Mathematical foundations of relativistic hydrodynamics

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Outline

Lecture 1: Relativistic Hydrodynamics

Lecture 2: Numerical methods

F. Banyuls et al, "Numerical 3+1 general relativistic hydrodynamics: a local characteristic approach", Astrophysical Journal, 476, 221 (1997)

J.M. Martí & E. Müller, "Numerical hydrodynamics in special relativity", Living Reviews in Computational Physics (2017)

J.A. Font, "Numerical hydrodynamics and magnetohydrodynamics in general relativity", Living Reviews in Relativity (2008)

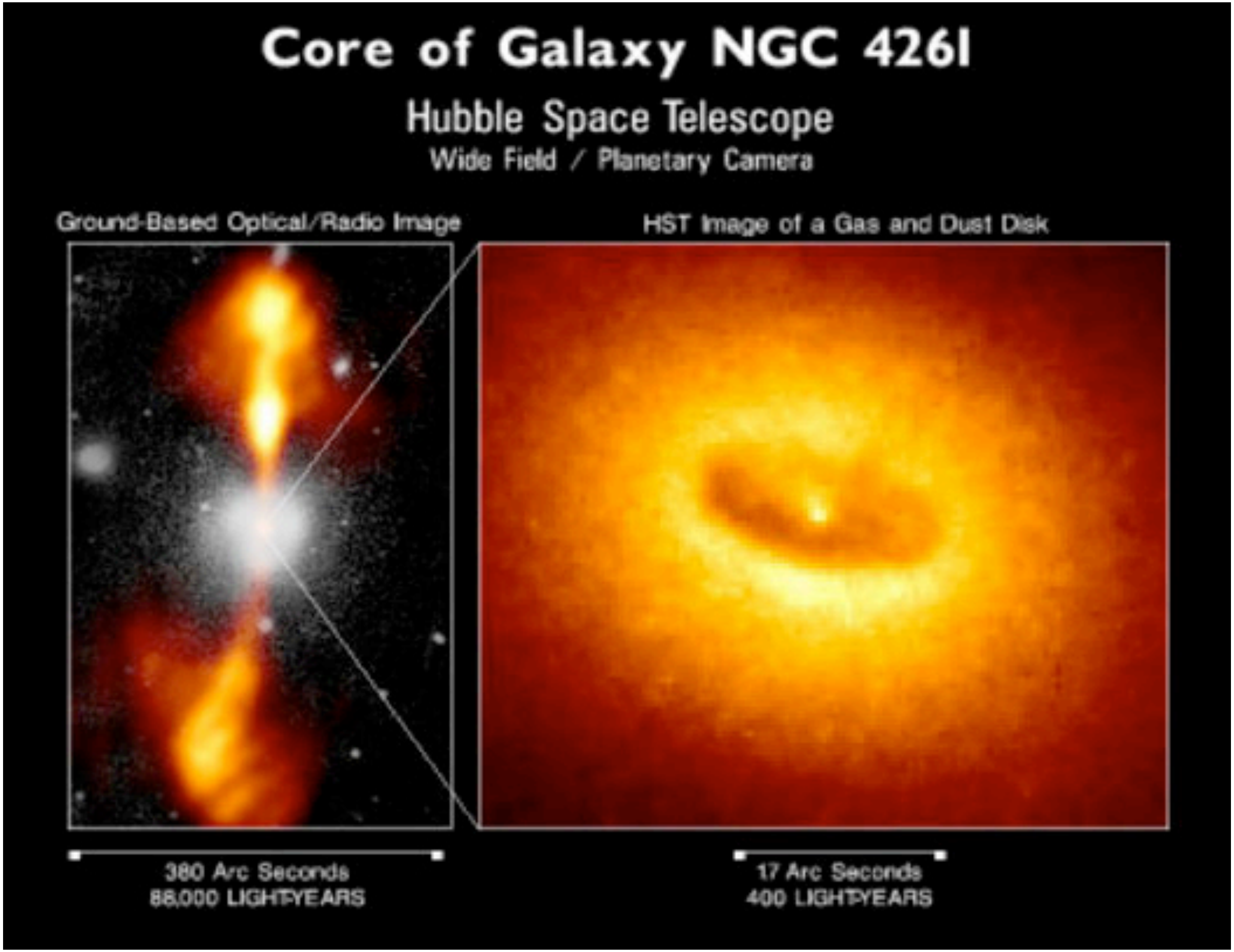
A.M. Anile, "Relativistic fluids and magneto-fluids", Cambridge University Press (1989)

L. Rezzolla & O. Zanotti, "Relativistic hydrodynamics", Oxford University Press (2013)

Lecture 1: Relativistic Hydrodynamics

Classical fluid dynamics

Relativistic fluid dynamics



General relativity and relativistic hydrodynamics play a major role in the description of gravitational collapse leading to the formation of compact objects (neutron stars and black holes).

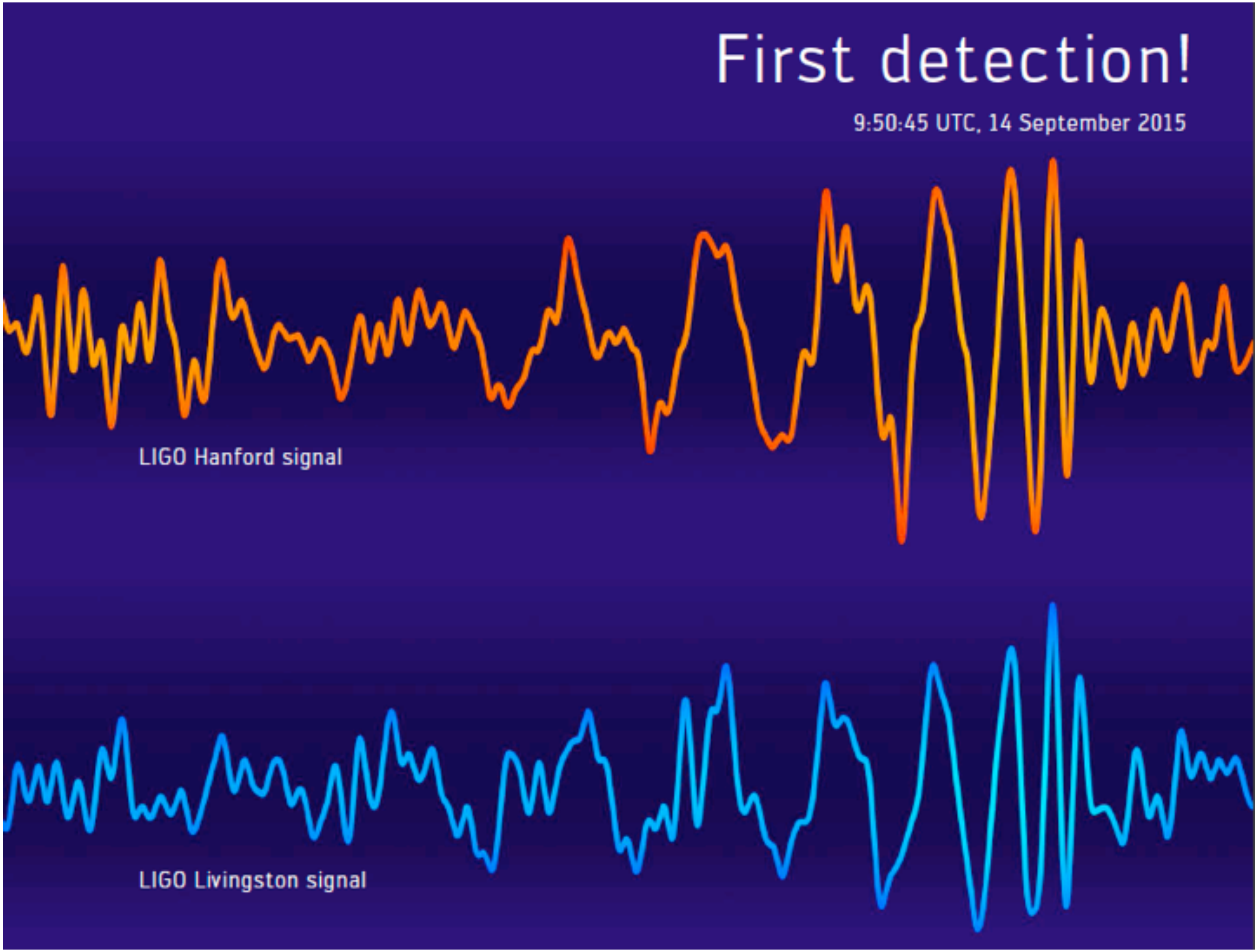
Prime Sources of Gravitational Radiation.

