CERN Summer School 2017
Introduction to Accelerator Physics

Part II

by

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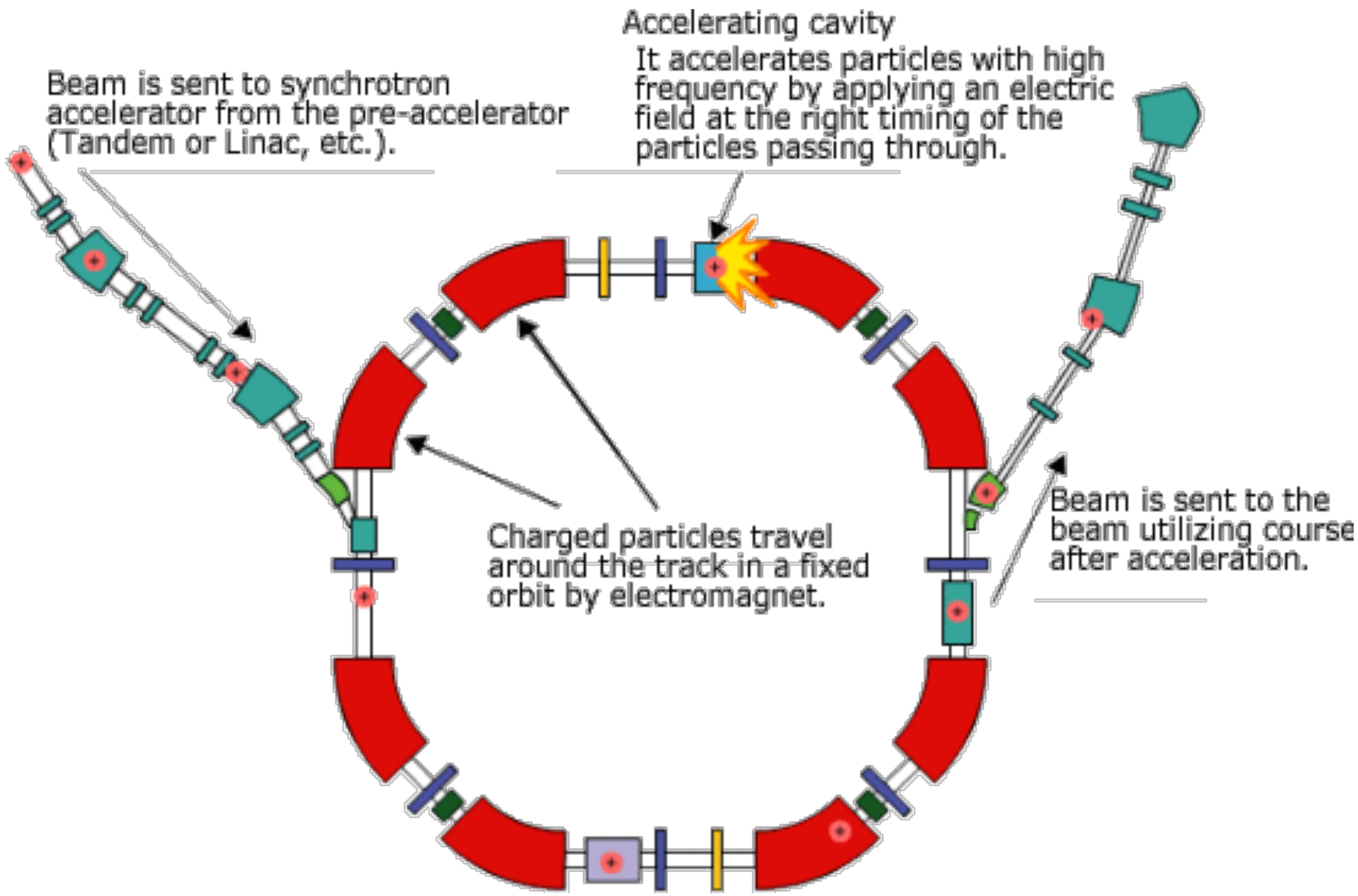
Contents of today's lecture



Contents of today's lecture

How can we keep the particles on a circular trajectory?

How can we keep the particles on a circular trajectory for 1000s of turns?

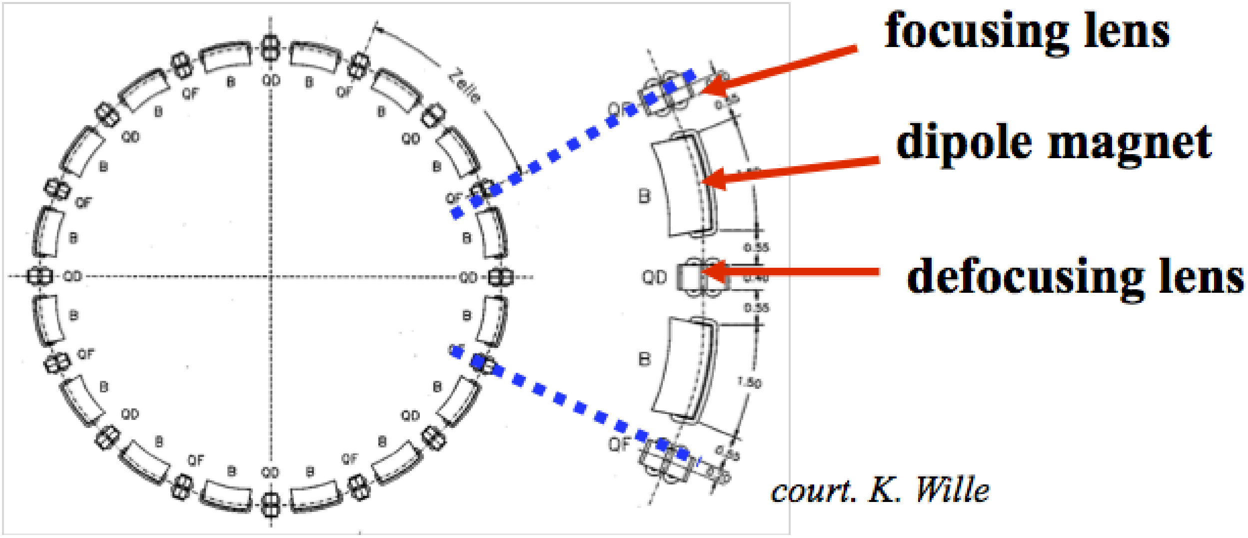
How can we influence the beam size?

How can we describe the motion of a particle in an alternating gradient storage ring?

What parameters are of importance?

Contents of today's lecture

Beam dynamics in the transverse plane



How can we keep the particles on a circular trajectory?

Usually use only magnetic fields for transverse control

$$\boxed{\vec{F} = q \cdot (\vec{E} + \vec{v} \times \vec{B})}$$ Lorentz Force

What is the equivalent E field of B = 1 T?

