_{\lambda\alpha} {}^{(3)}\Gamma^{\lambda}_{\nu\beta} - {}^{(3)}\Gamma^{\mu}_{\lambda\beta} {}^{(3)}\Gamma^{\lambda}_{\nu\alpha}$$

$^{(3)}R^{\mu}{}_{\nu\alpha\beta}$ is purely spatial $\quad {}^{(3)}R^{\mu}{}_{\nu\alpha\beta}\, n_\mu = 0$

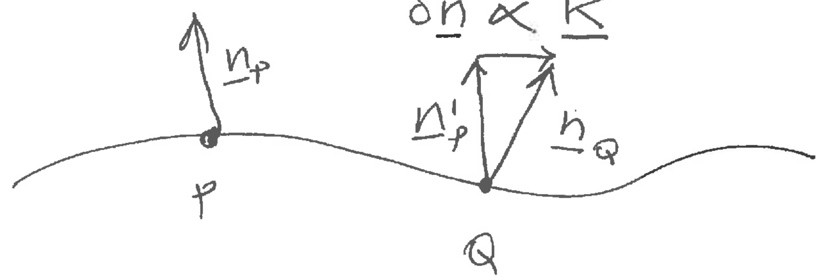
$\Longrightarrow {}^{(3)}R^{\mu}{}_{\alpha\mu\beta} = {}^{(3)}R_{\alpha\beta}$: spatial 3D <u>Ricci tensor</u>

$^{(3)}R^{\alpha}{}_{\alpha} = {}^{(3)}R$: " 3D <u>Ricci scalar</u>

Note that in going from a 4D curvature tensor $\underline{R}$ to a 3D curvature tensor ${}^{(3)}\underline{R}$ we clearly lose some information, namely the information about how the spatial hypersurface Σ_t is "bent" relative to the embedding 4D spacetime: ie <u>extrinsic curvature</u>

We will understand this concept better with some examples but let's first learn how to measure the extrinsic curvature ${}^{(3)}\underline{K} = \underline{K}$

It is quite intuitive that a way of measuring the extrinsic curvature is one in which we compare the difference in normals on the hypersurface Σ_t



Let $\underline{n}_P$ be the normal 4vector in P and $\underline{n}_Q$ the equivalent in Q.

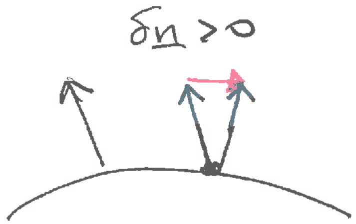
Let $\underline{n}'_P$ the 4vector $\underline{n}_P$ parallel-transported at Q. We can compare $\underline{n}'_P$ and $\underline{n}_Q$

Of course we are interested in the projection of $\delta\underline{n}$ on Σ_t

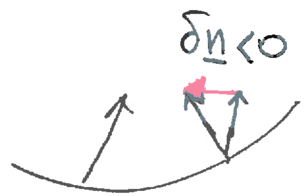
$$\boxed{\underline{K} = -\underline{\gamma} \cdot \delta \underline{n}} \iff \boxed{K_{\mu\nu} = -\gamma^{\alpha}_{\mu} \nabla_{\nu} n_{\alpha}}$$

Question: why do we choose the negative sign?

Answer: the two signs distinguish "convex" and "concave" hypersurfaces



negative extrinsic curvature



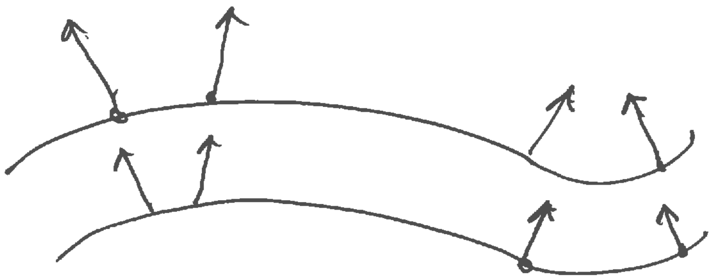
positive extrinsic curvature

Note that this is not the only way of defining the extrinsic curvature.

