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The Standard Model of particle physics

Michael Krämer (RWTH Aachen University)

Summary of 1st lecture: QED as a gauge theory

We start with the Lagrangian for a free Dirac field,

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi,$$

and observe that it is invariant under a phase transformation:

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$\mathcal{L} = \overline{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi$ is not invariant under local $U(1)$ transformations:

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} = \mathcal{L} \ + \overline{\psi}\gamma^{\mu}[\partial_{\mu}\omega(x)]\psi\,,$$

where we consider infinitesimal transformations

$$\psi \to \psi + \delta\psi = \psi - i\omega(x)\psi \quad \text{and} \quad \overline{\psi} \to \overline{\psi} + \delta\overline{\psi} = \overline{\psi} + i\omega(x)\overline{\psi}\,.$$

We can restore invariance under local $U(1)$ transformations if we introduce a vector field $A_\mu(x)$ with the interaction

$$-e\overline{\psi}\gamma^\mu A_\mu \psi \,,$$

so that the Lagrangian density becomes

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\psi \,.$$

The new Lagrangian is invariant under local $U(1)$ transformations if we require

$$A_\mu \to A_\mu + \delta A_\mu = A_\mu + \frac{1}{e}[\partial_\mu \omega(x)] \,.$$

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We need to add a Lorentz- and gauge invariant kinetic term for the field A_μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\psi\,,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

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$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

A mass term for the new field $\propto m_A^2 A_\mu A^\mu$ is not invariant under gauge transformations,

$$\delta\mathcal{L} = \frac{2m_A^2}{e} A^\mu \partial_\mu \omega(x) \neq 0 \,,$$

and thus not allowed.

It is useful to introduce the concept of a "covariant derivative" D_μ as

$$D_\mu = \partial_\mu + ieA_\mu \, .$$

With

$$\psi \to \psi + \delta\psi = \psi - i\omega(x)\psi \quad \text{and} \quad A_\mu \to A_\mu + \delta A_\mu = A_\mu + \frac{1}{e}[\partial_\mu \omega(x)]$$

one finds

$$D_\mu\psi \to D_\mu\psi + \delta(D_\mu\psi) = D_\mu\psi - i\omega(x)D_\mu\psi$$

so that

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

is gauge invariant.

One can express $F_{\mu\nu}$ in terms of the covariant derivative:

$$F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = \ldots = \partial_\mu A_\nu - \partial_\nu A_\mu \, .$$

Gauge transformations: Summary

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi$$

- The Dirac Lagrangian is invariant under local $U(1)$ transformations if we add a vector field A_μ and an interaction $-e\overline{\psi}\gamma^\mu A_\mu \psi$.
- The interaction is obtained by replacing the derivative ∂_μ with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.
- The gauge-invariant kinetic term for the vector field is $\propto F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} \propto [D_\mu, D_\nu]$.
- The new vector (gauge) field is massless, since a term $\propto A_\mu A^\mu$ is not gauge-invariant.
- The Lagrangian resulting from local $U(1)$ gauge-invariance is identical to that of QED.

Questions: Why does the term $\propto A^\mu A_\mu$ correspond to a mass?

Consider free scalar field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \,.$$

From the Euler-Lagrange equations, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$, we find

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \,.$$

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For fermions we consider the Dirac-Lagrangian,

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi\,.$$

From the Euler-Lagrange equations, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0$, we find

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0\,.$$

For vector particles we consider

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu \, .$$

From the Euler-Lagrange equations, $\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$, we find

$$\partial_\nu F^{\mu\nu} = m^2 A^\mu \, .$$

Thus

$$\partial_\mu \partial_\nu F^{\mu\nu} = m^2 \partial_\mu A^\mu \quad \Rightarrow \quad \partial_\mu A^\mu = 0$$

and

$$\partial_\nu F^{\mu\nu} = \partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = -\partial_\nu \partial^\nu A^\mu = m^2 A^\mu \, ,$$

i.e. each component of A^μ obeys the Klein-Gordon equation with mass m:

$$(\partial_\mu \partial^\mu + m^2) A^\mu = 0 \, .$$

The coefficients of the Lagrangian terms quadratic in the fields correspond to the particle masses.