– Ultra-relativistic: $|\vec{v}| \approx c \approx 3 \times 10^8 m/s$

$$\begin{aligned} F &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot 1T \\ &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot \frac{Vs}{m^2} \\ &= q \cdot 300 \frac{MV}{m} \end{aligned}$$

Equivalent electric field!!:

$$|\vec{E}| = 300 \frac{MV}{m}$$

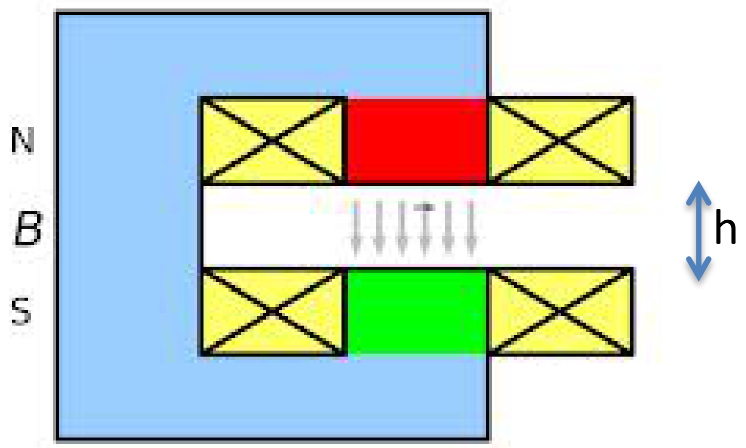
➔ To guide the particles we use magnetic fields from electro-magnets.

Dipole magnets: guiding magnets

Vertical magnetic field to bend in the horizontal plane

Dipole electro-magnets:

$$\boxed{\vec{F} = q \cdot \vec{v} \times \vec{B}}$$



$$B = \frac{\mu_0 n I}{h}$$

Dipole magnets: guiding magnets

Circular accelerator: Lorentz Force = Centrifugal Force

$$F_L = qvB$$
$$F_{centr} = \frac{mv^2}{\rho} \longrightarrow \frac{mv^2}{\rho} = qvB$$

$$\boxed{\frac{p}{q} = B\rho}$$

$B\rho$ Beam rigidity

Useful formula: $$\frac{1}{\rho[m]} \approx 0.3\frac{B[T]}{p[GeV/c]}$$

Example for the LHC

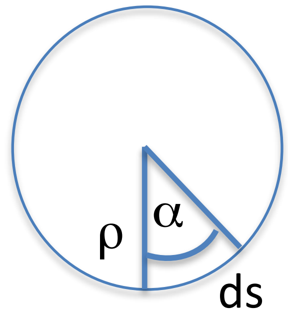
- p^+ @ 7 TeV/c
- 8.3 T

$$\frac{1}{2.53\ km} = 0.3\frac{8.3}{7000}$$

Define design trajectory (orbit)

Length of dipole magnet and field define total bending angle of magnet:

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{B\,dl}{B\rho}$$



Circular accelerator: total bending angle:= 2π

$$\alpha = 2\pi = \frac{\int B\ \ dl}{B\rho} = \frac{\int B\ \ dl}{\frac{p}{q}}$$

How many dipole magnets do we need in the LHC?

- Dipole length = 15 m
- Field 8.3 T

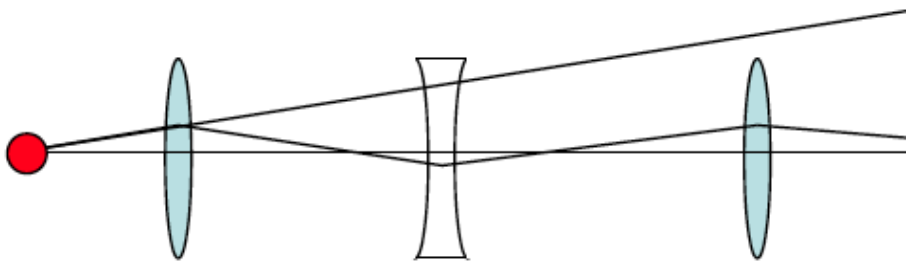
$$\int B\ dl \approx N\ l\ B = 2\pi\frac{p}{q}$$

$$N = \frac{2\pi\ \ 7000\ \ 10^9 eV}{8.3T\ \ 15m\ \ 3{\cdot}10^8\,\frac{m}{s}\,e} = 1232$$

Focusing is mandatory for stability

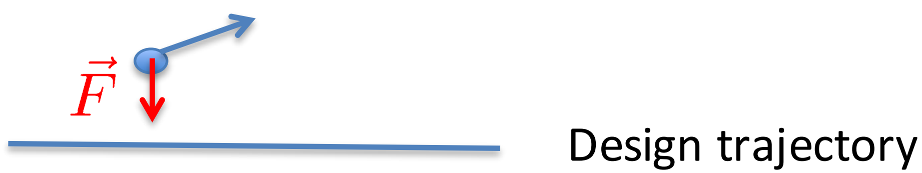
Define design trajectory with dipole magnets

Trajectories of particles in beam will deviate from design trajectory



➔ Focusing

- Particles should feel restoring force when deviating from design trajectory horizontally or vertically

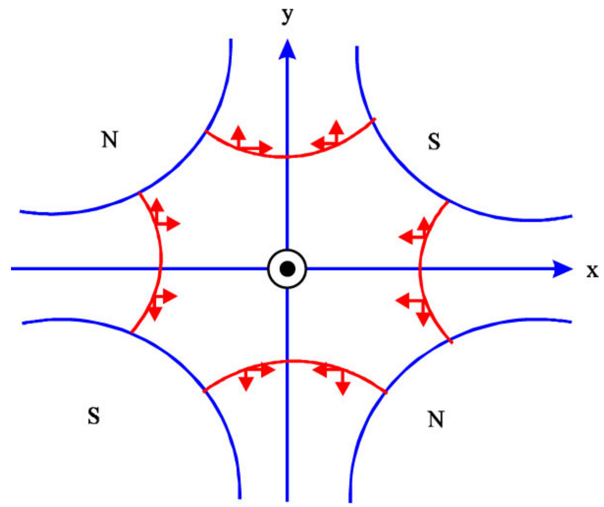


Focusing with Quadrupole Magnets

Requirement: Lorentz force increases as a function of distance from design trajectory

E.g. in the horizontal plane

$$F(x) = q \cdot v \cdot B(x)$$



The red arrows show the direction of the force on the particle

We want a magnetic field that

$$B_y = g \cdot x \qquad B_x = g \cdot y$$

➔ Quadrupole magnet

Gradient of quadrupole

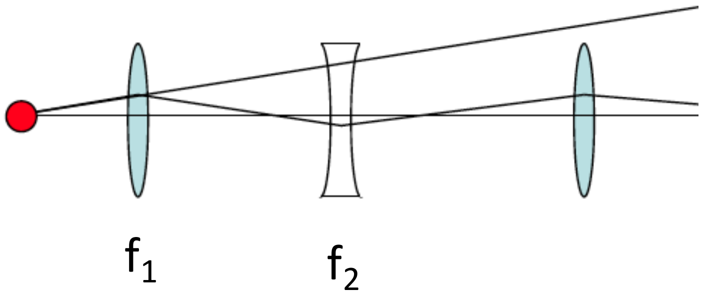
$$g = \frac{2\mu_0 nI}{r^2}\left[\frac{T}{m}\right]$$

Normalized gradient, focusing strength

$$k = \frac{g}{p/q}\left[m^{-2}\right]$$

Strong focusing

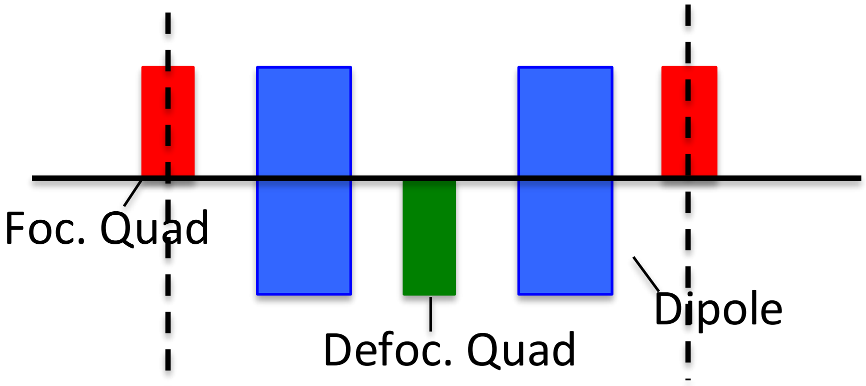
Light lenses:



