Transverse phase-space evolution in MICE Step IV

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Challenges with the 2016/04-05 data

Due to the loss of M1D and the absence of M2D in the 2016/04-05 data:

- → Low transmission through the cooling channel
- → Beam nonlinearities, particularly in TKD

Dealing with low transmission:

- → Select a **narrow beam** upstream that is efficiently transported, good option to study the material but unlikely to yield cooling
- → Focus on the **core density** increase by means of fractional emittance of phase-space volume estimation rather than 4D RMS emittance

Dealing with nonlinearites:

- → Focus on the **linear core** rather than include the tails
- → Use **non-parametric density estimation** techniques, calculate the volume of phase-space probability contours

Categories of phase-space presented

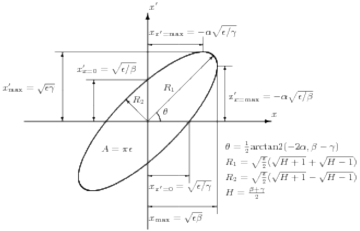
- **Toy Monte Carlo**
  - Takes any input distribution (x, y, p_x, p_y, p_z) (see **A**)
  - Deterministic Bethe-Bloch energy loss for given Toy absorber
  - Gaussian scattering $\mathcal{N}(0, \theta_0)$, $\theta_0 = \frac{13.6}{pc\beta}\sqrt{x/X_0}(1 + 0.038\ln(x/X_0))$
  - Measures the beam directly "downstream" of the absorber
- **MAUS Monte Carlo Simulation**
  - Generates beam of input normalised emittance ϵ_i and momentum p_i inside the upstream solenoid, given the field
  - Passes through the simulated TKU and TKD stations, the MICE absorber (LiH currently) and a virtual plane every 5 cm.
  - Standard physics processes
- **MICE Data**
  - Beam sampled in the hall at every TKU and TKD stations
  - Particle species selection currently using TOF01
  - No transmission selection in the analyses, but 140 ± 5 MeV/c input
  - **2016/04 setting 1.2** as a test case (10 mm–140 MeV/c, 880 mm $\beta_\perp$)

$\rightarrow$ 8645, 8653, 8677, 8680, 8683, 8685, 8687, 9689, 8691, 8692, 8693 (462k @TOF2)

Normalised RMS transverse emittance

4D normalised RMS emittance:

$$\epsilon_n = \frac{1}{m}\sqrt[4]{D} \qquad (1)$$



with D the determinant of the covariance matrix defined as

$$D = \det\begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{p_xx} & \sigma_{p_xp_x} & \sigma_{p_xy} & \sigma_{p_xp_y} \\ \sigma_{yx} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{p_yx} & \sigma_{p_yp_x} & \sigma_{p_yy} & \sigma_{p_yp_y} \end{pmatrix} \qquad (2)$$

with $\sigma_{\alpha\beta}$ the covariance of α and β, i.e.

$$\sigma_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \langle\alpha\rangle)(\beta_i - \langle\beta\rangle) = \langle\alpha\beta\rangle - \langle\alpha\rangle\langle\beta\rangle, \qquad (3)$$

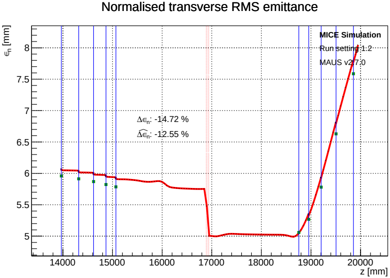
Emittance evolution in Step IV 2016/04 setting 1.2

In this case, only the the particles that make it through at selected:

- The RMS emittance reduction across the absorber is far greater than what is expected (∼7–8%) across the absorber
- Major non-linearities in the DS tracker, to such an extent in data that is drowns the cooling signal entirely (no p spread in this MC)

→ **Transmission loss bias and non-linear transport**

Simulation



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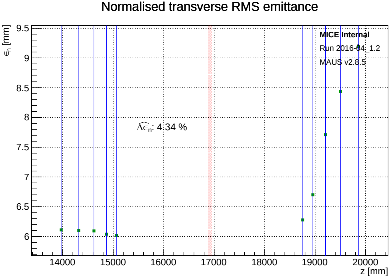
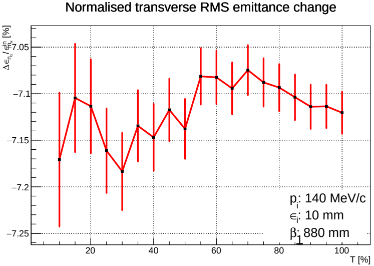


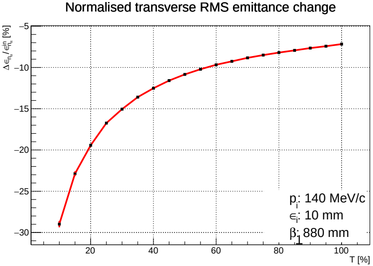
Bias caused by transmission loss

Toy

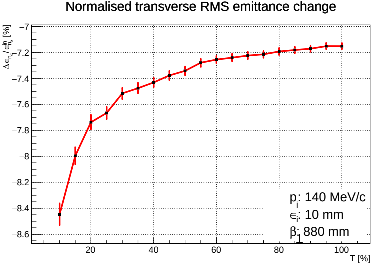
Large R rejection



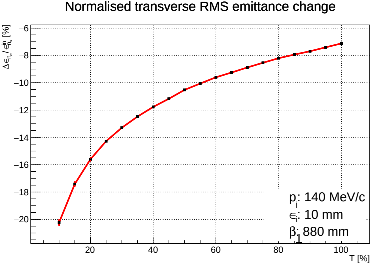
Large R rejection after drift



Large R rejection after solenoid



Large amplitude rejection



Alternatives to RMS emittance

For large input beams, that cool significantly in the MICE channel, the low transmission and the drift space between the FC and TKD introduces a **large bias on RMS emittance**, even for a beam selected downstream.