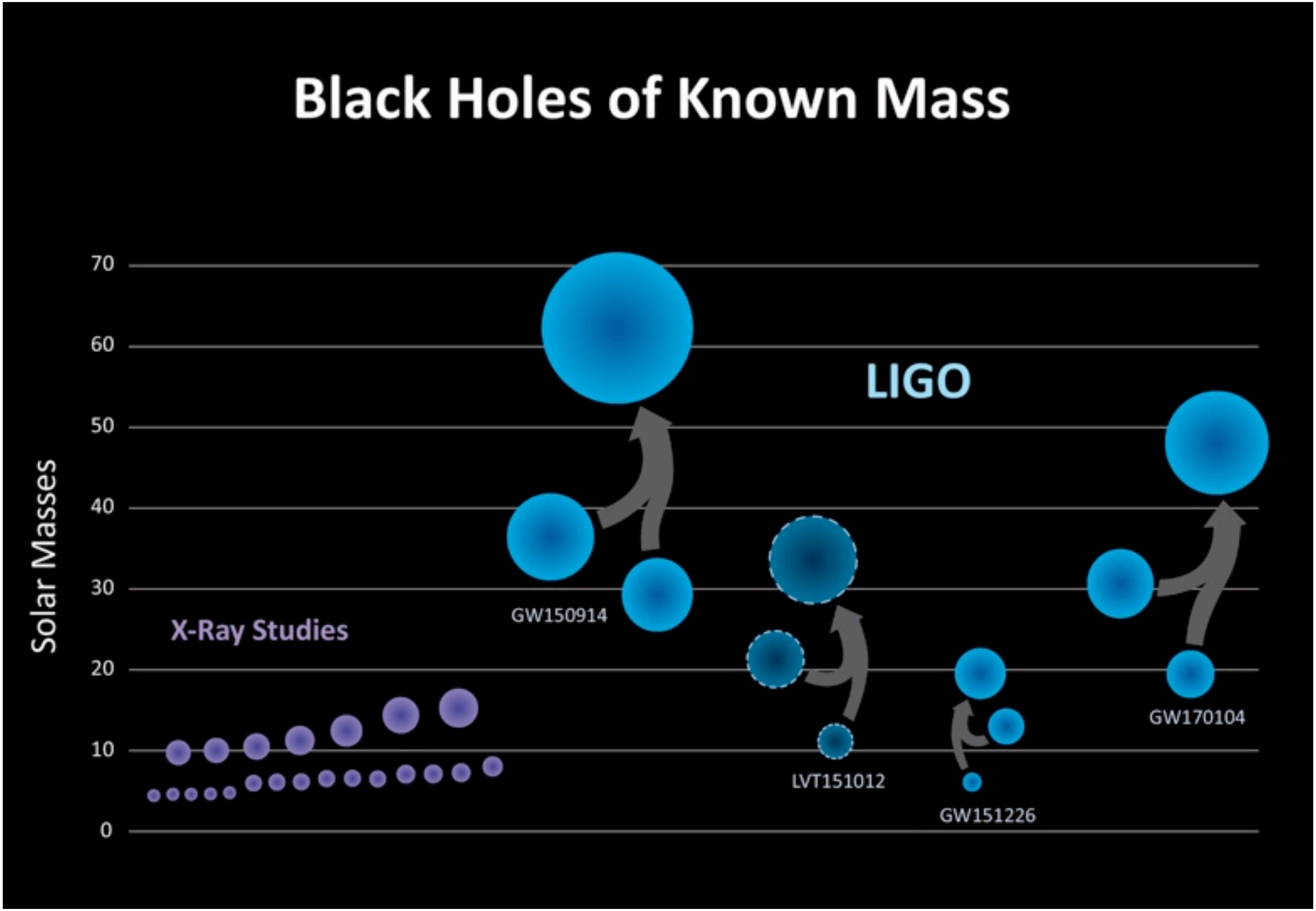
Time-dependent evolutions of fluid flow coupled to the spacetime geometry (Einstein’s equations) possible through accurate, large-scale numerical simulations.

Some scenarios can be described in the test-fluid approximation: GRHD/GRMHD computations in fixed curved backgrounds.

GRHD/GRMHD equations are nonlinear hyperbolic systems. Solid mathematical foundations and accurate numerical methodology imported from CFD. A “preferred” choice: high-resolution shock-capturing schemes written in conservation form.

Numerical Relativity provides gravitational waveforms (templates): main driver of the field.





Black Holes of Known Mass
70
60
50
40
30
20
10
0
Solar Masses
LIGO
X-Ray Studies
GW150914
LVT151012
GW151226
GW170104

Fluid dynamics

Fluid dynamics deals with the **behaviour of matter in the large** (average quantities per unit volume), on a macroscopic scale large compared with the distance between molecules, $l >> d_0 \sim 3\text{-}4 \times 10^{-8}$ cm, not taking into account the molecular structure of fluids.

Macroscopic behaviour of fluids assumed to be continuous in structure, and physical quantities such as mass, density, or momentum contained within a given small volume are regarded as uniformly spread over that volume.