$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

Consider f_1=f, f_2 = - f ➔ $\boxed{F = f^2/d > 0}$

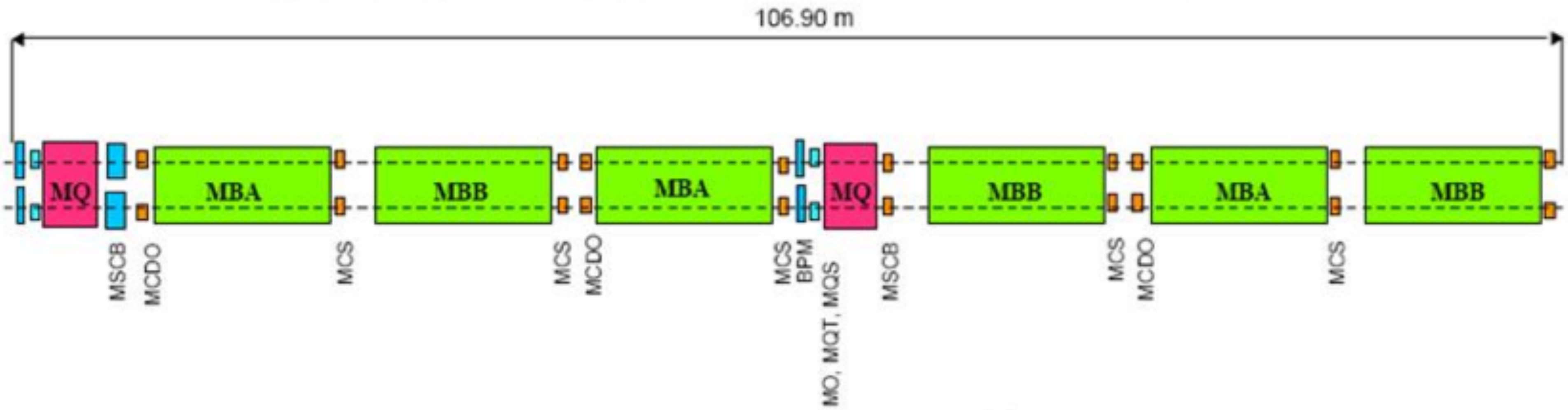
In a synchrotron: the lenses are the quadrupoles



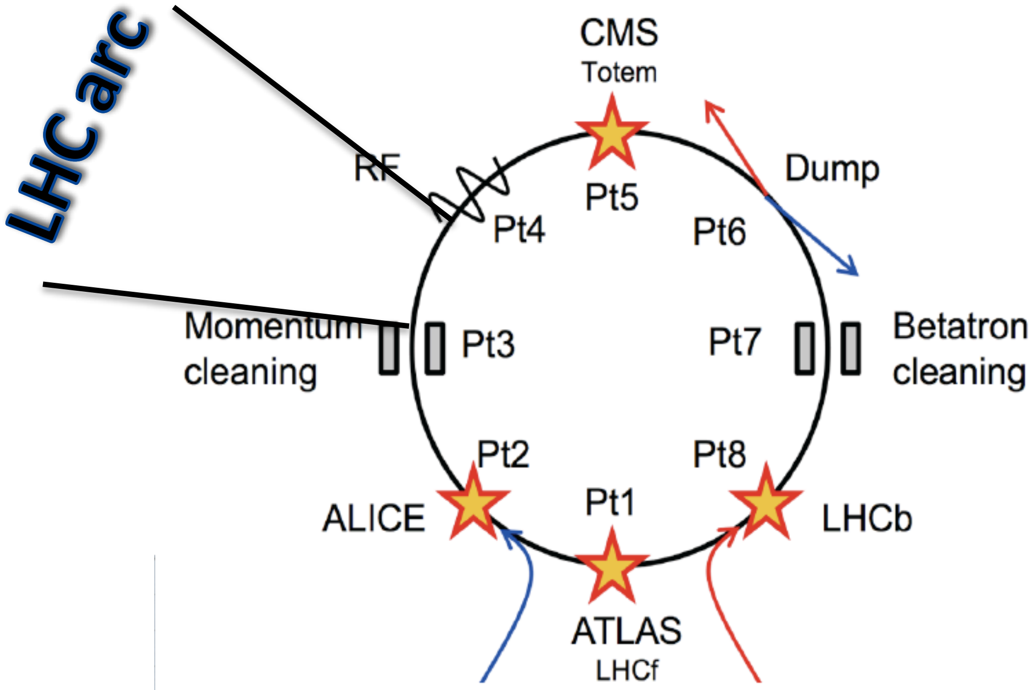
FODO Cell: F = Focusing, 0=nothing (bend, RF,..),
D = Defocusing

