INCLUSIVE $\tau$ DETERMINATION(S) OF $|V_{us}|$

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with


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• Systematic issues in the conventional BF-only FB FESR implementation (and the related inclusive $\tau V_{us}$ puzzle)

• An alternate FB FESR implementation solution

• A new, lattice-inclusive $us V+A \tau$ approach (advantages, results, and future prospects)
• **Non-τ vs inclusive τ determinations**

| $|V_{us}|$       | Source                                                                 |
|----------------|----------------------------------------------------------------------|
| 0.2257(9)(?)   | 3-family unitarity, HT16 $|V_{ud}|$                                                                           |
| 0.2233(4)_{exp}(4)_{latt} | $K_{\ell 3}$, 2+1+1 lattice $f_+(0)$                                      |
| 0.2254(3)_{exp}(8)_{latt} | $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, lattice $f_K/f_\pi$            |

c.f. “conventional” (BFs-only) inclusive τ FB FESR analysis result: **0.2195(19)** (Lusiani, CKM 2018)

• **Interesting (if true) given hints of lepton universality breakdown in $R[D^{(*)}]$** [e.g. Nature 546 (2017) 227]
HADRONIC $\tau$ DECAY, FB FESR BASICS

• $R_{ij;V/A} \equiv \Gamma[\tau \to \nu_\tau\text{hadrons}_{ij;V/A}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^-\bar{\nu}_e(\gamma)]$

• With $y_\tau \equiv s/m_\tau^2$, flavor $ij$ decays in SM [Tsai PRD4 (1971) 2821]

$$\frac{dR_{ij;V+A}}{ds} = \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \left[1 - y_\tau\right]^2 \tilde{\rho}_{ij;V+A}(s)$$

$$\tilde{\rho}_{ij;V+A}(s) \equiv \left[(1 + 2y_\tau) \rho_{ij;V+A}^{(J=1)}(s) + \rho_{ij;V+A}^{(J=0)}(s)\right]$$

kinematic weight: $w_\tau(y) = (1 - y)^2(1 + 2y)$
• FESRs for $\Pi = \Pi_{ud-us; V+A}^{(J=0+1)}(Q^2)$, $\rho = \rho_{ud-us; V+A}^{(J=0+1)}(s)$

$$\int_{s_{th}}^{s_0} ds w(s)\rho(s) = \frac{-1}{2\pi i} \oint_{|s|=s_0} ds w(s)\Pi(s)$$

with $|V_{i,j}|^2 \rho_{i,j; V/A}^{(0+1)}(s)$ from $dR_{i,j; V/A}/ds$

• $R_{i,j; V/A}^{w}(s_0)$: re-weighted $R_{i,j; V/A}^{(0+1)}$ analogue

$$R_{i,j; V/A}^{w}(s_0) \sim \int_{s_{th}}^{s_0} ds \frac{dR_{i,j; V/A}^{(0+1)}}{ds} \frac{w(s/s_0)}{w_T(s/m_T^2)}$$

• FB differences $\delta R^{w}(s_0) \equiv \frac{R_{ud; V+A}^{w}(s_0)}{|V_{ud}|^2} - \frac{R_{us; V+A}^{w}(s_0)}{|V_{us}|^2}$
• FESR, OPE for $\delta R^w(s_0)$, input $|V_{ud}| \Rightarrow$

\[
|V_{us}| = \frac{R^w_{us;V+A}(s_0)}{\sqrt{\frac{R^w_{ud;V+A}(s_0)}{|V_{ud}|^2} - [\delta R^w(s_0)]^{OPE}}}
\]

Self-consistency: $|V_{us}|$ independent of $s_0$, $w$

• The conventional implementation [Gamiz et al. JHEP03(2003)060]

  ○ $s_0 = m_T^2$, $w = w_\tau$ only [spectral integrals from inclusive $ud$, $us$ BF\textsc{s}, but no self-consistency tests]

  ○ $w_\tau$ degree 3 $\Rightarrow$ OPE to $D = 8$

  ○ Assumptions needed for $D = 6$ (VSA), $D = 8$ ($\sim 0$)
• Variable-$s_0$, -$w$ FB FESR tests with same $D = 6, 8$ assumptions show [KM et al. PLB781 (2018) 206]