other ways to measure the extrinsic curvature are offered by the "evolution" of the unit normals, ie by the acceleration of <u>normal observers</u>

(acceleration of fluid): $\tilde{a}_\mu = u^\nu \nabla_\nu u_\mu$

(" " normal observers): $a_\mu = n^\nu \nabla_\nu n_\mu$

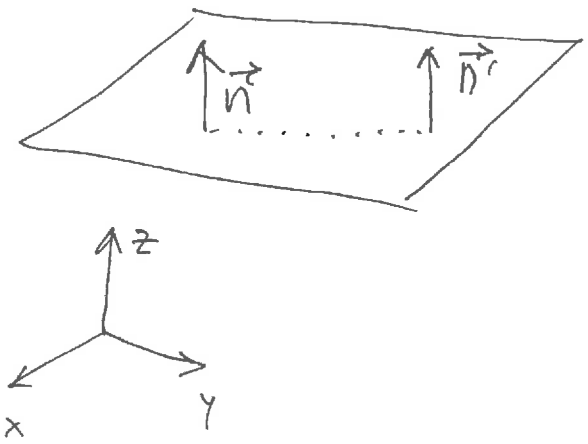


$$\boxed{K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu}$$

The extrinsic curvature tells us about how the hypersurface is curved ("bent") with respect to the 4D manifold.

This is not easy for us to picture, but quite familiar if we restric to 2D surfaces embedded in a Euclid 3D space (no time here...)

The first example is a plane in Euclidean (flat) $\mathbb{R}^3$



Consider a Cartesian coordinate system $X^i = (x, y, z)$ so that plane is surface $z = 0$; the scalar function t defining Σ_t is simply $t = z$

The spatial metric γ_{ij} induced by $g_{\alpha\beta}$ is diagonal with components

$$\gamma_{ij} = \mathrm{diag}\,(1, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2$$

$\boxed{\underline{R} = \mathrm{Riem} = 0}$

obviously the Riemann tensor is zero ; The normal 3-vector is $n^i = (0, 0, 1)$; $n_i = (0, 0, 1)$ $\vec{n} \cdot \vec{n} = 1$

The extrinsic curvature is then

$$K_{ij} = -\gamma^k_j \nabla_i n_k \qquad i = 1, 2 ;$$

$$= -\gamma^3_j \underbrace{\nabla_i n_3}_{\partial_x z = \partial_y z = 0}$$

$$= 0$$

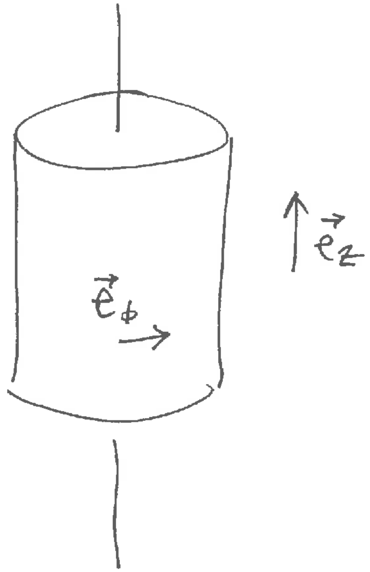
$\boxed{\mathrm{Riem} = 0 = \underline{K}}$

The extrinsic curvature of a plane is zero : normals can be parallel transported and do not change

We can move on with complexity:

cylinder in $\mathbb{R}^3$

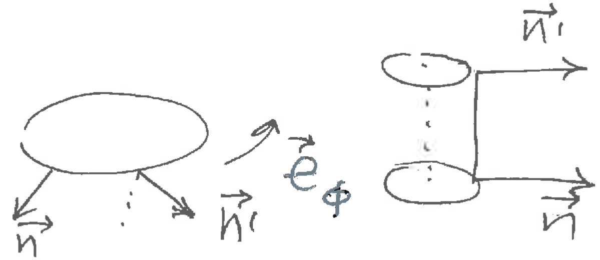
(ρ, ϕ, z)



It is not difficult to show that Riem $= 0$

(you can "cut" a cylinder and lay on a plane without wrinkles)

What about the extrinsic curvature?