Focal length of quadrupole $f = \frac{1}{k \cdot l_Q}$

The LHC FODO cell

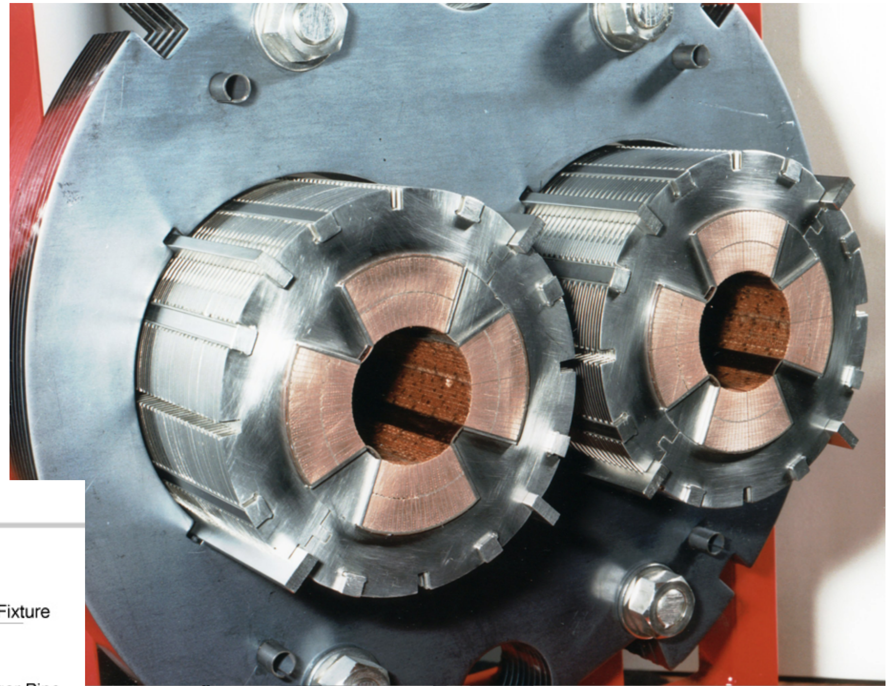


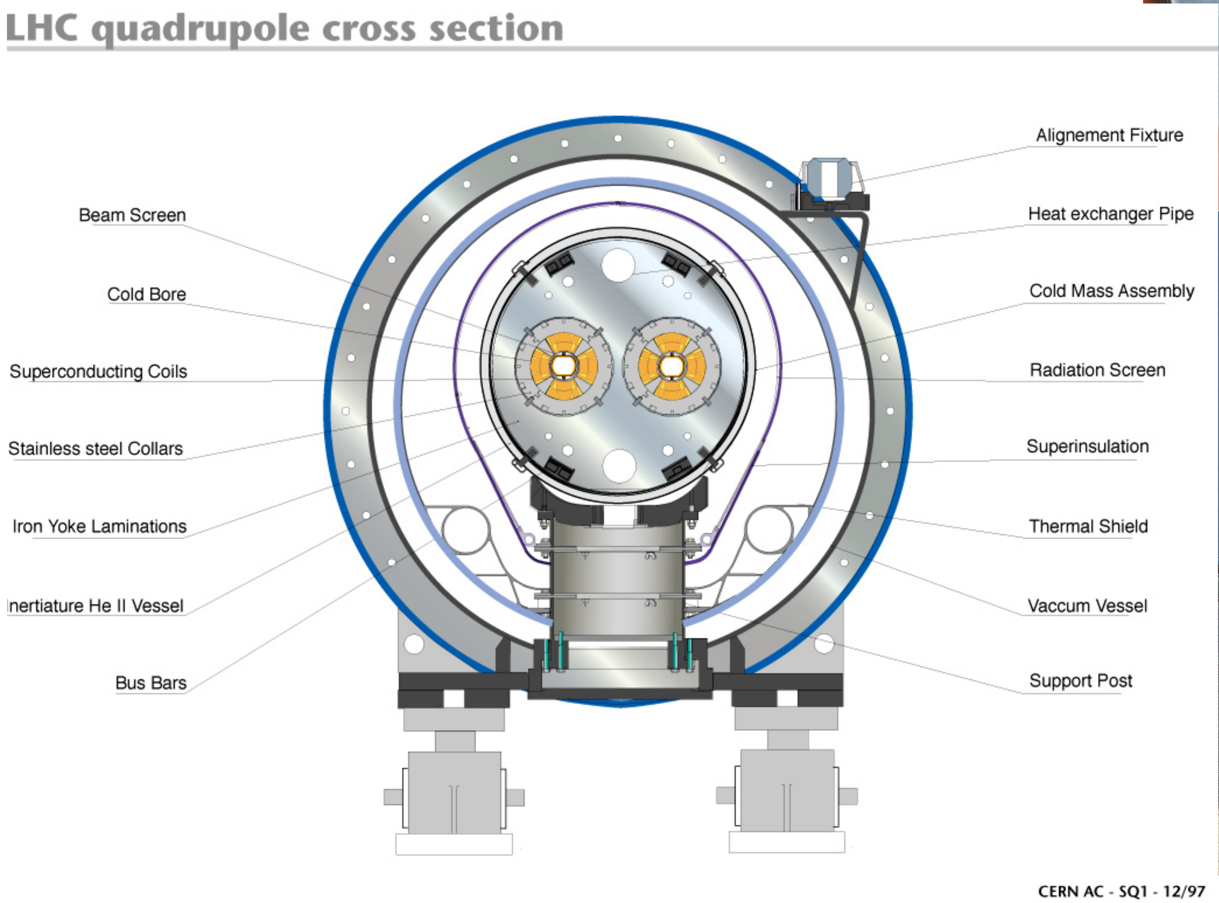
Each LHC arc consists of 23 FODO cells



The LHC main quadrupole magnet

Length = 3.2 m
Gradient = 223 T/m
Peak field 6.83 T
Total number in LHC: 392





Towards the Equation of Motion

And now a bit of theory to see how we can calculate trajectories through dipoles and quadrupoles.

Taylor series expansion of B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}x^3 + \ldots$$

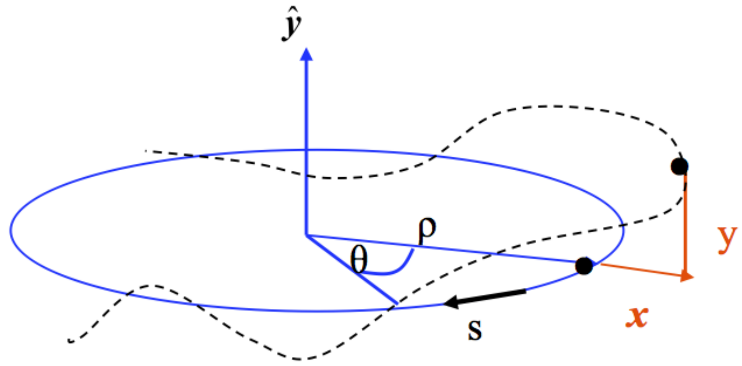
Normalize and keep only terms linear in x

$$\frac{B_y(x)}{p/q} = \frac{1}{\rho} + kx + \frac{1}{2}\cancel{m}x^2 + \frac{1}{3!}\cancel{n}x^3 + \ldots$$

$$\frac{B_y(x)}{p/q} \approx \frac{1}{\rho} + kx$$

Towards Equation of Motion

Use different coordinate system: Frenet-Serret rotating frame



The ideal particle stays on "design" trajectory. (x=0, y=0)

And: $x, y \ll \rho$

The design particle has momentum $p_0 = m_0\gamma v$.

$\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p}$ .... relative momentum offset of a particle

Towards Equation of Motion

Replace time ‘t’ free parameter by path length ‘s’:

$$x' = \frac{dx}{ds}$$

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \mathrm{v}\cdot\frac{d}{ds} \rightarrow x' = \frac{p_x}{p_0}$$

$$m\ddot{x} = F_x \rightarrow mx'' = \frac{F_x}{\mathrm{v}^2} \rightarrow \boxed{x'' = \frac{F_x}{\mathrm{v}\cdot p}}$$

And F_x is the Lorentz force the particle feels in the magnet...

The Equation of Motion

All we have to do now is to insert F_x of e.g. a quadrupole magnet

$$F_x = qB_y\mathrm{v} = q \cdot g \cdot x \cdot \mathrm{v}$$

$$= -k$$

$$x'' = \frac{q \cdot g \cdot x \cdot \mathrm{v}}{p \cdot \mathrm{v}}$$

after a bit of maths: <u>the equations of motion</u>

$$\boxed{\begin{aligned} x'' + kx &= 0 \\ y'' - ky &= 0 \end{aligned}}$$

Quadrupole field changes sign between x and y

Solution of Equation of Motion

Equation of motion in horizontal plane*:

$$x'' + Kx = 0$$

Equation of the **harmonic oscillator** with spring constant K

Solution can be found with ansatz

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

Insert ansatz in equation ➔

For K > 0: focusing $\omega = \sqrt{K}$

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

*: for completeness: $K := \frac{1}{\rho^2} + k$ take focusing of dipole field into account

Solution of Equation of Motion

a_1 and a_2 through boundary conditions:

$$s = 0 \rightarrow \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, K > 0:

$$x(s) = x_0 \cos(\sqrt{K} s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K} s)$$

$$x'(s) = -x_0 \sqrt{K} \sin(\sqrt{K} s) + x'_0 \cos(\sqrt{K} s)$$

Use matrix formalism: TRANSFER MATRIX $\begin{pmatrix} x \\ x' \end{pmatrix} = M_{foc} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

$$\boxed{M_{foc} = \begin{pmatrix} \cos(\sqrt{K} s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} s) \\ -\sqrt{K} \sin(\sqrt{K} s) & \cos(\sqrt{K} s) \end{pmatrix}}$$

Solution of Equation for Defocusing Quadrupole

Solution of equation of motion with K < 0:

$$x'' + Kx = 0$$

New ansatz is:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

And the transfer matrix

$$\boxed{M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \\ \sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}}$$

