From Compact Objects to Quasi-Normal Modes and Back

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Introduction

Studying **compact relativistic objects** is of great interest these days. The **groundbreaking discovery of gravitational waves** by LIGO [1] opens not only a completely new window to relativistic astrophysics, but also to novel fundamental tests of gravity [2]. **Compact objects** are the most promising gravitational wave sources and therefore the **ideal laboratory for theoretical and observational studies**.

My **PhD project** is dedicated to the semi-analytic study of such objects, with focus on the **quasi-normal modes** of their spacetime. They are an interesting tool to **connect theory with future observations**.

Overview

1. Compact Objects

- Most studied compact objects are black holes and neutron stars in general relativity, but **there could be room for more**:
  - Alternative models for ultra compact objects, e.g. gravastars.
  - Alternative theories of gravity or quantum gravitational modifications.
  - Recent possible detection of "echoes" in LIGO data?
- Many of them make **predictions in the gravitational wave sector**.

2. Quasi-Normal Modes

- **Quasi-normal modes** are a special class of perturbations [3].
- For gravitational waves study metric perturbations: $g_{\mu\nu}^{\text{new}} = g_{\mu\nu}^{\text{known}} + h_{\mu\nu}^{\text{pert.}}$.
- Insert perturbed metric $g_{\mu\nu}^{\text{new}}$ in linearized field equations.
- For non-rotating and spherically symmetric systems one finds

$$\frac{\mathrm{d}^2}{\mathrm{d}r^{*2}}\Psi(r^*) + \left(E_n - V(r^*)\right)\Psi(r^*) = 0,$$

  with $\omega_n = \sqrt{E_n}$ being the eigenvalues and $V(r^*)$ an **effective potential**.
- **Quasi-normal modes** ω_n are defined by special boundary conditions.
- $V(r^*)$ and ω_n of black holes differ significantly from alternative objects.

Direct Problem: $V(x) \rightarrow E_n$

- **Direct problem:** get the spectrum E_n from the potential $V(x)$.
- **Bohr-Sommerfeld (BS)** methods can be used for analytic studies.
- Not exact, but powerful and simplified analytic framework.
- For **quasi-stationary states** it is given by

$$\int_{x_0}^{x_1} \sqrt{E_n - V(x)}\,\mathrm{d}x = \pi\left(n + \frac{1}{2}\right) - \frac{i}{4}\exp\left(2i\int_{x_1}^{x_2}\sqrt{E_n - V(x)}\,\mathrm{d}x\right),$$

  with x_i being the classical turning points, $E_n = V(x_i)$.

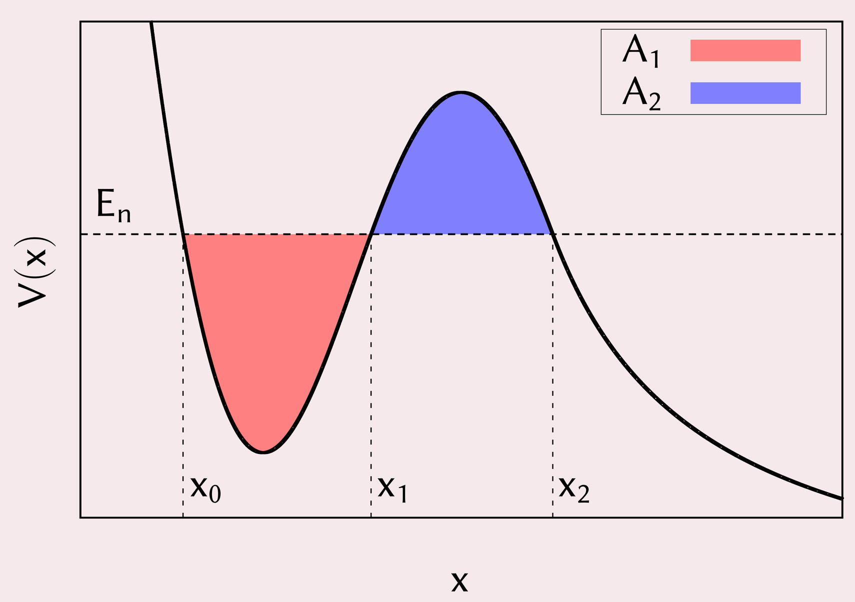

Fig. 1 : Typical potential for quasi-stationary states, areas are "related" to BS integrals.