The quantities that characterize a fluid (in the continuum limit) are functions of time and position:

$$\rho \;:\; (t,\vec{r}) \in \mathbb{R}^4 \;\longrightarrow\; \rho(t,\vec{r}) \in \mathbb{R}$$ density (scalar field)

$$\vec{v} \;:\; (t,\vec{r}) \in \mathbb{R}^4 \;\longrightarrow\; \vec{v}(t,\vec{r}) \in \mathbb{R}^3$$ velocity (vector field)

$$\Pi \;:\; (t,\vec{r}) \in \mathbb{R}^4 \;\longrightarrow\; \Pi(t,\vec{r}) \in \mathbb{R}^9$$ pressure tensor (tensor field)

Transport theorems:

Scalar field

$$\frac{d}{dt}\int_{V_t} f\,dV = \int_{V_t}\left[\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v})\right] dV, \qquad f = f(t,\vec{r})$$

$$\frac{d}{dt}\int_{V_t} \vec{F}\,dV = \int_{V_t}\left[\frac{\partial \vec{F}}{\partial t} + (\vec{v}\cdot\nabla)\vec{F} + \vec{F}(\nabla\cdot\vec{v})\right] dV, \quad \vec{F} = \vec{F}(t,\vec{r})$$

Vector field

V_t is a volume which moves with the fluid (Lagrangian description).

Mass conservation (continuity equation)

Let V_t be a volume which moves with the fluid; its **mass** is given by:

$$m(V_t) = \int_{V_t} \rho(t,\vec{r})\, dV$$

Principle of conservation of mass enclosed within that volume:

$$\frac{d}{dt} m(V_t) = \frac{d}{dt} \int_{V_t} \rho(t,\vec{r})\, dV = 0$$

Applying the transport theorem for the density (scalar field):

$$0 = \frac{d}{dt} \int_{V_t} \rho(t,\vec{r})\, dV = \int_{V_t} \left[\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) \right] dV$$

where the convective derivative is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

Since the previous equation must hold for any volume V_t we obtain the **continuity equation**:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) = 0 \Rightarrow \frac{D\log\rho}{Dt} = -\Theta \Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0}$$

Corolary:

$$-\frac{\partial}{\partial t}\int_V \rho \, dV = \int_{\partial V} \rho\vec{v} \cdot d\vec{\Sigma}$$

the variation of the mass enclosed in a fixed volume V is equal to the flux of mass across the surface at the boundary of the volume.

Incompressible fluid: $\nabla \cdot \vec{v} = 0 \Leftrightarrow \dfrac{D\rho}{Dt} = 0$

Momentum balance (Euler's equation)

"the variation of momentum of a given portion of fluid is equal to the net force (stresses plus external forces) exerted on it" (Newton's 2nd law):

$$\frac{d}{dt}\int_{V_t}\rho\vec{v}\,dV = -\int_{\partial V_t} p\,d\vec{\Sigma} + \int_{V_t}\vec{G}\,dV = \int_{V_t}[\vec{G}-\nabla p]\,dV$$

Applying the transport theorem on the l.h.s. of the above equation:

$$\int_{V_t}\left[\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v})\right]\,dV = \int_{V_t}[\vec{G}-\nabla p]\,dV$$

which must be valid for any volume V_t, hence:

$$\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v}) = \vec{G}-\nabla p$$

After some algebra and using the continuity eq. we obtain **Euler's eq.**:

$$\boxed{\rho\frac{D\vec{v}}{Dt} = \vec{G}-\nabla p \;\Leftrightarrow\; \rho\vec{a} = \vec{G}-\nabla p}$$

Energy conservation

Let E be the total energy of the fluid, sum of its kinetic energy and internal energy:

$$E = E_{\mathrm{K}} + E_{\mathrm{int}} = \frac{1}{2}\int_{V_t} \rho \vec{v}^2 \, dV + \int_{V_t} \rho \varepsilon \, dV$$

Principle of energy conservation: "the variation in time of the total energy of a portion of fluid is equal to the work done per unit time over the system by the stresses (internal forces) and the external forces".

$$\frac{dE}{dt} = \frac{d}{dt}\int_{V_t} \left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) dV = -\int_{\partial V_t} p\vec{v}\cdot d\vec{\Sigma} + \int_{V_t} \vec{G}\cdot\vec{v}\, dV$$

After some algebra (transport theorem, divergence theorem) we obtain:

$$\int_{V_t} \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right]\right) dV = \int_{V_t} \rho\vec{g}\cdot\vec{v}\, dV \qquad \vec{g} = \frac{\vec{G}}{\rho}$$

which, as must be satisfied for any given volume, implies:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right] = \rho\vec{g}\cdot\vec{v}}$$

Hyperbolic system of conservation laws

The equations of perfect fluid dynamics are a nonlinear hyperbolic system of conservation laws:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f^i}}{\partial x^i} = \vec{s}(\vec{u})}$$

$$\begin{aligned}
\vec{u} &= (\rho, \rho v^j, e) \\
\vec{f^i} &= (\rho v^i, \rho v^i v^j + p\delta^{ij}, (e+p)v^i) \\
\vec{s} &= (0, \rho g^i, \rho v^i g^i)
\end{aligned}$$

The concept of hyperbolicity

Let us consider a first-order system of evolution equations:

$$\partial_t u + M^i \partial_i u = s(u)$$

where M^i are $n \times n$ matrices and $i = 1, 2, 3$

Let us consider an arbitrary unit vector n_i and let us build the matrix $P = M^i n_i$ the so-called system's principal symbol.

The system can be classified as:

- Strongly hyperbolic: if P has real eigenvalues and there exists a complete set of eigenvectors for any n_i .

- Weakly hyperbolic: if P has real eigenvalues but there does not exist a complete set of eigenvectors.

- Symmetric hyperbolic: if P is a symmetric matrix regardless of n_i . They are, therefore, strongly hyperbolic.

Only strong and symmetric hyperbolic systems are well-posed.

Eigenvalues represent propagation speeds of the system.

Hyperbolic system of conservation laws

The equations of perfect fluid dynamics are a nonlinear hyperbolic system of conservation laws:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}^i}{\partial x^i} = \vec{s}(\vec{u})}$$

state vector $\quad \vec{u} = (\rho, \rho v^j, e)$

fluxes $\quad \vec{f}^i = (\rho v^i . \rho v^i v^i + p\delta^{ij}, (e+p)v^i)$

source terms $\quad \vec{s} = \left(0, -\rho\frac{\partial \Phi}{\partial x^j} + Q^j_M, -\rho v^i \frac{\partial \Phi}{\partial x^i} + Q_E + v^i Q^i_M\right)$

$\vec{g}$ is a conservative external force field (e.g. gravitational field):

$$\vec{g} = -\nabla\Phi \qquad \Delta\Phi = 4\pi G\rho$$

Q_M^i, Q_E are source terms in the momentum and energy equations, respectively, due to coupling between matter and radiation (when transport phenomena are also taken into account).

Hyperbolic equations have finite propagation speed: information can travel with limited speed, at most that given by the largest characteristic curves of the system.

The range of influence of the solution is bounded by the eigenvalues of the Jacobian matrix of the system.

$$A = \frac{\partial \vec{f}^{i}}{\partial \vec{u}} \Rightarrow \lambda_0 = v_i, \; \lambda_\pm = v_i \pm c_s$$

(link with numerical schemes in Lecture 2)

A bit on viscous fluids

A perfect fluid can be defined as that for which the force across the surface separating two fluid particles is normal to that surface.