  ○ strong unphysical $s_0$ dependence

  ○ strong unphysical $w$ dependence even at $s_0 = m_\tau^2$

  ○ apparent convergence for all $w$ beyond $s_0 = m_\tau^2$

• Results ⇒

  ○ Sizeable theoretical systematic in conventional (BF-only) implementation (more below)

  ○ Use of conventional implementation (unfortunately) must be abandoned going forward
Conventional implementation $|V_{us}|$ from $w_\tau$, $\hat{w}$ FESRs

\[ w_\tau(y) = 1 - 3y^2 + 2y^3, \quad \hat{w}(y) = 1 - 3y + 3y^2 - y^3 \]

D=6 contributions identical (up to sign)
D=8 contributions differ by a factor of $-1/2$
Correlated differences, $|V_{us}|$ from conventional implementation $w_\tau$, $\tilde{w}$ FESRs

Differences should be zero within errors if D=6,8 assumptions are reliable

$$w_\tau(y) = 1 - 3y^2 + 2y^3, \quad \tilde{w}(y) = 1 - 3y + 3y^2 - y^3$$
AN ALTERNATE FB FESR IMPLEMENTATION

- Theory side
  - No $D > 4$ assumptions: effective condensates $C_{D>4}$ from fits to data (N.B. requires variable $s_0$)
  - 3-loop-truncated FOPT $D = 2$, standard $D = 2 + 4$ error estimates [from comparison to lattice data]
  - $C_{2N+2}$ ($D = 2N + 2$ condensate), $|V_{us}|$ both from
    $w_N(y) = 1 - \frac{y}{N-1} + \frac{y^N}{N-1}$ FESR ($y = s/s_0$)
  - $|V_{us}|$ from different $w_N$ as self-consistency check
- Experimental input (also for new lattice approach)

  - Updated/corrected 2013 ALEPH for $ud$ V+A
  - $us$ V+A from sum over exclusive modes
    * $K$ from $K_{\mu 2}$ or $B[\tau \to K\nu_{\tau}]$
    * $K\pi$, $K^-\pi^+\pi^-$, $\bar{K}^0\pi^-\pi^0$: BaBar, Belle unit-normalized distributions
      [Note: Normalized using BF from HFLAV 2017 combined with near-completed BaBar Adametz thesis $B[K^-n\pi^0]$ results (Lusiani CKM 2018)]
    * Remaining ("residual modes"): 1999 ALEPH, with exclusive-mode, Lusiani CKM 2018 BF rescalings ($\sim 25\%$ errors, some MC)
NEW FB FESR RESULTS

• With fitted $D > 4$ condensates
  - unphysical $s_0$, $w(y)$-dependence problems resolved
  - $|V_{us}|$ increased by $\sim 0.0020$: $0.2195(19)(?) \rightarrow 0.2212(23)$ ($0.2219(22)$ if use $K_{\mu2}$+SM for $K$)

• Favorable ($\sim 0.0005$) theory error situation

• $us$ spectral integral uncertainty strongly dominant

• Sizeable 25% residual mode error $\Rightarrow$ significant near-term $us$ error improvement highly unlikely
Conventional implementation $w_\tau, w_2, w_3, w_4$ results
Alternate vs conventional implementation results

\[ s_0 [\text{GeV}^2] \]
A LATTICE $\tau$-BASED ALTERNATIVE


- Basic idea: generalized dispersion relations for products of combination $\tilde{\Pi}$ of $J = 0, 1$ $us$ $V+A$ polarizations with weights having poles at Euclidean $Q^2$
  - $\tilde{\Pi}(Q^2)$: polarization sum with spectral function $\tilde{\rho}(s)$ (experimental $dR_{us;V+A}/ds$)
  - Theory: Lattice $us$ 2-point function data (no OPE)
  - Weights tunable, allow suppression of larger-error, higher-multiplicity $us$ spectral contributions
More on the lattice-inclusive \( us \ \tau \) approach