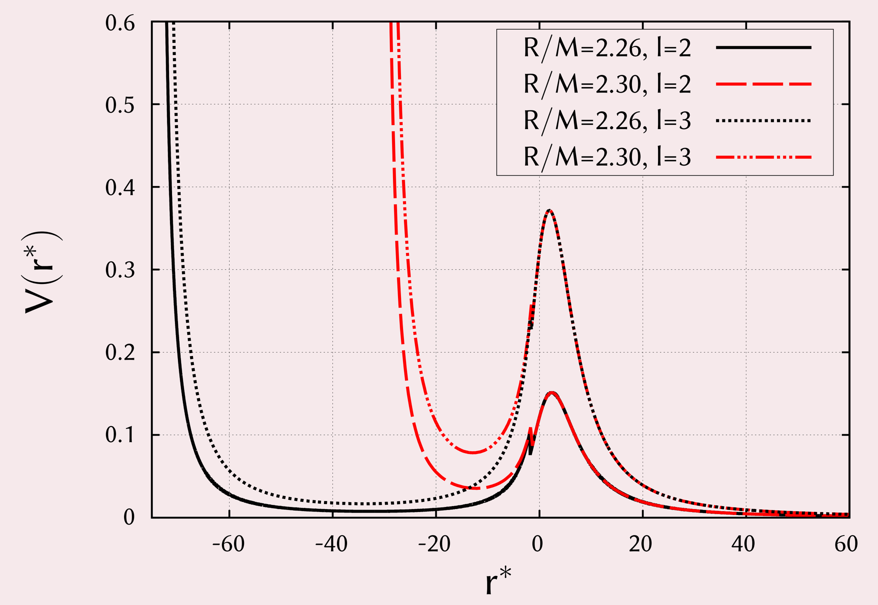

Fig. 2 : Axial mode potential for constant density stars with different R/M and l.

Inverse Problem: $E_n \rightarrow V(x)$

- **Inverse problem:** reconstruct the potential $V(x)$ from the spectrum E_n.
- Usually much harder to solve and seldom unique, if well posed at all.
- Combination of semi-classical techniques [4, 5] allow reconstruction of

$$\mathcal{L}_1(E) = 2\frac{\partial}{\partial E}\int_{E_{\min}}^{E}\frac{n(E') + 1/2}{\sqrt{E - E'}}\mathrm{d}E', \quad \mathcal{L}_2(E) = -\frac{1}{\pi}\int_{E}^{E_{\max}}\frac{(\mathrm{d}T(E')/\mathrm{d}E')}{T(E')\sqrt{E' - E}}\mathrm{d}E'.$$

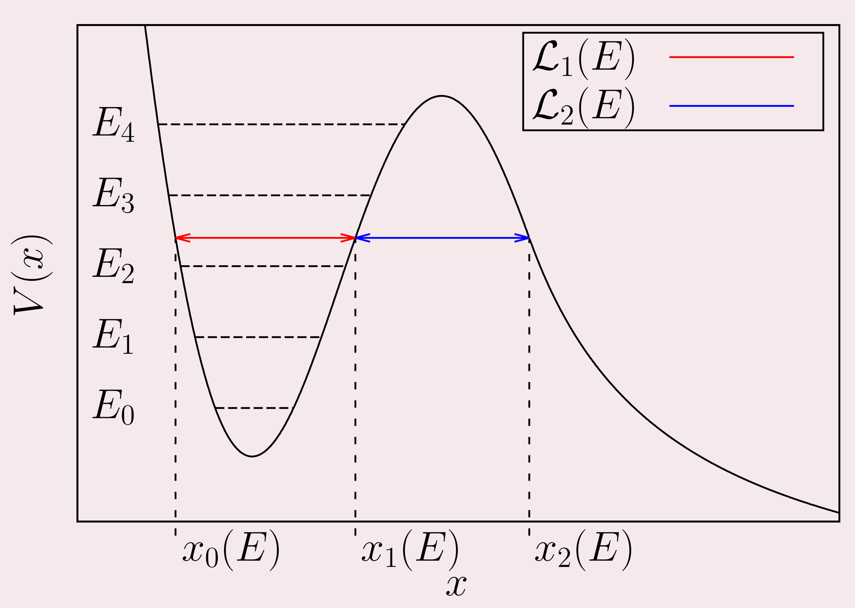

Fig. 3 : Potential with corresponding spectrum E_n.

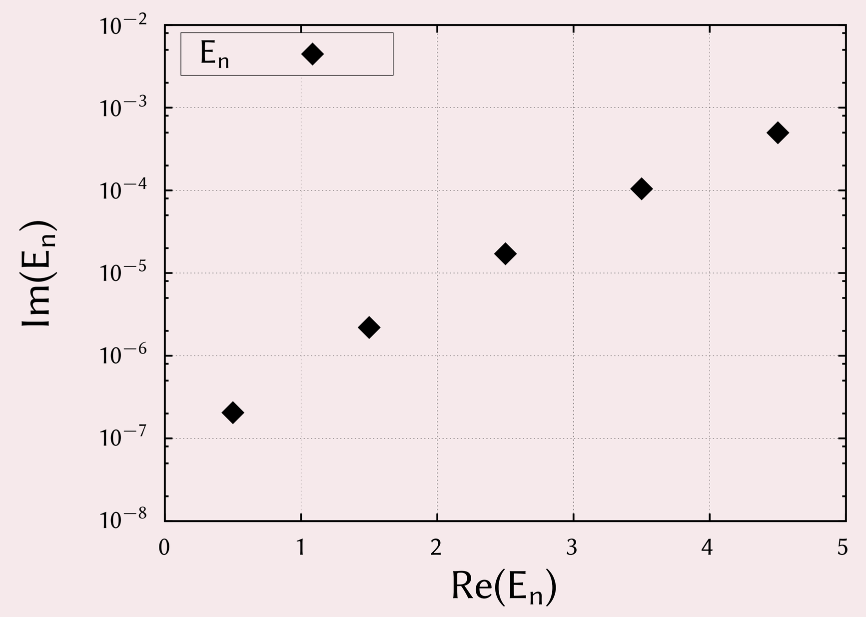

Fig. 4 : Spectrum for a potential shown in Fig. 3.