Summary of Transfer Matrices

Uncoupled motion in x and y

$$K = 1/\rho^2 - k$$ ....horizontal plane

$$K = k$$ ....vertical plane

Focusing quadrupole, K >0:

$$\boxed{M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}}$$

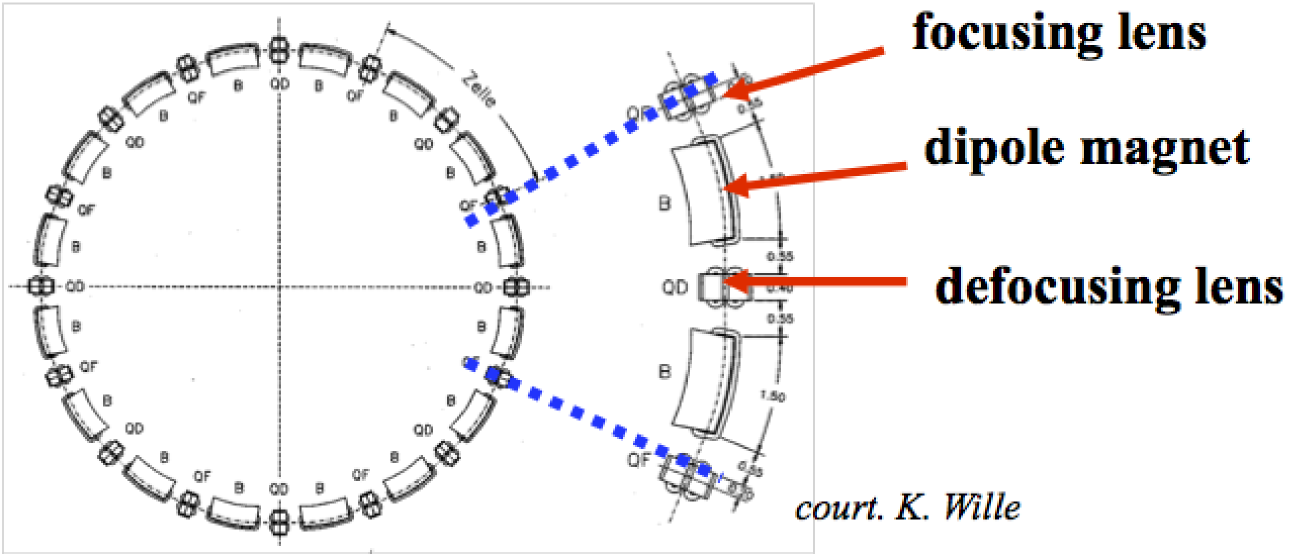
Defocusing quadrupole, K< 0:

$$\boxed{M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ \sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}}$$

Drift space: length of drift space L

$$\boxed{M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}}$$

Transfer matrix of synchrotron



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad M_{total} = M_{QF} \cdot M_D \cdot M_{Bend} \cdot M_D \cdot M_{QD} \cdot \ldots$$

A FEW IMPORTANT CONCEPTS

The Hill's Equation

We had...

$$x'' + Kx = 0$$

Around the accelerator K will not be constant, but will depend on s

$$x''(s) + K(s)x(s) = 0$$

Hill's equation

Where

- restoring force ≠ const, K(s) depends on the position s
- K(s+L) = K(s) periodic function, where L is the "lattice period"

General solution of Hill's equation:

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos(\psi(s) + \phi)$$

The Beta Function & Co

Solution of Hill's Equation is a quasi harmonic oscillation **(betatron oscillation)**: amplitude and phase depend on the position s in the ring.

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos(\psi(s)+\phi)$$

integration constants: determined by initial conditions

The beta function is a periodic function determined by the focusing properties of the lattice: i.e. quadrupoles

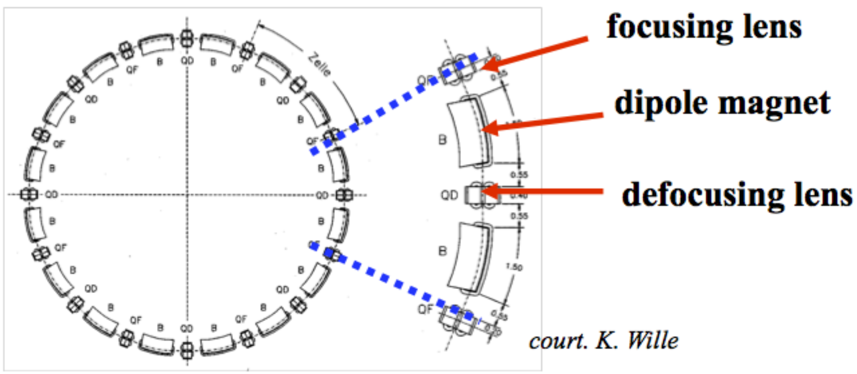
$$\beta(s+L) = \beta(s)$$

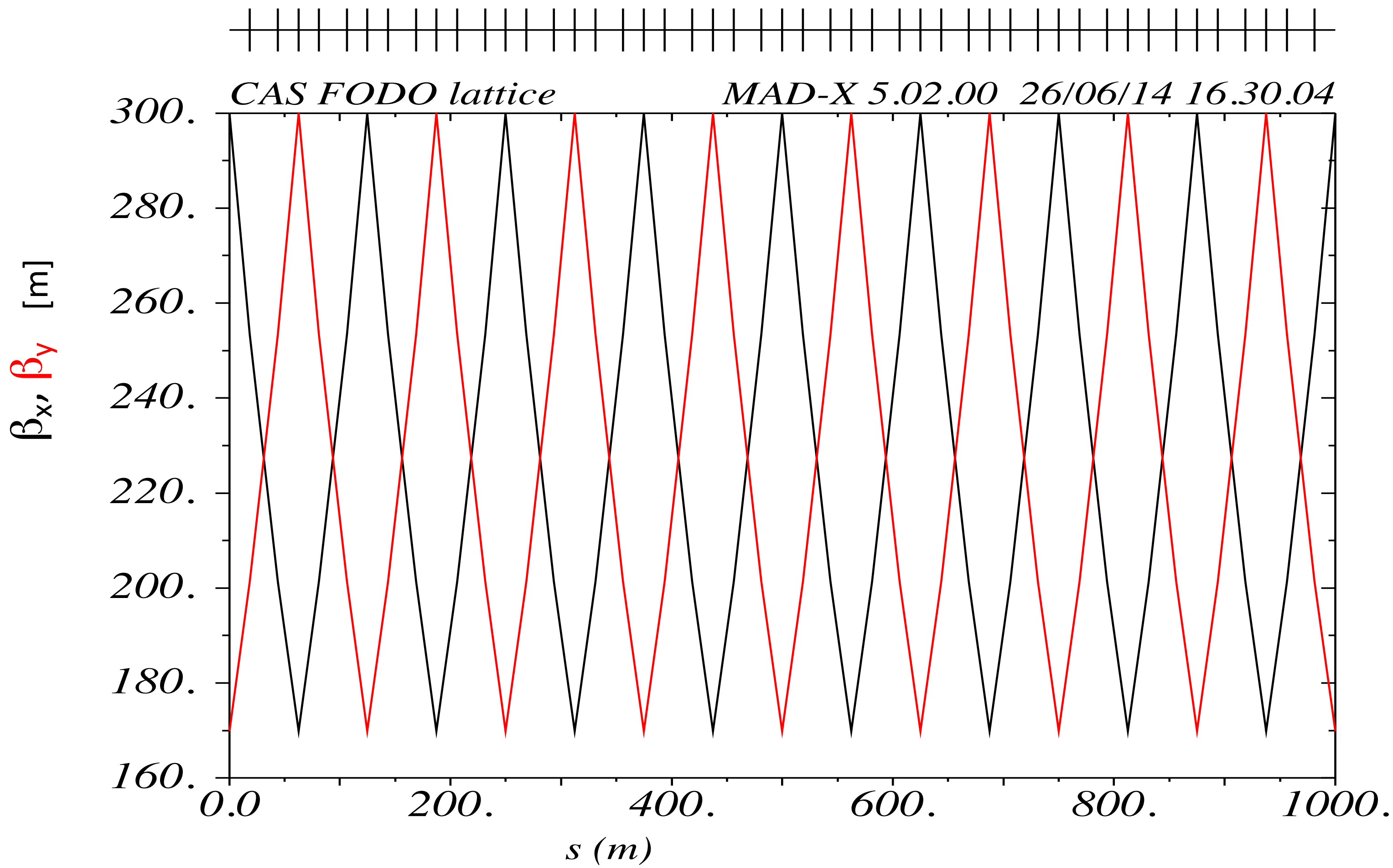
The "phase advance" of the oscillation between the point 0 and point s in the lattice.