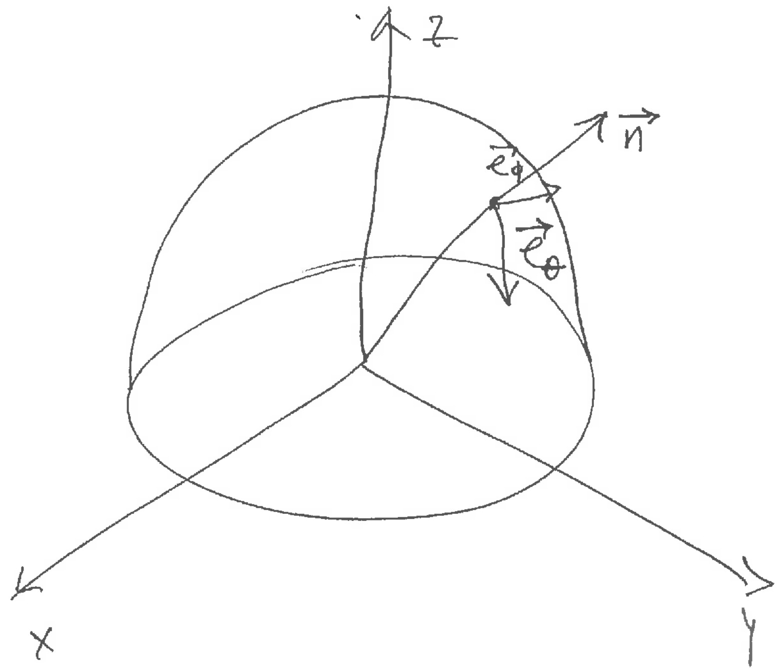


Exercise will show that extrinsic curvature vanishes in the $\vec{e}_z$ direction but is nonzero in the $\vec{e}_\phi$ direction

What about a 2-sphere in $\mathbb{R}^3$?

You will see that Riem $\neq 0$

$K_{ij} \neq 0$ in $\vec{e}_\phi$, $\vec{e}_\theta$ directions

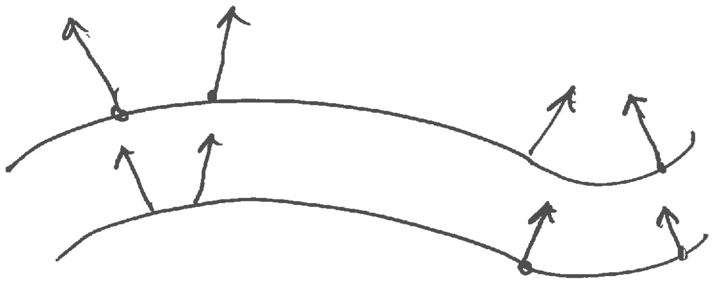


<u>Exercise</u>

other ways to measure the extrinsic curvature are offered by the "evolution" of the unit normals, ie by the acceleration of <u>normal observers</u>

(acceleration of fluid): $\tilde{a}_\mu = u^\nu \nabla_\nu u_\mu$

(" " normal observers): $a_\mu = n^\nu \nabla_\nu n_\mu$



$$\boxed{K_{\mu\nu} = - \nabla_\mu n_\nu - n_\mu a_\nu}$$

This definition has a marked physical meaning: Eulerian observers are passive tracers and evolve following the curvature of spacetime: they will converge if curvature is positive and viceversa.

Let's brush up a bit the concept of Lie derivative to discuss yet another way of computing the extrinsic curv.

$\mathcal{L}_{\underline{V}} \underline{U} = \nabla_{\underline{V}} \underline{U} - \nabla_{\underline{U}} \underline{V}$: cov. derivative of a vector field relative to another " "

$= -[\underline{U}, \underline{V}] = [\underline{V}, \underline{U}]$

$(\mathcal{L}_{\underline{V}} \underline{U})^{\mu} = V^{\nu} \partial_{\nu} U^{\mu} - U^{\nu} \partial_{\nu} V^{\mu}$; $(\mathcal{L}_{\underline{V}} \underline{U})_{\mu} = V^{\nu} \partial_{\nu} U_{\mu} + U^{\nu} \partial_{\nu} V_{\mu}$