Kinetic theory tells us that the existence of velocity gradients implies the appearance of a force tangent to the surface separating two fluid layers (across which there is molecular difusion).

$$d\vec{F} = -p\,d\vec{\Sigma} \Rightarrow \boxed{d\vec{F} = -\Pi\,d\vec{\Sigma}}$$

where Π is the pressure tensor which depends on pressure and velocity gradients.

$\Pi = p\mathrm{I} - \mathrm{S}$ where S is the **stress tensor** given by:

$$\mathcal{S} = 2\mu \underbrace{\left(D - \frac{1}{3}\Theta\mathbb{I}\right)}_{\text{distortion}} + \xi\underbrace{\Theta\mathbb{I}}_{\text{expansion}}$$

shear and bulk viscosities

Using the pressure tensor in the previous derivation of the Euler eq. and of the energy eq. yields their viscous versions:

$$\rho\frac{D\vec{v}}{Dt} = \vec{G} - \nabla p + \mu\Delta\vec{v} + \left(\xi + \frac{1}{3}\mu\right)\nabla\cdot(\nabla\cdot\vec{v})$$ Navier-Stokes eq.

$$\rho\frac{D\left(\frac{1}{2}\vec{v}^2 + \varepsilon\right)}{Dt} = \rho\vec{g}\cdot\vec{v} - \nabla\cdot(p\vec{v}) + \nabla\cdot(\mathcal{S}\cdot\vec{v}) - \nabla\cdot\vec{Q}$$ Energy eq.

General relativistic hydrodynamics

The general relativistic hydrodynamics equations are obtained from the **local conservation laws of the stress-energy tensor**, $T^{\mu\nu}$ (the Bianchi identities), **and of the matter current density** J^{μ} (the continuity equation):

$$\nabla_{\mu}(\rho u^{\mu}) = 0 \qquad \nabla_{\mu} T^{\mu\nu} = 0$$

Equations of motion

$$(\mu = 0, \cdots, 3)$$

∇_{μ} covariant derivative associated with the four dimensional spacetime metric $g_{\mu\nu}$

The density current is given by $J^{\mu} = \rho u^{\mu}$

u^{μ} is the fluid 4-velocity and ρ is the rest-mass density in a locally inertial reference frame.

The stress-energy tensor for a **non-perfect fluid** is defined as:

$$T^{\mu\nu} = \rho(1+\varepsilon)u^\mu u^\nu + (p - \mu\Theta)h^{\mu\nu} - 2\xi\sigma^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu$$

where ε is the specific internal energy density of the fluid, p is the pressure, and $h^{\mu\nu}$ is the spatial projection tensor, $h^{\mu\nu}=u^\mu u^\nu+g^{\mu\nu}$.

In addition, μ and ξ are the shear and bulk viscosity coefficients.

The expansion, Θ, describing the divergence or convergence of the fluid world lines is defined as $\Theta=\nabla_\mu u^\nu$. The symmetric, trace-free, and spatial shear tensor $\sigma^{\mu\nu}$ is defined by:

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla_\alpha u^\mu h^{\alpha\nu} + \nabla_\alpha u^\nu h^{\alpha\mu}) - \frac{1}{3}\Theta h^{\mu\nu}$$

Finally q^μ is the energy flux vector.

In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, assuming the stress-energy tensor to be that of a **perfect fluid:**

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

where we have introduced the relativistic specific enthalpy,

$$h = 1 + \varepsilon + \frac{p}{\rho}$$

Conservation laws with respect to an explicit coordinate chart

$x^\mu = (x^0, x^i)$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\ \frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \end{aligned}}$$

where the scalar x^0 represents a foliation of spacetime with hypersurfaces (with coordinates x^i). Moreover, $g=\det(g_{\mu\nu})$, and $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols.

The system formed by the eqs of motion and the continuity eq must be supplemented with an **equation of state** (EOS) relating the pressure to some fundamental thermodynamical quantities, e.g.

$$p = p(\rho, \varepsilon)$$

Perfect fluid: $p = (\Gamma - 1)\rho\varepsilon$

Polytrope: $p = \kappa\rho^{\Gamma}\,, \quad \Gamma = 1 + \frac{1}{N}$

In the "test-fluid" approximation (fluid's self-gravity neglected), the dynamics of the matter fields is fully described by the previous conservation laws and the EOS.

When such approximation does not hold, the previous equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime:

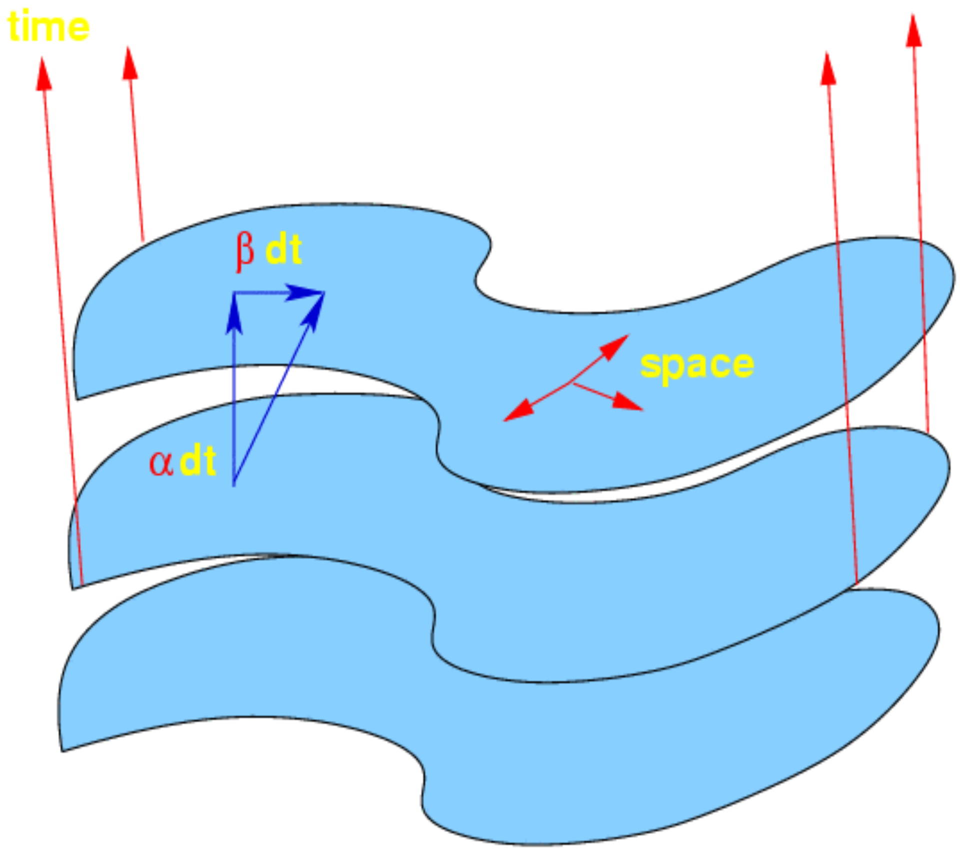
$$\nabla_\mu(\rho u^\mu) = 0$$
$$\nabla_\mu T^{\mu\nu} = 0$$
$$p = p(\rho, \varepsilon)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Einstein's equations

(Newtonian analogy: Euler's equation + Poisson's equation)

(cf. Rezzolla's Lectures)



The most widely used approach to solve Einstein's equations in Numerical Relativity is the 3+1 formulation.