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

The beta functions around the machine

An example...





Courant-Snyder Parameters: β (s), α (s), γ (s)

Definition: $\alpha(s) = -\frac{1}{2}\beta'(s) \qquad \gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos(\psi(s) + \phi)$$

$$x'(s) = -\sqrt{\frac{2J_x}{\beta(s)}}(\alpha(s)\cos(\psi(s) + \phi) + \sin(\psi(s) + \phi))$$

Let's assume for s(0) = s_0, $\psi(0) = 0$.

Defines ϕ from x_0 and x'_0, β_0 and α_0.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

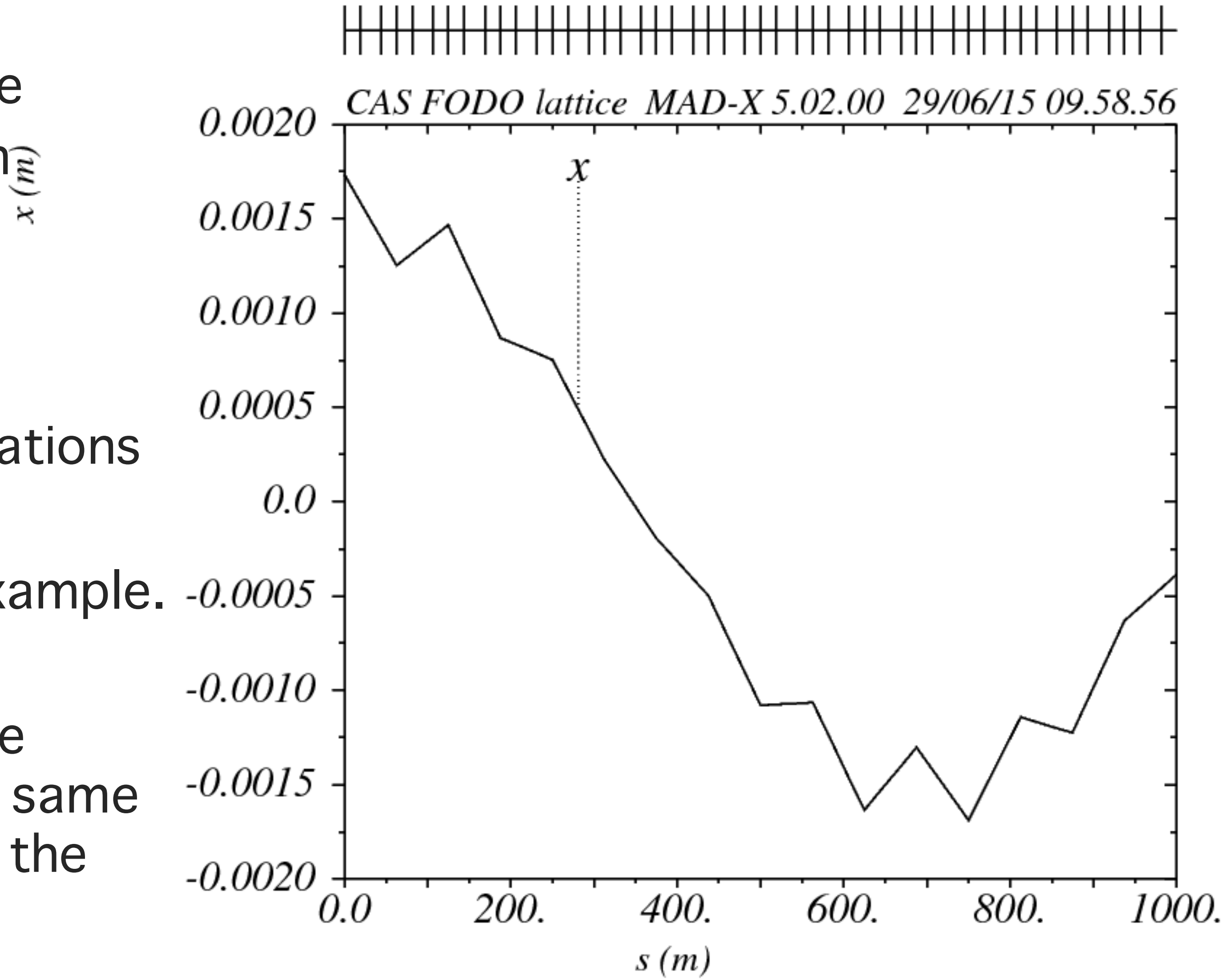
Can compute the single particle trajectories between two locations if we know α, β at these positions!

$$M = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ \frac{(\alpha_0-\alpha)\cos\psi-(1+\alpha\alpha_0)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi)) \end{pmatrix}$$

The trajectory around the ring

Whereas the beta functions are several 100 m...

...the trajectories are
in the order of ~ mm



The number of oscillations around the ring is less than 1 in this example.

The periodicity of the oscillation is not the same as the periodicity of the magnetic structure

The Tune

The number of oscillations per turn is called "tune"

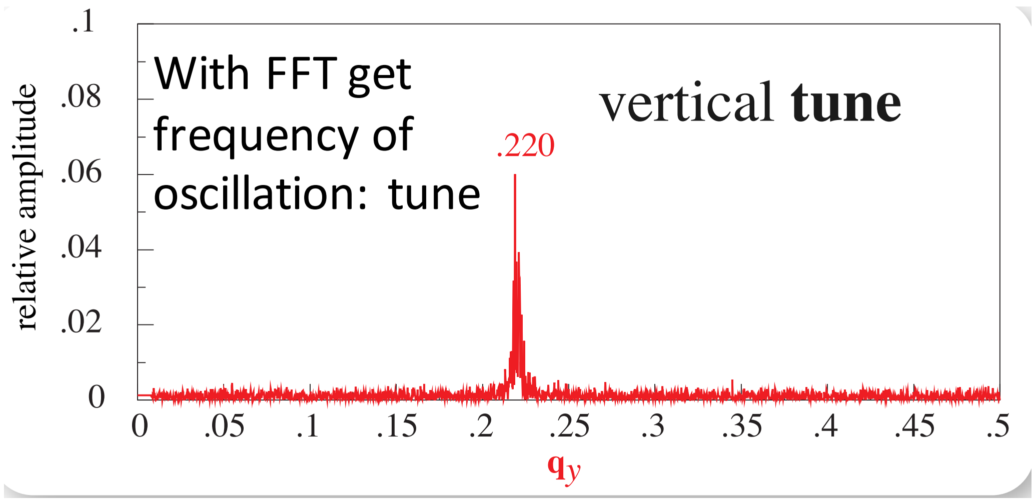
$$Q = \frac{\psi(L_{turn})}{2\pi} = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

The tune is an important parameter for the stability of motion over many turns.