Properties

$\mathcal{L}_{\phi \underline{V}} \underline{T} = \phi \mathcal{L}_{\underline{V}} \underline{T} - \underline{V} \mathcal{L}_{\underline{T}} \phi$

$\mathcal{L}_{\underline{V}} \phi = V^{\alpha} \partial_{\nu} \phi$

$\mathcal{L}_{\underline{V}} (a Y^{\alpha\nu} + b Z^{\beta\nu}) = a \mathcal{L}_{\underline{V}} Y^{\alpha\nu} + b \mathcal{L}_{\underline{V}} Z^{\beta\nu}$

$\mathcal{L}_{\underline{V}} (Z^{\mu\nu} Y_{\alpha\beta}) = \mathcal{L}_{\underline{V}} (Z^{\mu\nu}) Y_{\alpha\beta} + Z^{\mu\nu} \mathcal{L}_{\underline{V}} Y_{\alpha\beta}$

$$\mathcal{L}_{\underline{v}} T^{\alpha}{}_{\beta} = V^{\mu} \partial_{\mu} T^{\alpha}{}_{\beta} - T_{\beta}{}^{\mu} \partial_{\mu} V^{\alpha} + T^{\alpha}{}_{\mu} \partial_{\beta} V^{\mu} .$$

we can now apply the Lie derivative of the spatial metric relative to the normal vector, ie

$$\mathcal{L}_{\underline{n}} \gamma_{\mu\nu} = n^{\alpha} \nabla_{\alpha} \gamma_{\mu\nu} + \gamma_{\mu\alpha} \nabla_{\nu} n^{\alpha} + \gamma_{\nu\alpha} \nabla_{\mu} n^{\alpha}$$

| Exercise

$$= -2 k_{\mu\nu}$$

from which we obtain

$$\boxed{k_{ij} = -\frac{1}{2} \mathcal{L}_{\underline{n}} \gamma_{ij}}$$

Recalling now that $\underline{t} = \alpha \underline{n} + \underline{\beta}$

$$\mathcal{L}_{\underline{n}} = \frac{1}{\alpha} \mathcal{L}_{\alpha \underline{n}} = \frac{1}{\alpha} \left(\mathcal{L}_{\underline{t}} - \mathcal{L}_{\underline{\beta}} \right) = \frac{1}{\alpha} \left(\partial_t - \mathcal{L}_{\underline{\beta}} \right) \Rightarrow$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\underline{\beta}} \gamma_{ij}$$

$$= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2 D_{(i} \beta_{j)}}$$

Note that this can be seen both as a definition of K_{ij} and a kinematical description of the coordinates.

(extrinsic curvature) = (time derivative of coordinates measured by Eulerian observers)

In what follows we will use a number of identities derived well before Einstein's theory of general relativity and that are generic equations of differential geometry resulting from the different possible combinations of projections that can be applied to the Riemann tensor.

- We start with the Gauss- Codazzi equations

$$\boxed{\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\gamma} \cdot \underline{Riem}}$$

$$\gamma^{\mu}{}_{\alpha} \gamma^{\nu}{}_{\beta} \gamma^{\rho}{}_{\delta} \gamma^{\sigma}{}_{\lambda} R_{\mu\nu\rho\sigma} =$$

$$= {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta} K_{\beta\lambda} - K_{\alpha\lambda} K_{\beta\delta}$$

• Next, we consider the <u>Codazzi–Mainardi</u> equations

$$\boxed{\underline{\gamma}\cdot\ \underline{\gamma}\cdot\ \underline{\gamma}\cdot\ \underline{n}\cdot\ \underline{\mathrm{Riem}}} \iff$$

$$\gamma^{\mu}_{\ \alpha}\gamma^{\nu}_{\ \beta}\gamma^{\rho}_{\ \lambda}\, n^{\sigma}\, R_{\mu\nu\rho\sigma} = D_{\alpha} K_{\beta\lambda} - D_{\beta} K_{\alpha\lambda}$$