- \( |V_{us}|^2 \tilde{\rho}_{us;V+A}(s) \) from experimental \( dR_{us;V+A}/ds \)

\[
\tilde{\rho}_{us;V+A}(s) \equiv \left(1 + 2\frac{s}{m_\tau^2}\right) \rho^{(J=1)}_{us;V+A}(s) + \rho^{(J=0)}_{us;V+A}(s)
\]

(no continuum \( us \) \( J = 0 \) subtraction required)

- Associated (kinematic-singularity-free) polarization

\[
\tilde{\Pi}_{us;V+A}(Q^2) \equiv \left(1 - 2\frac{Q^2}{m_\tau^2}\right) \Pi^{(J=1)}_{us;V+A}(Q^2) + \Pi^{(J=0)}_{us;V+A}(Q^2)
\]

- \( \tilde{\rho}_{us;V+A}(s) \sim s \) as \( s \to \infty \)
• For weights \( w_N(s) \equiv \frac{1}{\prod_{k=1}^{N}(s+Q_k^2)} \), \( N \geq 3 \), obtain convergent, unsubtracted 'dispersion relation'

\[
\int_{th}^{\infty} ds \ w_N(s) \tilde{\rho}_{us;V+A}(s) = \sum_{k=1}^{N} \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j \neq k}(Q_j^2 - Q_k^2)} \equiv \tilde{F}_{w_N}
\]

- Lattice data for \( \tilde{\Pi}_{us;V+A}(Q_k^2) \) on RHS
- LHS from experimental \( dR_{us;V+A}/ds \), up to \(|V_{us}|^2\)
- \( w_N(s) \): rapid fall-off if all \( Q_k^2 < 1 \ \text{GeV}^2 \)
  \( \Rightarrow K, K\pi \) dominate LHS, near-endpoint multi-particle, \( s > m_T^2 \) contributions strongly suppressed
- Optimization: increasing \( \{Q_k^2\} \) decreases RHS lattice error, increases LHS experimental error
• A few details:

  ○ Near-physical-point RBC/UKQCD $n_f = 2 + 1$ DWF ensembles

    * $48^3 \times 96, \ 1/a = 1.73 \text{ GeV}, \ m_\pi = 0.139 \text{ GeV}, \ m_K = 0.499 \text{ GeV}$

    * $64^3 \times 128, \ 1/a = 2.36 \text{ GeV}, \ m_\pi = 0.139 \text{ GeV}, \ m_K = 0.508 \text{ GeV}$

    * (Small) retuning to physical valence masses [PQ]

  ○ Time-momentum representation for $\tilde{\Pi}_{us;V + A(Q_k^2)}$ entering sum in $\tilde{F}_{wN}$
Weighted spectral integrals (include $|V_{us}|^2$ factor)

\[ \tilde{R}_{w_N} \equiv \int_0^{m_T^2} \frac{m_T^2}{12\pi^2 S_{EW} (1 - y_T)^2} \frac{dR_{us;V + A(s)}}{ds} w_N(s) ds \]

\[ \tilde{F}^{pQCD}_{w_N} \equiv \int_{m_T^2}^{\infty} \tilde{\rho}_{us}(s)^{pQCD} w_N(s) ds \]

\[ |V_{us}| = \sqrt{\tilde{R}_{us;w_N} / [\tilde{F}_{w_N} - \tilde{F}^{pQCD}_{w_N}]} \]

Expect continuum spectral $A, J = 0$ channel contributions negligible for $w_N$ employed (confirmed by lattice data), hence exclusive $A, J = 0$ analysis also possible (using only $K$ spectral contribution)

$w_N$ below: uniform pole spacing $\Delta$, centroid $C$
Reweighting of exclusive mode distributions

Left panel: un-re-weighted experimental data; right panel: \( N = 4, C = 0.5 \text{ GeV}^2 \) re-weighted version
Inclusive determinations

Inclusive $|V_{us}|$ (Lusiani CKM2018 BFs, including $B_K$)
INCLUSIVE ANALYSIS RESULTS SUMMARY

• Error optimized for \( N = 4, \ C \simeq 0.7 \ \text{GeV}^2 \)

• Optimized \(|V_{us}|\) results:
  