It has to be chosen appropriately, measured and corrected.

Measure beam position at one location turn by turn

Beam position will change with $\propto \cos(2\pi Q i)$



The importance of the Courant-Snyder Parameters: β (s), α (s), γ (s)

The general form of the transfer matrices that describe the one period cell:

$$M_x = I \cos \mu_x + S \cdot A_x \sin \mu_x$$

where I is the identity matrix S is the antisymmetric matrix and A_x is a symmetric matrix containing the Courant-Snyder parameters:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_x = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix} \quad A_x^{-1} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$$

Can calculate the Courant-Snyder parameters at another location:

$$\boxed{A_x(s_1)^{-1} = M_x(s_1, s_0) A_x(s_0)^{-1} M_x(s_1, s_0)^{\mathrm{T}}}$$

The importance of the Courant-Snyder Parameters: β (s), α (s), γ (s)

Construct quantity J_x from the phase-space coordinates x, x'

$$J_x = \tfrac{1}{2} \begin{pmatrix} x & x' \end{pmatrix} A_x \begin{pmatrix} x \\ x' \end{pmatrix}$$

We call it ***action*** variable.

Now we have a look at how J_x transforms through the accelerator:

$$J_x \rightarrow \tfrac{1}{2} \begin{pmatrix} x & x' \end{pmatrix} M^T \left((M^T)^{-1} A_x M^{-1}\right) M \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$= \tfrac{1}{2} \begin{pmatrix} x & x' \end{pmatrix} A_x \begin{pmatrix} x \\ x' \end{pmatrix} = J_x$$

J_x is an invariant of motion through the beam line, accelerator of repeated e.g. FODO cells,…

The importance of the Courant-Snyder Parameters: β (s), α (s), γ (s)

J_x can be written as:

$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2\right)$$

....the equation of an ellipse in the phase-space x, x'

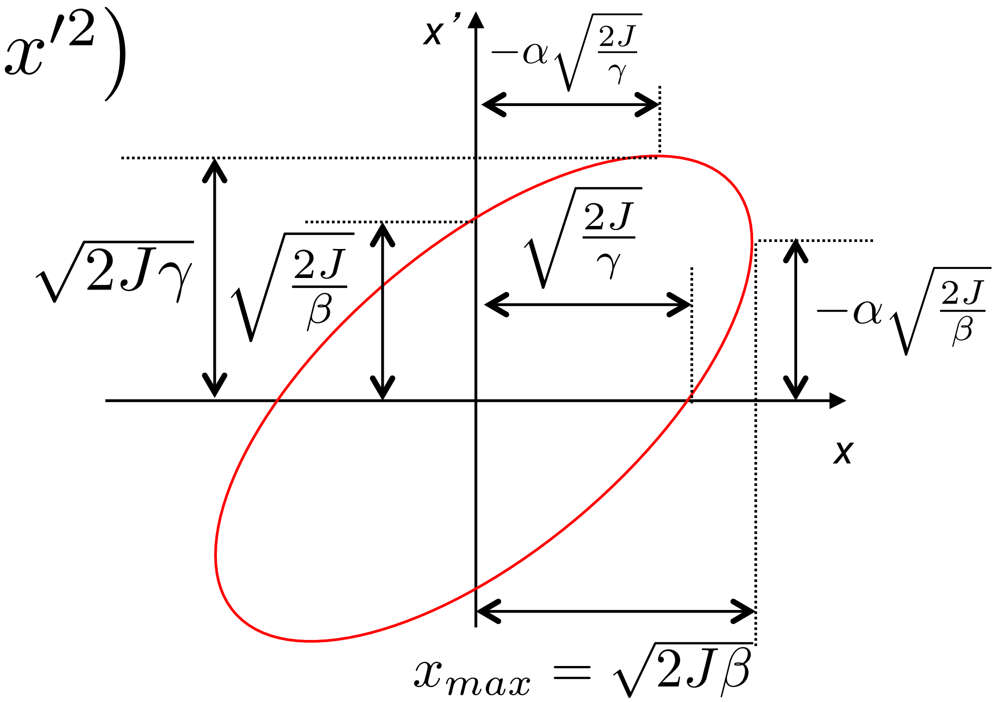
The area of the ellipse is

$$A = 2 \cdot \pi \cdot J_x$$

The area in phase space is invariant.

The shape and orientation are defined by the Courant-Snyder parameters.

phase space

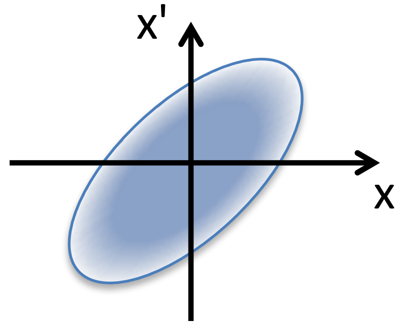


Area = 2πJ_x

Courant-Snyder Parameters and Particle Distribution

We had...

$$x = \sqrt{2\beta_x J_x} \cos \psi_x$$



The mean square value of x at a given location is

$$\langle x^2 \rangle = 2\beta_x \langle J_x \cos^2 \psi_x \rangle = \beta_x \langle J_x \rangle = \beta_x \epsilon_x$$

assume action and phase uncorrelated, and uniform distribution in phase from 0 to 2π

Define *emittance* of particle distribution; Invariant of motion

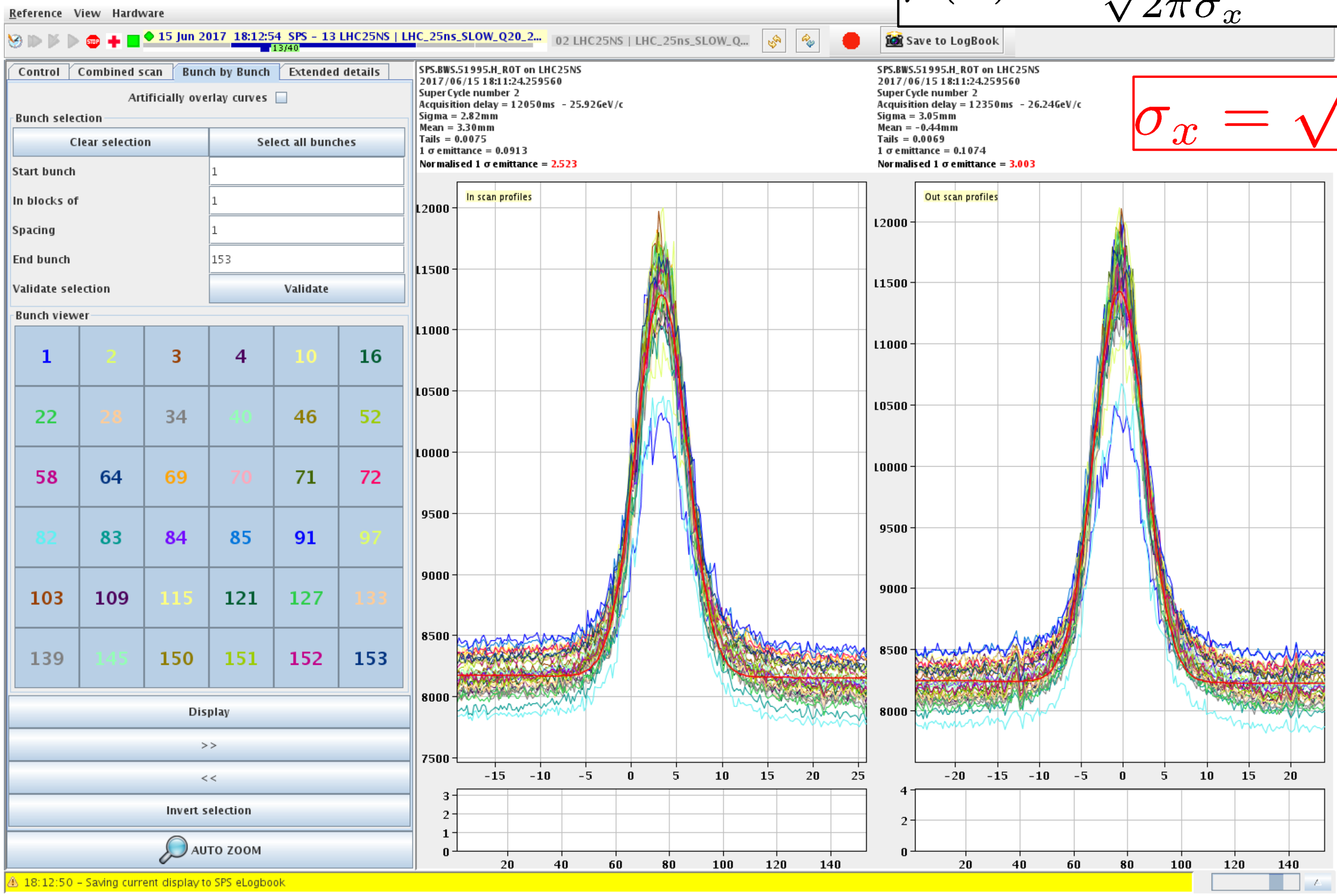
$$\langle J_x \rangle = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle} := \epsilon_x$$