$\square$

• <u>Ricci equations</u>

$$\boxed{\underline{\gamma}\cdot\ \underline{\gamma}\cdot\ \underline{n}\cdot\ \underline{n}\cdot\ \underline{\mathrm{Riem}}}$$

$$\gamma^{\alpha}_{\ \mu}\gamma^{\beta}_{\ \nu}\, n^{\delta} n^{\lambda}\, R_{\alpha\delta\beta\lambda} = {}^{(3)}R_{\mu\nu} + K K_{\mu\nu} - K^{\lambda}_{\ \nu} K_{\mu\lambda}$$

$$K = K^{\mu}_{\ \mu}$$

Putting things together one obtains

$$\partial_t K_{ij} = - D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{ij} \partial_k \beta^k$$

$$+ K_{il} \partial_j \beta^l$$

$$+ \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2 K_{il} K^l{}_j \right)$$

$$+ \underbrace{4\pi \alpha \left[\gamma_{ij} (S - E) - 2 S_{ij} \right]}$$

contributions related to the energy-momentum tensor and zero if $T = 0$.

Note analogy with kinematics/dynamics

$$\begin{cases} \partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots \\ \partial_t K_{ij} = - D_i D_j \alpha + \dots \end{cases}$$

$$\partial_t x = v$$

$$\partial_t v = \partial_t^2 x = a$$

The matter quantities are given respectively by

$$S_{\mu\nu} := \gamma^{\alpha}{}_{\mu} \gamma^{\beta}{}_{\nu} T_{\alpha\beta}$$: spatial part of energy momentum tensor

$$S_{\mu} := -\gamma^{\alpha}{}_{\mu} n^{\beta} T_{\alpha\beta}$$: momentum density

$$S := S^{\mu}{}_{\mu}$$: trace of S

$$E := n^{\alpha} n^{\beta} T_{\alpha\beta}$$: energy density measured by Eulerian observer

Q: How many evolution equations?
Let's go back to Einstein eqs. $G_{\mu\nu} = 8\pi T_{\mu\nu}$

$G_{\mu\nu}$: 4×4 matrix $\Rightarrow$ 16 components ; symmetric tensor $\Rightarrow$

10 components $\Rightarrow$ 10 2nd-order PDEs $\Rightarrow$ <u>20 1st-order PDEs</u>

Simplest 2nd-order PDE : wave equation

$\Box \phi = 0$

$\partial_t^2 \phi - v^2 \partial_x^2 \phi = 0$

Define $\partial_t \phi = X$ $\Rightarrow$
$\partial_x \phi = \Psi$

$\partial_t X = v^2 \partial_x \Psi$

$\partial_t \phi = X$

gone from 1 2nd-order eq. to 2 first-order eqs.

We have seen that we have derived evolution eqs.

$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots$ 6 eqs

$\partial_t K_{ij} = -D_i D_j \alpha + \dots$ 6 eqs

12 eqs.

There are still 8 equations that we have not yet accounted for.

There are some contractions that we have not yet considered

$\underline{\gamma} \cdot \underline{\gamma} \cdot \underline{\text{Riem}} \rightarrow \qquad \gamma^{\alpha\mu} \gamma^{\beta\nu} R_{\alpha\beta\mu\nu} = 2 G_{\mu\nu} n^\mu n^\nu$

(iii) $\qquad {}^{(3)}R + K^2 - K_{ij} K^{ij} = 16 \pi E \qquad$ (1 eq.)

$\underline{\gamma} \cdot \underline{n} \cdot \underline{G} \qquad \gamma^{\alpha\mu} n^\nu G_{\mu\nu} = \gamma^{\alpha\mu} n^\nu R_{\mu\nu} = D^\alpha K - D_\mu K^{\alpha\mu}$

(iv) $\qquad D_j \left(K^{ij} - \gamma^{ij} K \right) = 8 \pi S^i \qquad$ (3 eqs.)

In contrast to eqs. (i) and (ii), eqs (iii) and (iv) do not have time derivatives: they are constraint equations

The appearance of these two classes of eqs can be found in EM: Maxwell eqs in fact have a very similar split

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B} - 4\pi \vec{J} \qquad \partial_t E_i = \epsilon_{ijk} \partial^j B^k - 4\pi J_i$$

$$\partial_t \vec{B} = - \vec{\nabla} \times \vec{E} \qquad \partial_t B_i = - \epsilon_{ijk} \partial^j E^k$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \partial_i B^i = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e \qquad \partial_i E^i = 4\pi \rho_e$$

The analogies between the ADM and the Maxwell equations will help us understand some of the problems associated with the ADM eqs and suggest possible solutions.