  ○ With \( B[\tau \to K\nu_\tau] \) for \( K \):
    \[ 0.2240 \ (13)_{\text{exp}} \ (13)_{\text{th}} \]

  ○ With \( \Gamma[K_{\mu2}] + \text{SM} \) for \( K \):
    \[ 0.2254 \ (10)_{\text{exp}} \ (13)_{\text{th}} \]

• Residual mode sum error becoming relevant (\( K^-\pi^0\pi^0 \) distribution desirable)
Comparison to results from other methods

|V_{us}| 0.22 0.225 0.23 0.235 0.24

- $K_{13}$, PDG 2016
- $\Gamma[K_{\mu2}]$
- 3-family unitarity, HT14 $|V_{ud}|$
- $\tau$ FB FESR, CKM2018 input (problematic conventional implementation)
- $\tau$ FB FESR, CKM2018 input (new HLMZ17 implementation)
- $\tau$, lattice [N=3, C=0.3 GeV$^2$]
- $\tau$, lattice [N=4, C=0.7 GeV$^2$]
- $\tau$, lattice [N=5, C=0.9 GeV$^2$]

this work
Comments/Prospects

- Lattice analysis confirms larger $|V_{us}|$ from $\tau$ decay data [as per alternate HMLZ FB FESR implementation]

- Significant error reduction from lattice approach c.f. FB FESR determination employing same data

- Theory uncertainty under better control for lattice than for OPE

- Improved suppression of high-$s$, higher-error spectral contributions in lattice approach without blowing up theory errors
• Lattice thus superior to FB FESR approach and should replace it going forward

• Trend to $|V_{us}|$ fall-off for $N = 3$, larger $C$ compatible with missing high-$s$, higher-multiplicity spectral strength (larger impact on FB FESR than lattice results)

• Theory (lattice) errors straightforwardly reducible through improved statistics

• Significant experimental error reduction from improved $K, K\pi \tau$ BFs even without unit-normalized distribution improvements (e.g., Belle II)
SUMMARY

- Old $3 \sigma$ low inclusive FB $\tau$ FESR $|V_{us}|$ problem resolved
  - Conventional FB FESR implementation using only inclusive BFs no longer tenable
  - Alternate, no-assumptions implementation: $|V_{us}|$ higher by $\sim 0.0020$, compatible with other determinations
  - Near-term improvements via $u_s$ exclusive BFs
  - Highly favorable theoretical error situation
  - However, competitive $|V_{us}|$ needs improvements to old ALEPH higher-multiplicity, low-statistics data
Advantage of new lattice-inclusive $us \ V + A \ \tau$ approach

- **Theory:**
  * Lattice in place of OPE; no $us \ J = 0$ subtraction; improvement through increased statistics
  * *Parasitic on lattice $a_\mu$ effort (a major effort in the lattice community)*

- **Spectral integrals:**
  * Theory errors still small for weights strongly suppressing higher multiplicity contributions
  * Strong $K, K\pi$ dominance of spectral integral
  * Experimental improvements currently possible through improved $K$ BF, $K\pi$ BF, distributions
BACKUP SLIDES
- Relative exclusive mode $R_{us;V+A}^w$ contributions

<table>
<thead>
<tr>
<th>Wt</th>
<th>$s_0$ [GeV$^2$]</th>
<th>$K$</th>
<th>$K\pi$</th>
<th>$K\pi\pi$</th>
<th>Other (B-factory)</th>
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<td>$w_2$</td>
<td>2.15</td>
<td>0.496</td>
<td>0.426</td>
<td>0.062</td>
<td>0.010</td>
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<tr>
<td></td>
<td>3.15</td>
<td>0.360</td>
<td>0.414</td>
<td>0.162</td>
<td>0.065</td>
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<tr>
<td>$w_3$</td>
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<td>0.446</td>
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<tr>
<td></td>
<td>3.15</td>
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<td>0.415</td>
<td>0.182</td>
<td>0.074</td>
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<tr>
<td>$w_4$</td>
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<tr>
<td></td>
<td>3.15</td>
<td>0.314</td>
<td>0.411</td>
<td>0.194</td>
<td>0.081</td>
</tr>
</tbody>
</table>