Results

We tested the methods by studying each of the problems for constant density stars and gravastars in two separate works [6, 7].

1. Direct Problem

- Developed/solved a **full analytic toy model** for very simple and useful approximations of the trapped ω_n.
- Numerically studied a correction to the real part which also improves the imaginary part.
- Results agree very well with precise full numerical values.

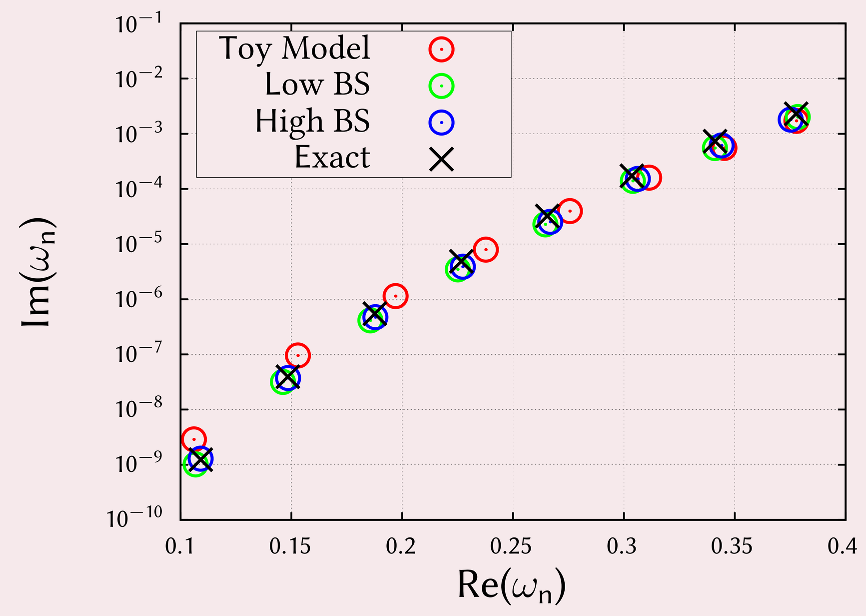

Fig. 5 : Trapped ω_n of a constant density star with $R/M = 2.26$ and $l = 2$ from different methods.

2. Inverse Problem

- Constructed the widths $\mathcal{L}_1(E)$ and $\mathcal{L}_2(E)$ from different spectra using inter-/extrapolation.
- **Birkhoff's theorem** yields unique solution for the reconstructed potential.
- Precision increases with the number of trapped modes ω_n in the potential.

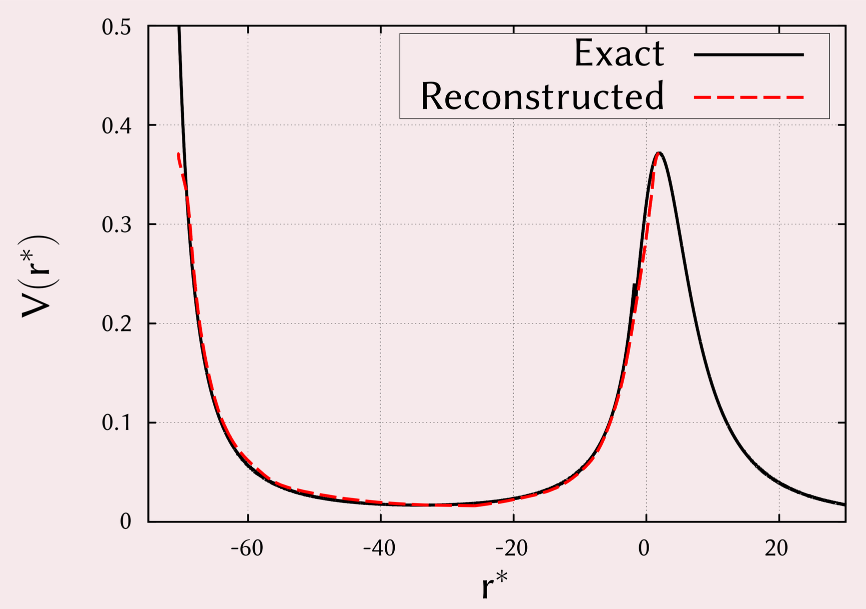

Fig. 6 : Reconstructed and exact axial mode potential of a constant density star with $R/M = 2.26$ and $l = 3$.

Conclusions

- **Bohr-Sommerfeld/WKB methods are useful tools in the semi-analytic study of gravitational perturbations from compact objects.**
- **They can be used to calculate the trapped quasi-normal modes [6] and to reconstruct the perturbation potential [7].**
- **A first parameter estimation of ultra compact objects using gravitational waves is discussed in [8].**

References

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