Questions: What about the anomalous magnetic moment?

The gyromagnetic ratio g (g-factor) is defined through $\vec{\mu}_{\mathrm{m}} = g\mu_{\mathrm{B}}\vec{S}$.

The Dirac equation predicts $g = 2$, which is also the leading-order prediction of QED. The deviation from the Dirac (or leading-order QED) prediction is called anomalous magnetic moment: $a_l \equiv (g_l - 2)/2$, where $l = e, \mu$.

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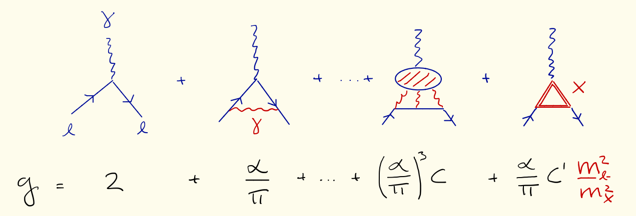
An anomalous magnetic moment is introduced by quantum fluctuations, including possibly new heavy particles:



$\rightarrow$ the anomalous magnetic moment of the muon is more sensitive to new physics than that of the electron by a factor $(m_\mu/m_e)^2 \approx 4 \times 10^4$.

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- Recall that the electromagnetic potentials are arbitrary up to "gauge transformations" of the form $A_\mu \to A_\mu + \delta A_\mu = A_\mu + (\partial_\mu \omega(x))/e$. Invariance of the QED Lagrangian with interactions, $\mathcal{L} \supset -e\overline{\psi}\gamma^\mu A_\mu \psi$, will be preserved if we simultaneously change the phase of the fermion field $\psi \to e^{-i\omega(x)}\psi, \quad \overline{\psi} \to e^{i\omega(x)}\overline{\psi}$.

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- At the quantum level, and considering theories with vector particles such as QED or the Standard Model, gauge invariance is crucial to reconcile unitarity and renormalizability.
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- At the quantum level, and considering theories with vector particles such as QED or the Standard Model, gauge invariance is crucial to reconcile unitarity and renormalizability.
- Gauge transformation are not physical in the sense that the symmetry transformation only changes our mathematical description but does not lead to a different physical situation. Thus some people prefer to talk about "gauge redundancy".
- Encouraged by the tremendous success of QED, and guided by the wish to construct renormalizable theories, we shall apply the idea of gauge invariance (i.e. local phase invariance) also to the other sectors of the Standard Model.

Outline

- QED as a gauge theory
- Quantum Chromodynamics
- Breaking gauge symmetries:

  the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism
- Exploring electroweak symmetry breaking at the LHC

Gauge transformations: non-abelian gauge groups

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We focus on the group $SU(n)$, i.e. the group of special unitary transformations.

To specify an $SU(n)$ matrix, we need $n^2 - 1$ real parameters, so we can write

$$e^{-i\omega^a T^a}$$

where the $\omega^a, a \in \{1, ..., n^2 - 1\}$ are real parameters, and the T^a are called generators of the group. [If you are unfamiliar with the concept of a group generator, you can think of the T^a as traceless, hermitian $n \times n$ matrices.]

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The crucial new feature is that the elements of $SU(n)$ do not commute,

$$e^{-i\omega_1^a T^a} e^{-i\omega_2^a T^a} \neq e^{-i\omega_2^a T^a} e^{-i\omega_1^a T^a} ,$$

because the generators do not commute:

$$[T^a, T^b] = if^{abc} T^c \neq 0 .$$

Recall that the $SU(n)$ transformations act on the fermion fields, so ψ carries an index i, with $i \in \{1, ..., n\}$:

$$\psi \rightarrow \left(e^{-i\omega^a T^a}\right)\psi \quad \text{or} \quad \psi_i \rightarrow \left(e^{-i\omega^a T^a}\right)_i^{\ j}\psi_j$$

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Considering infinitesimal transformations

$$\delta\psi_i = -i\omega^a (T^a)_i^j \psi_j \quad \text{and} \quad \delta\overline{\psi}^i = i\omega^a \overline{\psi}^j (T^a)_j^i$$

one finds that the Lagrangian is not invariant under local $SU(n)$ transformations:

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$$\delta\mathcal{L} = \overline{\psi}^i (T^a)_i^j \gamma^\mu (\partial_\mu \omega^a(x))\psi_j \, .$$

We can restore local $SU(n)$ gauge-invariance by introducing $n^2 - 1$ new vector particles A_μ^a, one for each generator of the group.