Spacetime is foliated with a set of non-intersecting spacelike hypersurfaces Σ. Within each surface distances are measured with the spatial 3-metric.

Two kinematical variables describe the evolution between each hypersurface: the lapse function, and the shift vector.

3+1 ADM system

$$
\begin{aligned}
\partial_t K_{ij} = & - D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K^{kj} + K K_{ij}) \\
& - 8\pi\alpha (R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_\beta K_{ij})
\end{aligned}
$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$R + K^2 - K_{ij} K^{ij} = 16\pi e$$

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

$$e = n^\mu n^\nu T_{\mu\nu} \qquad S_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta} \qquad S = S^\mu{}_\mu$$

$$j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} = -(e + p)(u_\mu + n_\mu)$$

These 6+6+3+1 equations are known as **ADM equations**. In practice only the evolution equations are solved and the constraint equations are used to monitor the quality of the numerical solution.

Cauchy (Initial Value) Problem

The ADM equations constitute a Cauchy problem where the PDEs are solved given certain initial conditions on the initial hypersurface.

- Specify the initial data γ_{ij} , K_{ij} at $t=0$ subject to the constraint equations.

- Specify the coordinates through the (freely specifiable) lapse function α and shift vector β^i

- Evolve the initial data to the next time step using the Einstein's equations and the definition of the extrinsic curvature K_{ij}

Original references:

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Reformulating the ADM system: the BBSN system

$$\mathcal{D}_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}$$

$$\mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t \phi = -\frac{1}{6}\alpha K$$

$$\mathcal{D}_t \tilde{A}_{ij} = e^{-4\phi}[-\nabla_i \nabla_j \alpha + \alpha(R_{ij} - S_{ij})]^{\mathrm{TF}} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j)$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i \nabla_j \alpha + \left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S)\right]$$

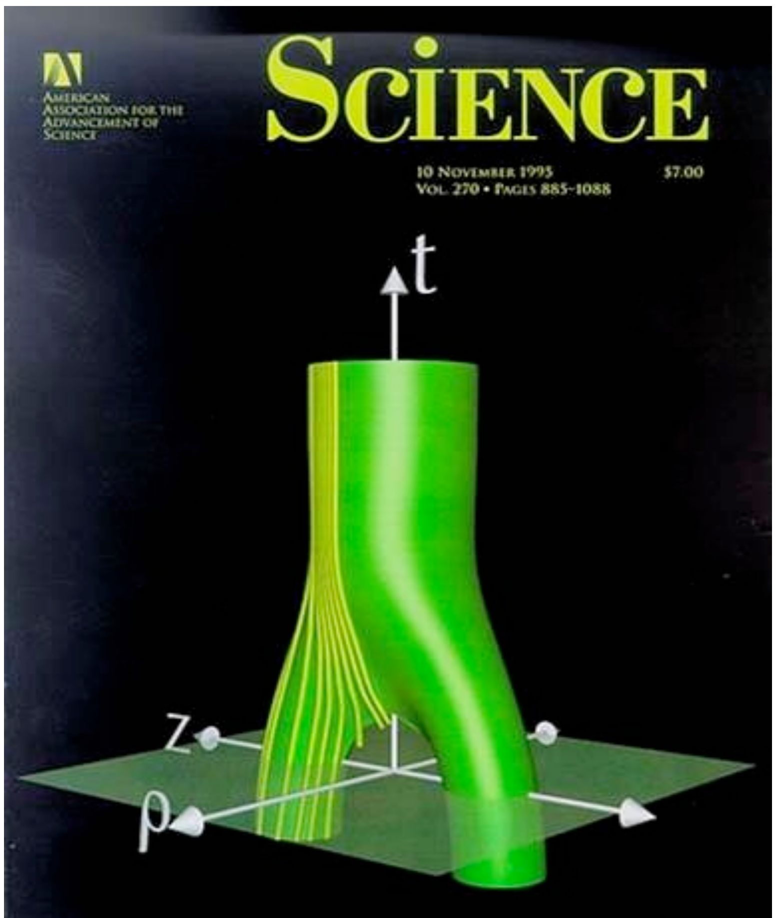
$$\mathcal{D}_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi\right)$$
$$-\partial_j\left(\beta^l \partial_l \tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m \beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l \beta^l\right)$$

These equations are known as BSSNOK equations or simply the conformal, traceless formulation of Einstein's equations.

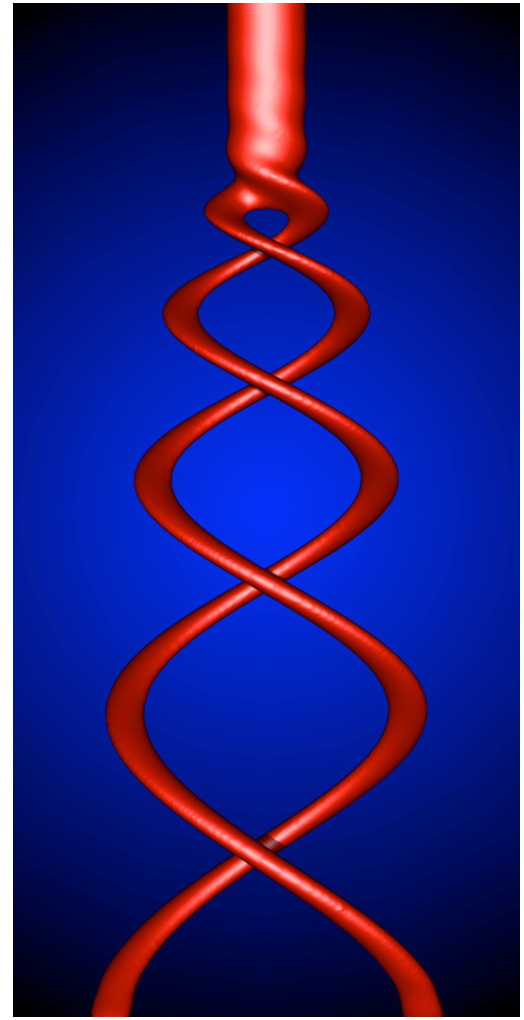
Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

BBH simulations: State of the art

1995: Pair of pants
(Head-on collision)



BBH Grand Challenge Alliance



FSU-Jena Numerical Relativity Group

2007: Pair of twisted pants
(spiral & merge)

3+1 GR Hydro equations: formulations

$$
\begin{aligned}
\frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\
\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda}
\end{aligned}
$$

Different formulations exist depending on:

