Higher-order QCD in hadronic tau decays from Padé approximants

Diogo Boito, *IFSC-Universidade de São Paulo, Brazil*
Pere Masjuan, *IFAE-Universitat Autònoma de Barcelona, Spain*
Fabio Oliani, *IFSC-Universidade de São Paulo, Brazil*

Abstract

We use the method of Padé approximants to obtain predictions for the higher order QCD corrections to the inclusive hadronic tau decay width. Our predictions are robust and model independent.

Hadronic tau decays

- The tau hadronic width can be parametrized as

pt. theory

$$R_{\tau,V/A} = \frac{N_c}{2} S_{\rm EW} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta_{\rm NP} + \delta_{\rm EW}\right]$$

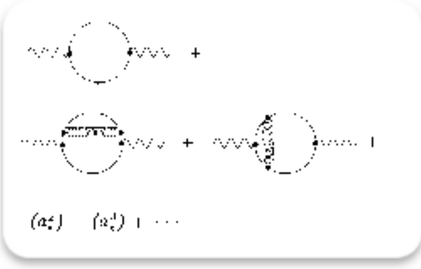
- The perturbative contribution is obtained, using analyticity, as a contour integral where the *Adler function* intervenes

$$\delta^{(0)} = \frac{1}{2\pi i}\oint_{|x|=1} \frac{dx}{x} W(x) \widehat{D}^{(1+0)}_{\rm pert}(m_\tau^2 x), \longrightarrow \widehat{D}_{\rm pert}(s) = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} \ln^{k-1}\left(\frac{-s}{\mu^2}\right)$$

Adler function

- The Adler function is known to five loops [1]

$$\widehat{D}(Q^2) = a_Q + 1.640\, a_Q^2 + 6.371\, a_Q^3 + 49.08\, a_Q^4 + \cdots$$



- There is, however, a remaining ambiguity in $\delta^{(0)}$

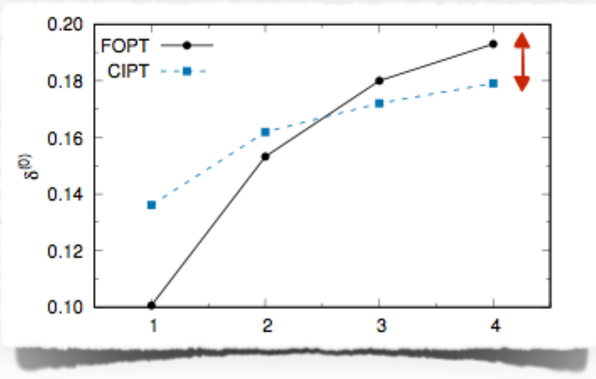
fixed renorm. scale

$$\alpha_s^1 \qquad \alpha_s^2 \qquad \alpha_s^3 \qquad \alpha_s^4$$

$$\delta^{(0)}_{\rm FO} = 0.1006 + 0.0526 + 0.0268 + 0.0130 = 0.1931$$

$$\delta^{(0)}_{\rm CI} = 0.1361 + 0.0258 + 0.0102 + 0.0071 = 0.1791$$

running renorm. scale



Ambiguity related to the (lack of) knowledge about higher orders

- But perturbation theory is only asymptotic (factorially divergent)and it is useful to work with the Borel transformed series defined as[4]

Borel sum

$$B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} \longrightarrow \widehat{D}(\alpha) \equiv \int_0^\infty dt e^{-t/\alpha} B[\widehat{D}](t)$$

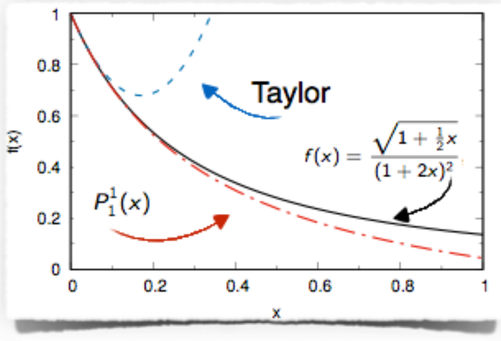
Padé approximants

- Padé approximants are ratios of two polynomials constructed to approximate a function $f(z)$ whose Taylor expansion is known