They should transform as

$$\delta A_\mu^a(x) = -f^{abc} A_\mu^b(x)\omega^c(x) + \frac{1}{g}\left[\partial_\mu \omega^a(x)\right] \, .$$

The interaction of the new vector particles and the fermions is obtained by replacing the ordinary derivative with the covariant derivative

$$D_\mu = (\partial_\mu + igT^a A^a_\mu)\,.$$

Note that in this case D_μ is a $n \times n$ matrix.

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The $SU(n)$ invariant Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \overline{\psi}^i \, (i\gamma^\mu D_\mu - m)_i^j \, \psi_j$$

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The field strength tensor is constructed from

$$F^{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$$

with $F^{\mu\nu} = T^a F^a_{\mu\nu}$.

This gives $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu$.

Non-abelian gauge transformations: Summary

- A non-abelian gauge theory ("Yang-Mills theory") is a theory in which the Lagrangian is invariant under local transformations of a non-abelian group.
- This invariance is achieved by introducing a gauge boson, A^a_μ, for each generator of the group. The interaction between the gauge bosons and the fermions is obtained by replacing the partial derivative ∂_μ with the covariant derivative $D_\mu = (\partial_\mu + igT^a A^a_\mu)$.
- The kinetic term for the vector field is $\propto F^a_{\mu\nu} F^{a\,\mu\nu}$, where $F^a_{\mu\nu}$ is constructed from the commutator of the covariant derivative.
- $F^a_{\mu\nu} F^{a\,\mu\nu}$ contains terms which are cubic and quartic in the gauge boson fields, indicating that the gauge bosons interact with each other.
- The gauge bosons are massless, since a term $\propto A^a_\mu A^{a\,\mu}$ is not invariant under local gauge transformations.

QCD as an SU(3) gauge theory

We start with the Dirac Lagrangian for a free quark q,

$$\mathcal{L} = \bar{q}\,(i\gamma^\mu\partial_\mu - m)\,q\,,$$

and require invariance under local $SU(3)$ transformations.

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Recall that $SU(3)$ is the group of special unitary transformations, i.e. the group of all 3×3 unitary matrices with determinant one. To specify an $SU(3)$ transformation, one needs $3^2 - 1 = 8$ real parameters, so we can write

$$e^{-i\omega^a T^a}$$

where the $\omega^a, a \in \{1, ..., 8\}$ are real parameters, and the T^a are the generators of the group.

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$SU(3)$ is a non-abelian group, i.e. its generators and thus its elements do not commute,

$$[T^a, T^b] = if^{abc}T^c \neq 0 \quad \text{and} \quad e^{-i\omega_1^a T^a} e^{-i\omega_2^b T^b} \neq e^{-i\omega_2^b T^b} e^{-i\omega_1^a T^a}\,.$$

The $SU(3)$ transformations act on the quark fields, so q carries an index i, with $i \in \{1, ..., 3\}$:

$$q \to \left(e^{-i\omega^a T^a}\right) q \quad \text{or} \quad q_i \to \left(e^{-i\omega^a T^a}\right)_i^{\ j} q_j$$

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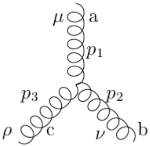
with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu$.

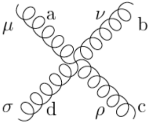
Because of the gauge symmetry, the gluon is massless.