1. Choice of slicing: level surfaces of x^0 spatial (3+1) or null
2. Choice of physical (primitive) variables (ρ, ε, u^i …)

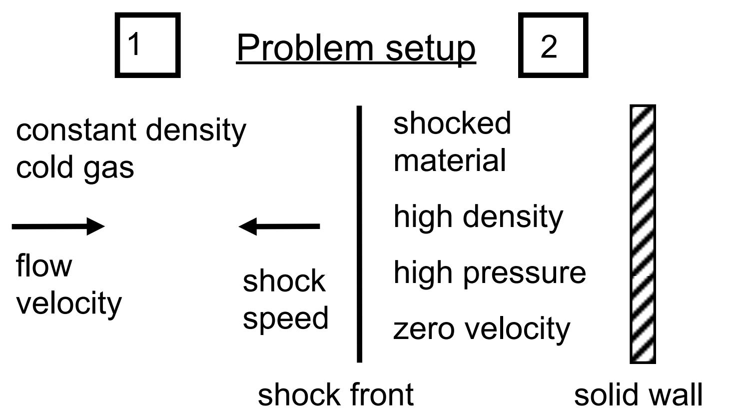
Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology appeared later:

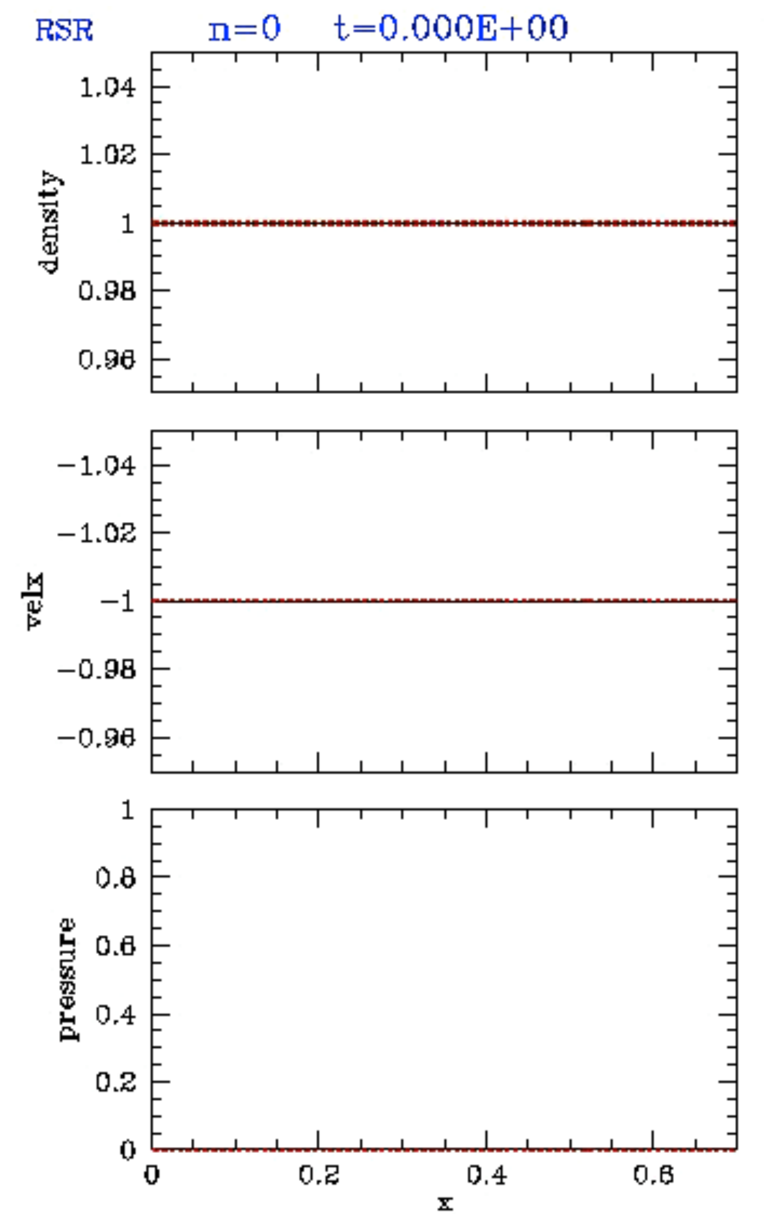
- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS
- Papadopoulos & Font (2000): covariant, general EOS

Relativistic shock reflection

The relativistic shock reflection problem is a demanding test involving the heating of a cold gas which impacts at relativistic speed with a solid wall creating a shock which propagates off the wall.



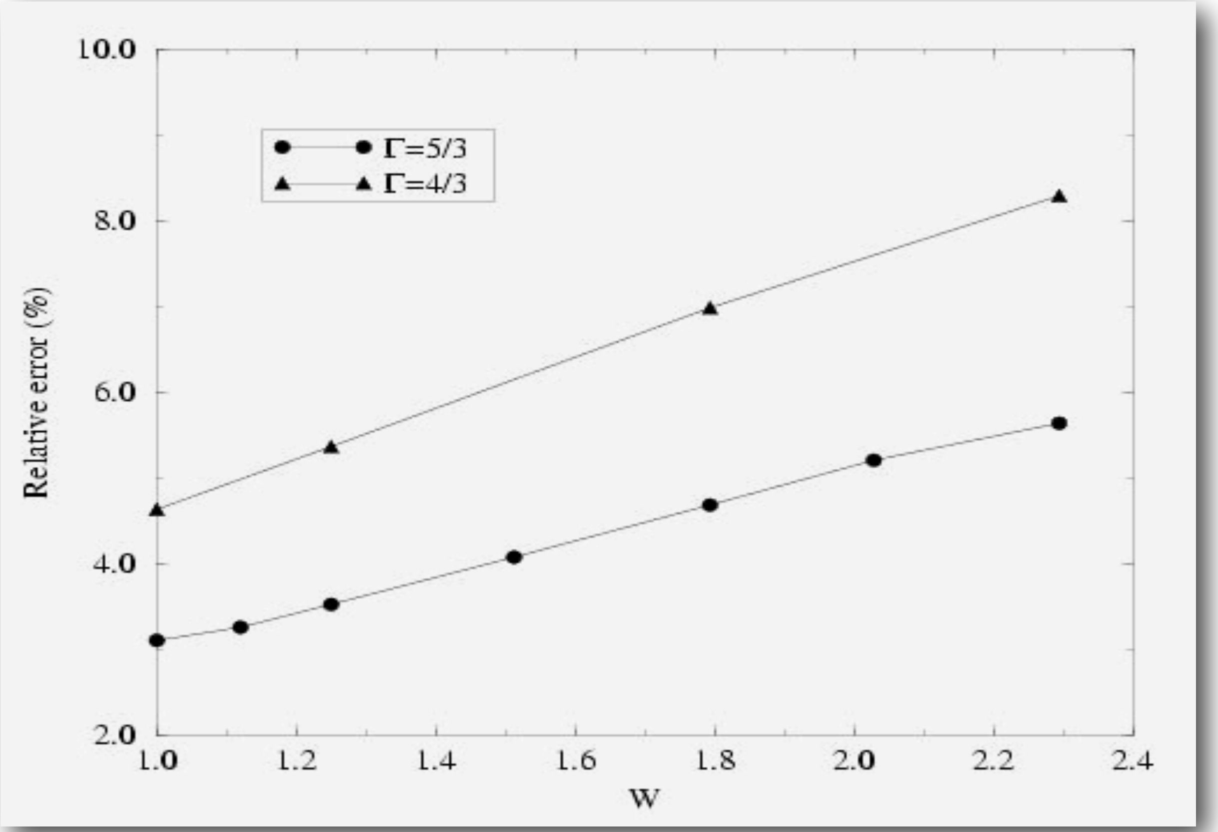
Analytic solution: $\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\varepsilon_2, \quad \varepsilon_2 = W_1 - 1$



(from Martí & Müller, 2003)

Non-conservative formulations show limitations when simulating ultrarelativistic flows (Centrella & Wilson 1984, Norman & Winkler 1986).