Courant-Snyder Parameters and Particle Distribution

Typically particles in accelerator have Gaussian particle distribution in position and angle.

$$\rho(x) = \frac{N}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{x^2}{2\sigma_x^2}}$$

$$\sigma_x = \sqrt{\varepsilon\beta_x}$$



Emittance during Acceleration

What happens to the emittance if the reference momentum P_0 changes?

Can write down transfer matrix for reference momentum change:

$$M_x = \begin{pmatrix} 1 & 0 \\ 0 & P_0/P_1 \end{pmatrix} \quad \longrightarrow \quad \epsilon_{x1} = \frac{P_0}{P_1}\epsilon_{x0}$$

The emittance shrinks with acceleration!

With $P = \beta\gamma mc$ where γ, β are the relativistic parameters

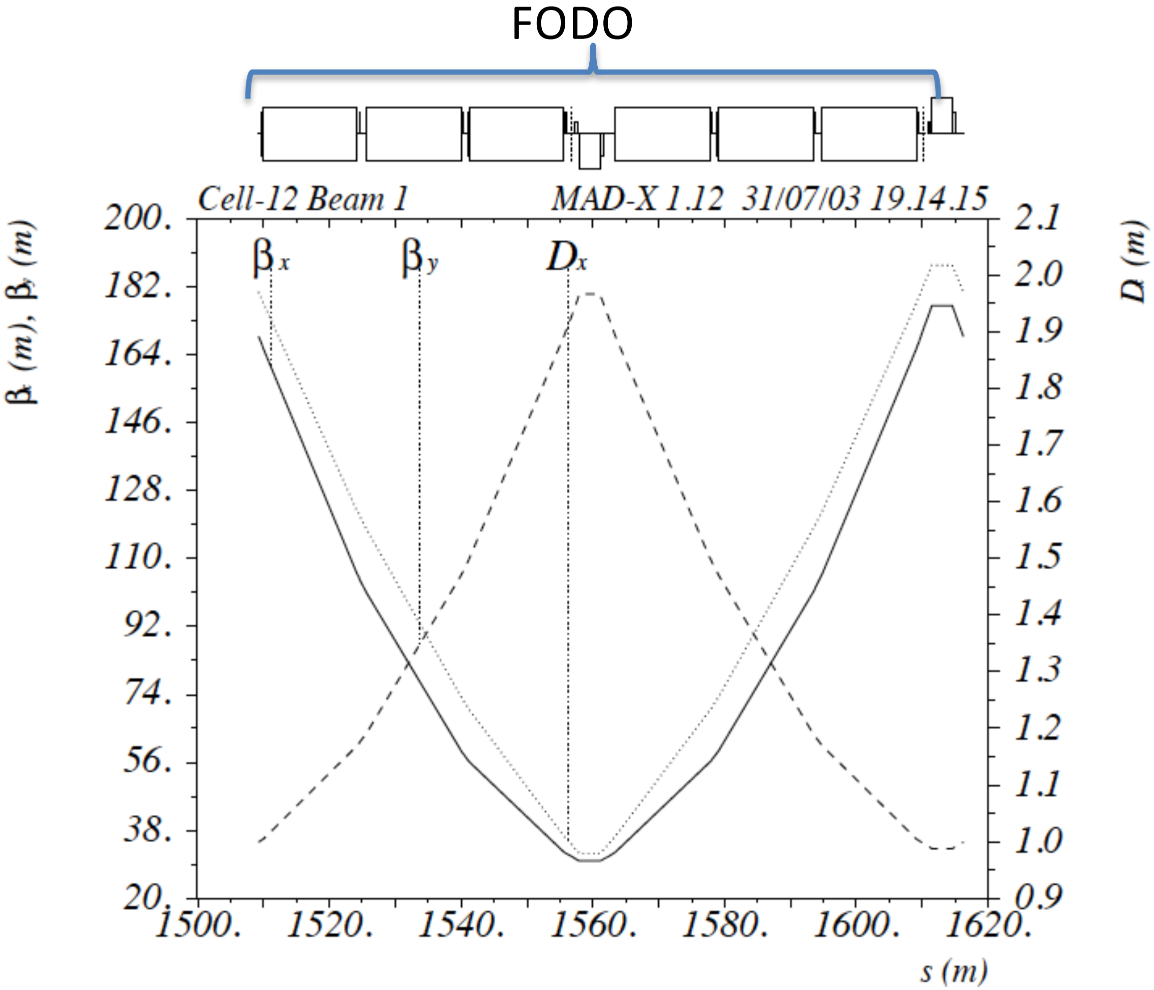
The conserved quantity is $\beta_1\gamma_1\epsilon_{x1} = \beta_0\gamma_0\epsilon_{x0}$

It is called *normalized emittance*.

Some numbers…

The LHC consists of 8 arcs. Each arc consists of 23(+2) FODO cells.

The regular FODO cell has the following characteristics:



Phase advance: 90°
Maximum beta: 180 m

$$\sigma = \sqrt{\varepsilon\beta}$$

The beam size changes along the cell!

Maximum horizontal beam size in the focusing quadrupoles

Maximum vertical beam size in the defocusing quadrupoles

Some numbers...

The emittance at LHC injection energy 450 GeV: ε = 7.3 nm

At 7 TeV: ε = 0.5 nm

$$\varepsilon_{7TeV} = \varepsilon_{450GeV}\frac{\gamma_{450GeV}}{\gamma_{7TeV}}$$

Normalized emittance: ε^* = 3.5 μm

Normalized emittance preserved during acceleration.

And for the beam sizes:

At the location with the maximum beta function (β_{max} = 180 m):

σ_{450GeV} = 1.1 mm

σ_{7TeV} = 300 μm

Aperture requirement: a > 10 σ

Vertical plane: 19 mm ~ 16 σ @ 450 GeV