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \longrightarrow P_N^M(z) = \frac{Q_M(z)}{R_N(z)} \approx f_0 + f_1 z + f_2 z^2 + \cdots + f_{M+N} z^{N+M} + \mathcal{O}\left(z^{M+N+1}\right)$$

- efficient approximation ✓
- partial reconstruction of analytic properties ✓
- good predictions of higher orders ✓
- model independent results ✓

$$P_N^M(x) = \frac{Q_M(x)}{R_N(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_M x^M}{1 + b_1 x + b_2 x^2 + \cdots + b_N x^N}$$



D-log Padé approximants

- It is more costly (in terms of coefficients) to reproduce functions with cuts on the complex plane. *Strategy* → D-log Padé approximants

$$f(x) = \frac{A(x)}{(\mu - x)^\gamma} + B(x) \xrightarrow[\text{by Padés}]{\text{approximate } F(x)} F(z) = \frac{\rm d}{{\rm d}z}\log[f(z)] \sim \frac{\gamma}{\mu - z}$$

One can then obtain a new approximation to the original function [2]

$$\mathrm{Dlog}_N^M(z) = f(0) e^{\int dz \frac{Q_M(z)}{R_N(z)}} \approx f_0 + f_1 z + f_2 z^2 + \cdots$$

Partial approximants: constructed imposing the (perhaps partial) knowledge of the analytical structure of the function: poles etc… → $\mathbb{P}_{N,K}^M(z) = \frac{Q_M(z)}{R_N(z) T_K(z)}$

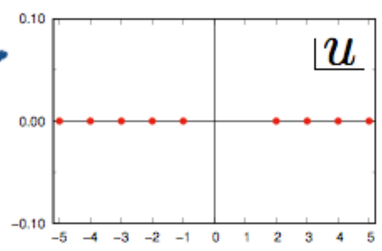
fixed polynomial [3]

Testing the method: large-β_0 limit

- The results are known to *all orders* in the large-β_0 limit of QCD.

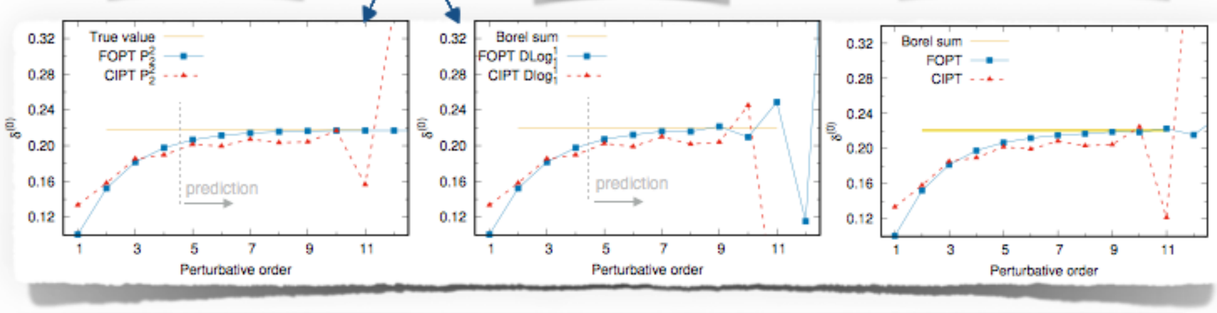
$$B[\widehat{D}](u) = \frac{32}{3\pi}\frac{e^{(C+5/3)u}}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$

renormalon singularities [4]



$$\delta^{(0)}_{{\rm FO},L\beta}(a_Q) = a_Q + 5.119\, a_Q^2 + 28.78\, a_Q^3 + 156.7\, a_Q^4 + 900.8\, a_Q^5 + 4867\, a_Q^6 \cdots$$