Relativistic shock reflection test relative errors as a function of the fluid's Lorentz factor W. For W≈2 (v≈0.86c), error ~ 5-7% (depends on the adiabatic index of the EOS) and shows a linear increase with W.



Ultrarelativistic flows could only be handled once conservative formulations were adopted (Martí, Ibáñez & Miralles 1991; Marquina et al 1992)

Valencia's conservative formulation (Banyuls et al 1997)

Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system, translating to relativistic hydrodynamics existing tools of CFD.

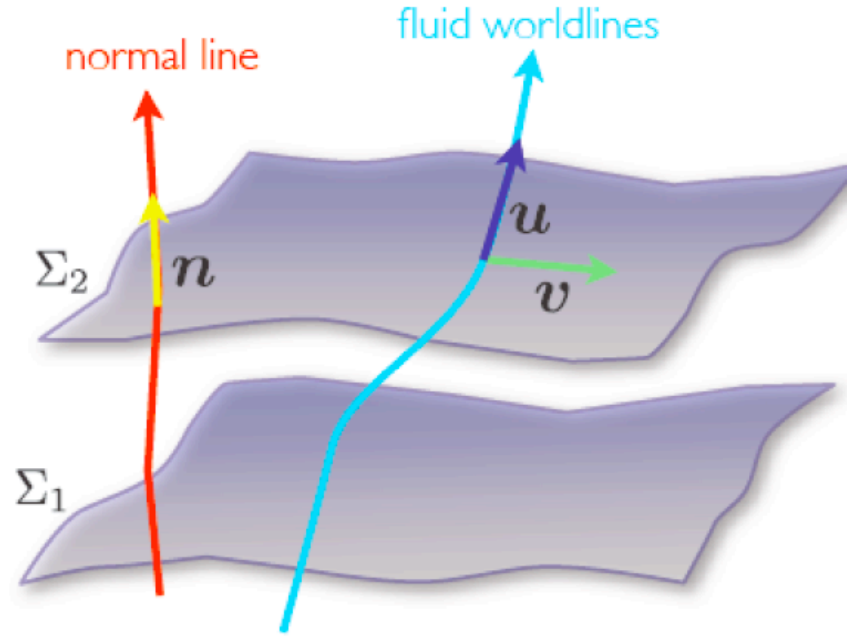
3+1: foliation of spacetime with spatial hypersurfaces Σ_t with constant t.

Line element:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Eulerian observer: at rest on the hypersurface; moves from Σ_t to $\Sigma_{t+\Delta t}$ along the unit normal vector. Speed given by:

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$



Hyperbolic system:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial x^0}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

$$\begin{aligned}
\mathbf{U} &= (D, S_j, \tau) \\
\mathbf{F}^i &= \left(D\left(v^i-\frac{\beta^i}{\alpha}\right), S_j\left(v^i-\frac{\beta^i}{\alpha}\right)+p\delta^i_j, \tau\left(v^i-\frac{\beta^i}{\alpha}\right)+pv^i\right) \\
\mathbf{S} &= \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right)
\end{aligned}$$

First-order flux-conservative hyperbolic system

$$\begin{aligned}
D &= \rho W \\
S_j &= \rho h W^2 v_j \\
\tau &= \rho h W^2 - p - D
\end{aligned}$$

$$W^2 = \frac{1}{1-v^j v_j}$$

Lorentz factor

$$h = 1+\varepsilon+\frac{p}{\rho}$$

specific enthalpy

Recovering special relativistic and Newtonian limits

Full GR

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho W}{\partial t}+\frac{\partial\sqrt{-g}\rho W v^i}{\partial x^i}\right)=0$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho h W^2 v^j}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}\right)=T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right)$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2-\rho W)v^i}{\partial x^i}\right)=\alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)$$

Minkowski

$$\frac{\partial\rho W}{\partial t}+\frac{\partial\rho W v^i}{\partial x^i}=0$$

$$\frac{\partial\rho h W^2 v^j}{\partial t}+\frac{\partial(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}=0$$

$$\frac{\partial(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial(\rho h W^2-\rho W)v^i}{\partial x^i}=0$$

Newton

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v^i}{\partial x^i}=0$$

$$\frac{\partial\rho v^j}{\partial t}+\frac{\partial(\rho v^i v^j+p\delta^{ij})}{\partial x^i}=0$$

$$\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2+p\right)v^i}{\partial x^i}=0$$

(quiz: prove it!)

Eigenvalues (characteristic speeds)

Numerical schemes based on Riemann solvers use the local characteristic structure of the hyperbolic system of equations.

The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exists. The above system satisfies, hence, the definition of hyperbolicity.

Eigenvalues (along the x direction)

$$\begin{aligned}\lambda_0 &= \alpha v^x - \beta^x \quad \text{(triple)} \\ \lambda_\pm &= \frac{\alpha}{1 - v^2 c_s^2}\left\{ v^x(1 - c_s^2) \pm c_s \sqrt{(1 - v^2)[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)]} \right\} - \beta^x\end{aligned}$$