In practice, the ADM equations have not been used in 3D applications and their use has been abandoned when it has become clear that they are <u>weakly hyperbolic</u>.

To appreciate the implications of this statement we need a small digression.

A large class of equations in mathematical physics (eg. EFEs, eqs. of hydrodynamics and MHD) can be written in a compact form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S} \qquad (*)$$

$$\partial_t U_J + (A^i)_{JK} \nabla_i U_K = S_J$$

where $\underline{U} = \{U_1, U_2, \dots U_J\}$: state vector

$\underline{S} = \{S_1, S_2, \dots S_J\}$: source term

A : matrix of coefficients and

$(A^i)_{JK}$: i different matrices, one for each direction

$(A^x)_{JK}$, $(A^y)_{JK}$, $(A^z)_{JK}$

The properties of the system (*) depend on the properties of A, and S.

(i) $\left\{\begin{array}{l} a_{JK} \text{ : elements of } A \\ a_{JK} = \text{const.} \;;\; S_J = \text{const} \end{array}\right\}$ (*) is a LINEAR system of equations with constant coefficients

(ii) $\left\{ a_{JK} = a_{JK}(x,t) \;;\; S_J = S_J(x,t) \right\}$ (*) is a LINEAR system with variable coefficients

(iii) $A = A(\underline{u})$ (*) is a non LINEAR system (often referred to as quasi-linear)

More importantly, the system (*) is said to be (strongly) HYPERBOLIC if A is diagonalizable with a set of real eigenvalues $\lambda_1, \dots \lambda_N$ and a set of N linearly independent right eigenvectors, ie if

$$\Lambda := R^{-1} A R = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_N)$$

R: matrix of right eigenvectors $R^{(i)}$

$$A R^{(i)} = \lambda_i R^{(i)}$$

$\lambda_i \in \mathbb{R}$: real eigenvalues

(*) is said to be <u>STRICTLY HYPERBOLIC</u> if $\lambda_{(i)}$ are real and distinct

(*) is said to be <u>SYMMETRIC HYPERBOLIC</u> if A is symmetric, ie $A = A^T$

(*) is said to be <u>WEAKLY HYPERBOLIC</u> if A is not diagonalizable

Examples of hyperbolic equations are

- advection equation $\partial_t u + v\, \partial_x u = 0$
- wave equation $\partial_t^2 u - v^2 \partial_x^2 u = 0$
- hydrodynamic equations (inviscid)
- Einstein equations (e.g., in harmonic coordinates $\Box x^\alpha = 0$)

The importance of hyperbolicity is strictly related with that of <u>WELL POSEDENESS</u> of the Cauchy initial-value problem.

$\underline{u}(x,0)$: initial data

$\underline{u}(x,t)$: solution of set (*) at time t

(*) is well posed if

$$\| u(x,t) \| \leq k\, e^{at}\, \| u(x,0) \|$$

$k, a \in \mathbb{R}$ constants.

In other words the solution is always bounded by some exponential of the initial data ("it does not blow up..!")

An important theorem of hyperbolic systems states

(*) a hyperbolic set of equations $\Rightarrow$ (*) is well posed

opposite implication not true.

It follows that a weakly hyperbolic system is not guaranteed to be well-posed and indeed the numerical solution leads to the growth of unstable modes ("codes crash"...)

In the case of the ADM equations the weak hyperbolicity comes from the mixed derivatives in the Ricci tensor for the evolution of the extrinsic curvature, ie

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} + \cdots \right)$$

$\gamma_l{}^i \gamma_m{}^j \partial_i \partial_j$ mixed second derivatives, eg $\partial_x \partial_y$, $\partial_x \partial_z$, $\partial_y \partial_z$

while one wishes to have diagonal second derivatives, ie

$\gamma^{ij} \partial_i \partial_j$: ∂^2_x, ∂^2_y, ∂^2_z

There are a number of ways around this problem and the easiest way to understand how this works is to look at Maxwell equations.