- "Other": 1999 ALEPH data/MC, $\sim$ 25% error

- $\Rightarrow$ sub-0.5% $|V_{us}|$ (sub-% $R_{us;V+A}^w$ error) requires experimentally (much) more challenging higher-multiplicity mode improvements
LHS: ACLP2013 $K\pi$ normalization; RHS: HFAG2016 $K\pi$ normalization
Lattice vs OPE results for $\Delta \Pi_T(Q^2)$

- Lattice data
- Central 3-loop $D=2 + D=4$ OPE
- 3-loop $D=2 + D=4$ OPE $\pm 1\sigma$
- 3-loop $D=2 + D=4 + VSA D=6$ OPE
Finite \( t \) behavior

- Current-current two-point function
  \[
  C_{V/A}^{\mu\nu}(t) = \sum_{\vec{x}} \langle J_{V/A}^\nu(\vec{x}, t)(J_{V/A}^\mu(0, 0))^\dagger \rangle
  \]

- \( J = 0 \), 1 components: \( C_{V/A}^{(1)}(t) = \frac{1}{3} \sum_{k=x,y,z} C_{V/A}^{kk}(t) \),
  \[
  C_{V/A}^{(0)}(t) = C_{V/A}^{tt}(t)
  \]

- \( Q^2 = 0 \)-subtracted \( J = 0 \), 1 polarizations

  \[
  \Pi_{V/A}^{(J)}(Q^2) - \Pi_{V/A}^{(J)}(0) = \sum_t K(Q, t) C_{V/A}^{(J)}(t)
  \]
  
  \[
  K(Q, t) = \frac{\cos \hat{Q}t - 1}{\hat{Q}^2} + \frac{1}{2} t^2
  \]
Resulting $J = 0, 1, V, A$ contributions to $\tilde{F}_{w_N}$

\[ \tilde{F}^{(J)}_{V/A; w_N} = \lim_{t \to \infty} L^{(J)}_{V/A; w_N}(t) \]

where

\[ L^{(J)}_{V/A; \omega_N}(t) = \sum_{l=-t}^{t} w^{(J)}_{N}(l) C^{(J)}_{V/A}(l) \]

with

\[ w^{(1)}_{N} = \sum_{k=1}^{N} K \left( \sqrt{Q_k^2}, t \right) \left( 1 - \frac{2Q_k^2}{m_T^2} \right) \text{Res} [w_N(s)]_{s=-Q_k^2} \]

\[ w^{(0)}_{N} = \sum_{k=1}^{N} K \left( \sqrt{Q_k^2}, t \right) \text{Res} [w_N(s)]_{s=-Q_k^2} \]
Large $t$ convergence behavior

48$^3 \times 96$ lattice data, $N=4$, $C=0.5$ [GeV$^2$]
Lattice error breakdown vs. $C$, $N = 4$
Error budget (updated BF at Charm 2018 by A. Lusiani)

<table>
<thead>
<tr>
<th>contribution</th>
<th>[N, C[GeV$^2$]]</th>
<th>3, 0.3</th>
<th>3, 1</th>
<th>4, 0.7</th>
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<td>0.20</td>
<td>0.34</td>
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<td>others, stat.</td>
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<td>0.34</td>
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<td>0.11</td>
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<td>0.05</td>
<td>0.26</td>
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<td>0.01</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>0.59</td>
<td>0.91</td>
<td>0.58</td>
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<tr>
<td>experiment</td>
<td>$K$</td>
<td>0.39</td>
<td>0.22</td>
<td>0.36</td>
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<td>$K\pi$</td>
<td></td>
<td>0.16</td>
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<td>$K^-\pi^+\pi^-$</td>
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<td>$\bar{K}^0\pi^-\pi^0$</td>
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<td>residual</td>
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<td>total</td>
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<td>1.39</td>
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<td>0.51</td>
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<tr>
<td>Combined</td>
<td>total</td>
<td>0.83</td>
<td>1.67</td>
<td>0.82</td>
<td>0.83</td>
</tr>
</tbody>
</table>

- Exp error reduces about 10%. Now theory and experimental errors are same for the preferred $N = 4$ value
Exclusive $A$, $J = 0$ channel determination