$P_2^2(\alpha_s)$ — Reconstruction with Padé approx. — $\mathrm{Dlog}_1^1(u)$ [5] — large-β_0 (exact)



- excellent reproduction of exact results up to order ~10 ✓
- results are equally good using Borel transform or the series in the coupling ✓
- completely model independent ✓

Results in QCD

- Having successfully tested the method, we can apply it to QCD [5]

Pade app. to the pt series

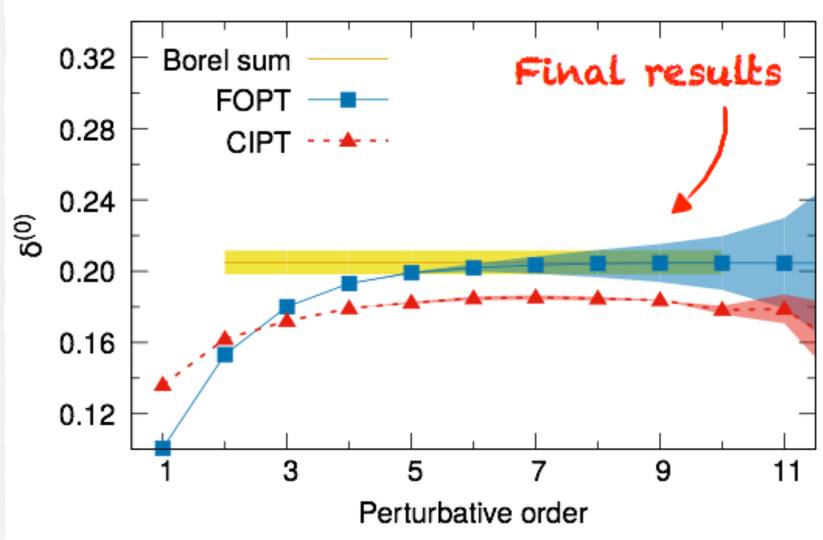
	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Padé sum
P_1^2	55.62	276.2	3865	1.952×10^4	4.288×10^5	1.289×10^6	0.2080
P_2^1	55.53	276.5	3855	1.959×10^4	4.272×10^5	1.307×10^6	0.2079
P_1^3	input	304.7	3171	2.442×10^4	3.149×10^5	2.633×10^6	0.2053
P_3^1	input	301.3	3189	2.391×10^4	3.193×10^5	2.521×10^6	0.2051

D-log Pade app. to the Borel transformed series

	$c_{4,1}$	$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$	$c_{9,1}$	Borel sum
DLog_0^1	51.90	272.6	3530	1.939×10^4	3.816×10^5	1.439×10^6	0.2050
DLog_1^0	52.08	273.7	3548	1.953×10^4	3.840×10^5	1.456×10^6	0.2052
DLog_0^2	input	254.1	3243	1.725×10^4	3.447×10^5	1.187×10^6	0.2012
DLog_2^0	input	256.4	3271	1.769×10^4	3.493×10^5	1.258×10^6	0.2019

Final values

$c_{5,1}$	$c_{6,1}$	$c_{7,1}$	$c_{8,1}$
277 ± 51	3460 ± 690	$(2.02 \pm 0.72) \times 10^4$	$(3.7 \pm 1.1) \times 10^5$



- model independent
- FOPT is favoured
- small errors
- reliable up to order ~11

$$c_{51} = 277 \pm 51$$

[5]

Main references and acknowledgements

[1] Baikov, Chetyrkin and Kühn, Phys. Rev. Lett. **101** (2008)
[2] Baker, *Essentials of Padé approximants*, (1975)
[3] Masjuan and Peris, JHEP **0705** 040 (2007)
[4] Beneke, Phys.Rept. **317** (1999)
[5] Boito, Masjuan, Oliani, JHEP **1808** 075 (2018).

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