Let's introduce the vector potential $\vec{A} := \vec{\nabla} \times \vec{B}$

$$A_\mu = (-\phi, A_i)$$

Then the Maxwell equations can be written as

$$(\blacktriangledown) \quad \partial_t A_i = -E_i - D_i \phi$$

$$(\blacktriangle) \quad \partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

Note

of course in classical EM $D_i \longleftrightarrow \partial_i$ but I'm keeping the D-notation to highlight the analogy with GR.

The term $D_i D^j A_j$ breaks hyperbolicity and we want to get rid of this term. There are different ways to do this.

1) <u>Lorenz gauge</u> : $\partial_t \phi = - D^j A_j$

This does the job because by taking another time derivative of $\partial_t A_i$ eq ($\blacktriangledown$)

$$\xrightarrow{} D_i(-D^j A_j)$$

$$-\partial_t^2 A_i + D^j D_j A_i - D_i D^j A_j = D_i (\partial_t \phi) - 4\pi J_i$$

$$-\partial_t^2 A_i + D^j D_j A_i = -4\pi J_i \iff \Box A_i = 4\pi J_i \quad (\boxtimes)$$

where $\Box$: Dalambertian

$$\Box \phi = (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\phi \quad \text{in Cartesian coordinates}$$

Equation ($\boxtimes$) is an hyperbolic equation and hence <u>well-posed</u>

2) The second route doesn't involve a gauge but an auxiliary quantity

$$\Gamma := D^i A_i \qquad : \text{scaler function}$$

Such that eq ($\blacktriangledown$) can be written as

$$\Box A_i = -D_i \Gamma - D_i \partial_t \phi + 4\pi J_i \qquad (\therefore)$$

Written in this way eq ($\therefore$) is again <u>hyperbolic</u> because the principal part (ie $\Box A_i$) is the same and the "disturbing" term appears now as a source on the RHS ($D_i \Gamma$).

To fix the ADM equations and remove weak hyperbolicity we do something very similar: ie we introduce new quantities. Befre doing this, we need also some other ingredients.

1) Introduce a conformally related metric,

ie

$$g_{\mu\nu} \longleftrightarrow \tilde{g}_{\mu\nu} = \phi^{n} g_{\mu\nu}$$

physical metric — conformally related metric

ϕ: conformal factor so that

$$ds^2 = \phi^{+n} g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

A conformal metric allows us to set some additional conditions on the properties of the determinant of the corresponding 3-metric. In particular

2) $$\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij} \qquad \tilde{\gamma}^{ij} = \phi^{-2} \gamma^{ij}$$

so that we can impose the volume element (•) to be

$$\tilde{\gamma} := \det(\tilde{\gamma}_{ij}) = 1$$

and the conformal factor is given by

$$\phi = (\det(\gamma_{ij}))^{-1/6} = \gamma^{-1/6} \quad (••)$$

(•) Recall that $V_\Omega = \int_\Omega \sqrt{\det(\gamma_{ij})}\, d^3x$	(••) $\tilde{\gamma} = \phi^6 \gamma = 1 \Rightarrow \phi^6 = \gamma^{-1}$

3) Introduce the conformally related Christoffel symbols (or Christoffel symbols of conformal three-metric)

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\left(\delta^i_j \partial_k \ln\phi + \delta^i_k \partial_j \ln\phi - \gamma_{jk}\gamma^{il}\partial_l \ln\phi\right)$$

4) Introduce the conformally related trace-free extrinsic curvature

$$\tilde{A}_{ij} = \phi^2 A_{ij} = \phi^2\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) ; \quad \hat{A}^{ij} = \phi^{-2}A^{ij}$$

so that

$$\tilde{A}_{ij}\,\gamma^{ij} = \hat{A}^i{}_i = \phi^2\left(K^i{}_i - \frac{1}{3}\gamma_{ij}\gamma^{ij}K\right) = 0$$

In other words the conformal extrinsic curvature is <u>traceless</u>

5) introduce additional variables to separate mixed derivatives : "Gammas"