Special relativistic limit (along x-direction)

$$\lambda_0 = v^x \quad \text{(triple)}$$

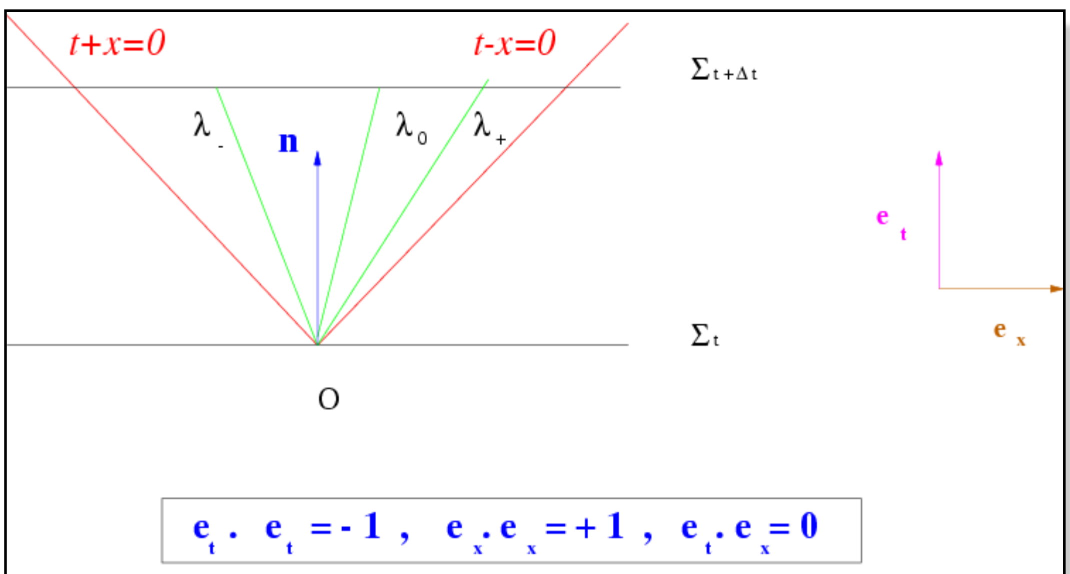
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}\left(v^x\left(1 - c_s^2\right) \pm c_s \sqrt{\left(1 - v^2\right)\left[1 - v^x v^x - \left(v^2 - v^x v^x\right) c_s^2\right]} \right)$$

coupling with transversal components of the velocity (important difference with Newtonian case)

Even in the purely 1D case:

$$\vec{v} = (v^x, 0, 0) \Rightarrow \lambda_0 = v^x, \; \lambda_\pm = \frac{v^x \pm c_s}{1 \pm v^x c_s}$$

For causal EOS sound cone lies within light cone



Recall Newtonian (1D) case:

$$\lambda_0 = v^x, \; \lambda_\pm = v^x \pm c_s$$

General relativistic MHD

Dynamics of relativistic, electrically conducting fluids in the presence of magnetic fields.

Ideal GRMHD: Absence of viscosity effects and heat conduction in the limit of infinite conductivity (perfect conductor fluid).

The stress-energy tensor includes contribution from the perfect fluid and from the magnetic field b^μ measured by observer comoving with the fluid.

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{PF}} + T^{\mu\nu}_{\mathrm{EM}} \longrightarrow \boxed{T^{\mu\nu} = \rho h^* u^\mu u^\nu + p^* g^{\mu\nu} - b^\mu b^\nu}$$

$$T^{\mu\nu}_{\mathrm{PF}} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$$T^{\mu\nu}_{\mathrm{EM}} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\delta}F_{\lambda\delta} = \left(u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}\right)b^2 - b^\mu b^\nu$$

$$F^{\mu\nu} = -\eta^{\mu\nu\lambda\delta}u_\lambda b_\delta$$

Ideal MHD condition: electric four-current must be finite.

$$F^{\mu\nu}u_\nu = 0$$

$$J^\mu = \rho_{\mathrm{q}} u^\mu + \sigma F^{\mu\nu}u_\nu \quad \sigma \to \infty$$

with the definitions:

$$b^2 = b^\nu b_\nu$$

$$p^* = p + \frac{b^2}{2}$$

$$h^* = h + \frac{b^2}{\rho}$$

General relativistic MHD: equations

Antón et al. (2006)

Conservation of mass: $\nabla_\mu(\rho u^\mu) = 0$

Conservation of energy and momentum: $\nabla_\mu T^{\mu\nu} = 0$

Maxwell's equations: $\nabla_\mu {}^*F^{\mu\nu} = 0$ $\quad {}^*F^{\mu\nu} = \frac{1}{W}(u^\mu B^\nu - u^\nu B^\mu)$

- Divergence-free constraint: $\vec{\nabla}\cdot\vec{B} = 0$
- Induction equation: $\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\vec{B}\right) = \vec{\nabla}\times\left[\left(\alpha\vec{v} - \vec{\beta}\right)\times\vec{B}\right]$

Adding all up: first-order, flux-conservative, hyperbolic system + constraint

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t} + \frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right) = \mathbf{S} \qquad \frac{\partial(\sqrt{\gamma}B^i)}{\partial x^i} = 0$$

$$D = \rho W \qquad S_j = \rho h^* W^2 v_j - \alpha b_j b^0 \qquad \tau = \rho h^* W^2 - p^* - \alpha^2 (b^0)^2 - D$$

$$\mathbf{U} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^k \end{bmatrix} \quad \mathbf{F}^i = \begin{bmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i + p^*\delta^i_j - b_j B^i/W \\ \tau\tilde{v}^i + p^* v^i - \alpha b^0 B^i/W \\ \tilde{v}^i B^k - \tilde{v}^k B^i \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 \\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{bmatrix}$$

MHD equations: hyperbolic structure

Wave structure classical MHD (Brio & Wu 1988): 7 physical waves

Two ALFVEN WAVES: $\lambda_{a_\pm} \Longrightarrow \lambda_a = v_x \pm v_a$

Two FAST MAGNETOSONIC WAVES: $\lambda_{f_\pm} \Longrightarrow \lambda_{f_\pm} = v_x \pm v_f$

Two SLOW MAGNETOSONIC WAVES: $\lambda_{s_\pm} \Longrightarrow \lambda_{s_\pm} = v_x \pm v_s$

One ENTROPY WAVE: $\lambda_e \Longrightarrow \lambda_e = v_x$

$$\boxed{\lambda_{f_-} \leq \lambda_{a_-} \leq \lambda_{s_-} \leq \lambda_e \leq \lambda_{s_+} \leq \lambda_{a_+} \leq \lambda_{f_+}}$$

$$\boxed{v_{f,s}^2 = \frac{1}{2}\left\{ c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho}\right)^2 - 4\left(\frac{B_x^2}{\rho}\right)c_s^2} \right\}, \quad v_a = \sqrt{\frac{B_x^2}{\rho}}}$$

Anile & Pennisi (1987), Anile (1989) (see also van Putten 1991) have studied the characteristic structure of the equations (eigenvalues, right/left eigenvectors) in the space of covariant variables (u^μ,b^μ,p,s).

Wave structure for relativistic MHD (Anile 1989): roots of the characteristic equation.

Only entropic waves and Alfvén waves are explicit.

Magnetosonic waves are given by the numerical solution of a quartic equation.

Augmented system of equations: Unphysical eigenvalues/eigenvectors (entropy & Alfvén) which must be removed numerically (Anile 1989, Komissarov 1999, Balsara 2001, Koldoba et al 2002).

Basic takeaway facts

- GR hydrodynamics equations (and MHD) can be cast as a hyperbolic system of conservation laws.

- Wave structure of equations known.

- High-resolution methods from CFD can be applied.

- When coupled to hyperbolic formulations of Einstein's equations (e.g. BSSN), numerical relativity simulations of non-vacuum spacetimes are feasible.

- Routinely employed nowadays - CCSNe, BNS, BH-NS, etc - providing gravitational waveforms in the strong-gravity regime.