The interactions of QCD follow from gauge invariance:

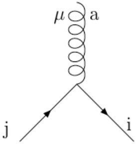
$$\mathcal{L}_{\text{interaction}} = g\, A^a_\mu\, \bar{q}\gamma^\mu T^a q - g\, f^{abc}(\partial_\mu A^a_\nu) A^{b\,\mu} A^{c\,\nu} - g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu} :$$



$$-g\, f^{abc}\left(g_{\mu\nu}\,(p_1-p_2)_\rho + g_{\nu\rho}\,(p_2-p_3)_\mu + g_{\rho\mu}\,(p_3-p_1)_\nu\right)$$



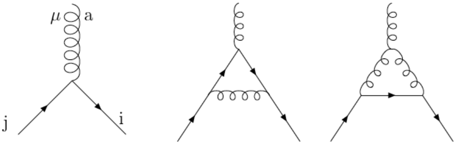
$$-i\,g^2 f^{eab} f^{ecd}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$-i\,g^2 f^{eac} f^{ebd}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$-i\,g^2 f^{ead} f^{ebc}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})$$



$$-i\,g\,\gamma^\mu\,(T^a)_{ij}$$

Consider a dimensionless physical observable R, e.g. the ratio of two cross sections, evaluated at some large energy scale Q. If $Q \gg m$, one can set $m \to 0$, and dimensional analysis implies that R should be independent of Q.

This is not true in quantum field theory. Quantum fluctuations change the value of the effective coupling:



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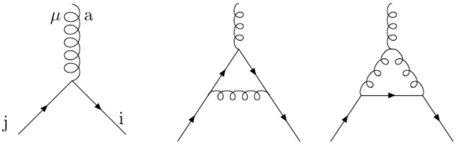
Consider a dimensionless physical observable R, e.g. the ratio of two cross sections, evaluated at some large energy scale Q. If $Q \gg m$, one can set $m \to 0$, and dimensional analysis implies that R should be independent of Q.

This is not true in quantum field theory. Quantum fluctuations change the value of the effective coupling:



The calculation of R as a perturbation series in the coupling $\alpha_s \equiv g/4\pi$ requires renormalization to remove ultraviolet contributions. This introduces a second mass scale μ – the point at which the UV contributions are subtracted. Thus

$$R = R(Q^2/\mu^2, \alpha_s(\mu^2)) \, .$$

However, a physical observable must not depend on the scale μ, i.e.

$$\mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, \alpha_s(\mu^2)) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] R = 0\,.$$

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Introducing

$$t = \ln\left(\frac{Q^2}{\mu^2}\right), \quad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

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The β-function has a perturbative expansion and can be extracted from an explicit calculation of higher-order loop-corrections to propagators and vertices.

The running of the coupling at one-loop is thus determined from

$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2)) \quad \text{and} \quad \beta(\alpha_s) = -b\alpha_s^2$$

which yields

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b \ln(Q^2/\mu^2)} \quad \text{with} \quad b = \frac{33 - 2n_f}{12\pi} \, .$$

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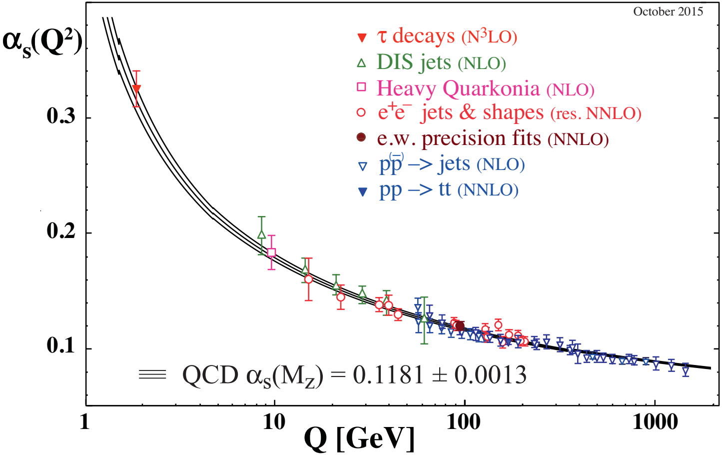
Note that in QED one finds $b = -1/3$ so that the QED coupling

$$\alpha_{\rm QED}(Q^2) = \frac{\alpha_{\rm QED}(\mu^2)}{1 - \frac{\alpha_{\rm QED}(\mu^2)}{3\pi} \ln(Q^2/\mu^2)}$$

increases with increasing Q^2.

The running QCD coupling

Dissertori, PDG



non-perturbative $\longleftrightarrow$ perturbative