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk} = \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \partial_l \tilde{\gamma}_{jk}$$

The resulting set of equation is then:

cf. ADM eqs.

$$\left[\begin{aligned} \partial_t \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k + \beta^k \partial_k \tilde{\gamma}_{ij} \\ \partial_t \tilde{A}_{ij} &= \phi^2 \left[-D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} - 8\pi S_{ij}\right)\right]^{TF} + \beta^k \partial_k \tilde{A}_{ij} + \dots \end{aligned}\right]$$

$$\partial_t \phi = \frac{1}{3}\phi \alpha K - \frac{1}{3}\partial_i \beta^i + \beta^k \partial_k \phi$$

$$\partial_t K = -D_i D^i \alpha + \alpha \left[\tilde{A}_{ij} \tilde{A}^{ij} + \dots\right.$$

$$\partial_t \tilde{\Gamma}^i = \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ik} \partial_k \partial_j \beta^j + \dots$$

where "TF" indicates that the trace-free part of the bracket is used, ie $\left({}^{(3)}R_{ij}\right)^{TF} \longrightarrow {}^{(3)}R_{ij} - \frac{1}{3}\gamma_{ij}\,{}^{(3)}R^k{}_k$

$\square$

These equations are those normally used and are referred to as the BSSNOK formulation[1], or conformally traceless.

When comparing with the ADM equations we have clearly gained 5 more evolution eqs: $\partial_t \phi$, $\partial_t K$, $\partial_t \tilde{\Gamma}^i$ but the system is now hyperbolic and indeed well behaved in numerical simulations.

The additional computational costs: 12 variables $\rightarrow$ 15 variables

$(12+5-2)$ $\tilde{\gamma}_{ij}$ $\hat{A}_{ij}$ are traceless or with known trace

(1) (Baumgarte, Shapiro, Shibata, Nakamura, Ohara, Kojima)

Similarly the constraint equations are then

$$\boxed{{}^{(3)}R = \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 + 16\pi E}$$

Proof

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

LHS :

$${}^{(3)}R + K^2 - \left(\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K\right)\left(\tilde{A}^{ij} + \frac{1}{3}\gamma^{ij}K\right) =$$

$$= {}^{(3)}R + K^2 - \left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}\cancel{\tilde{A}^{i}{}_{i}}K + \frac{1}{9}\cdot 3K^2\right)$$

$$\tilde{A}^{i}{}_{i} = \tilde{A}^{ij}\gamma_{ij} = \left(K^{ij} - \frac{1}{3}\gamma^{ij}K\right)\gamma_{ij} = K - K = 0$$

$$= {}^{(3)}R + K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{3}K^2$$

$$= {}^{(3)}R - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 16\pi E \quad \Longrightarrow \quad {}^{(3)}R = \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 + 16\pi E$$

while the momentum constraint equations are

$$D_j\left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i \quad \Longleftrightarrow$$

$$D_j\left(\hat{A}^{ij} + \frac{1}{3}\gamma^{ij} K - \gamma^{ij} K\right) = 8\pi S^i$$

$$D_j\left(\hat{A}^{ij} - \frac{2}{3}\gamma^{ij} K\right) = 8\pi S^i$$

$$\tilde{\gamma}^{ij} = \phi^{-2}\gamma^{ij}$$
$$\gamma^{ij} = \phi^{2}\tilde{\gamma}^{ij}$$

$$\boxed{D_j\left(\hat{A}^{ij} - \frac{2}{3}\phi^2 \tilde{\gamma}^{ij} K\right) = 8\pi S^i}$$

$\square$

$$\begin{cases} \mathcal{H} : \quad 0 = \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 - {}^{(3)}R + 16\pi E \\ \mathcal{M}^i : \quad 0 = D_j\left(\hat{A}^{ij} - \frac{2}{3}\phi^2\tilde{\gamma}^{ij} K\right) - 8\pi S^i \end{cases}$$

How are the constraints handled?

These are 3D nonlinear elliptic equations to be solved on each slice and very expensive to solve.

Solving one of them is more expensive than computing the full set of evolution equations.

In practice the constraints are monitored as a measure of the quality of the solution