All hope is not lost, however, as what must be shown is an **increase in phase-space density** going through the absorber. Two options:

- Single-particle amplitude approach
  - → Still uses the covariance matrix Σ as a metric
  - → Assumes a somewhat Gaussian core
  - → Allows to select low amplitude, i.e. high density regions
  - → Relates very easily to RMS emittance
- Density estimation approach
  - → Find a reliable estimator of the density in the 4D phase-space
  - → Compute the volume occupied by a constant fraction α of the particles
  - → A reduction in volume signifies an increase in density
  - → Can be related to RMS emittance

Transverse single-particle amplitude

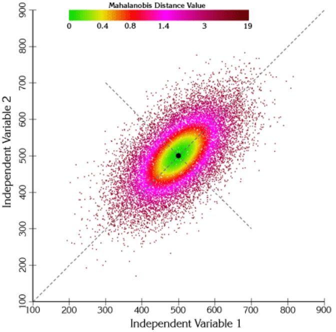
The **transverse amplitude** is defined as:

$$A_\perp \equiv \epsilon_n u_i^T \Sigma^{-1} u_i \tag{4}$$

with Σ the covariance matrix and u_i the phase-space column vector of the i^{th} particle. The u_i are centred so that $u_{i,\alpha} = \alpha_i - \langle\alpha\rangle$. For a gaussian beam, the amplitudes are distributed as a χ^2 distribution with 4 degrees of freedom. Its mean is $\langle A_\perp\rangle = \epsilon_n\langle\chi_4^2\rangle = 4\epsilon_n$.

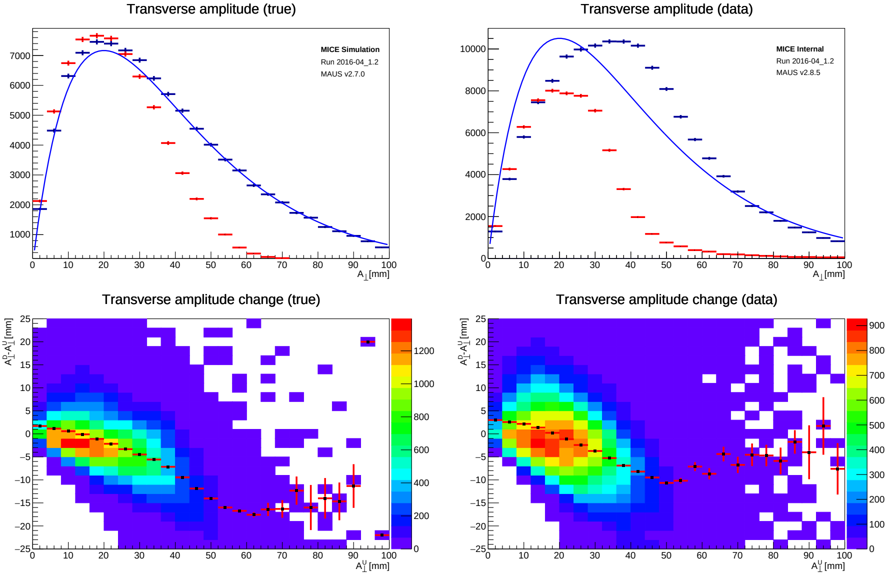
The amplitude gives a definition of a weight to select **any given fraction α of the beam**, rejecting the tails if need be. It is analogous to the squared **Mahalanobis distance**, d^2, scaled by the emittance, ϵ_n.

This basically represents the **Euclidean distance** of the particle to the centre of the distribution, in a **metric defined by $\boldsymbol{\Sigma}$**.



Amplitude distributions in Step IV 2016/04 setting 1.2

Simulation | **Data**



Using amplitudes to produce a subsample

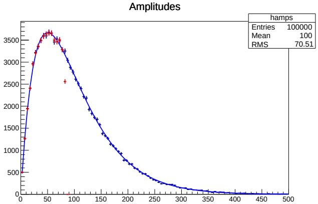
The method to produce an α-subsample is as follows:

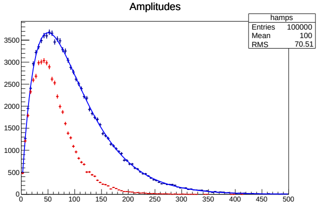
1. calculate the amplitudes, $A_\perp^i$, $i = 1, \ldots, N$, of every particle;
2. find a limit $A_\perp^\alpha$ so that the sample of all particles that verify $A_\perp < A_\perp^\alpha$ represents a fraction α of the entire population;
3. re-evaluate the covariance matrix Σ on the reduced sample;
4. repeat 1., 2. and 3. until we get convergence on the sample.

→ The RMS emittance of the subsample is the **subsample emittance**

→ The volume occupied by the subsample **fractional emittance**

→ Must select the **same amount of particles** up and downstream





Consequence of a subsample selection

Selecting a sample amplitude-wise out of the χ_4^2 distribution is equivalent to **truncating** the original distribution. It does **not** artificially increase the core density. For a fractional α, the cut-off is

$$\gamma(2, L/2\epsilon_n) = \alpha \quad \rightarrow \quad L/2\epsilon_n = -W\left(\frac{\alpha - 1}{e}\right) - 1, \tag{5}$$

with $W(\cdot)$, the product log function. This means that the expected subsample fractional change is

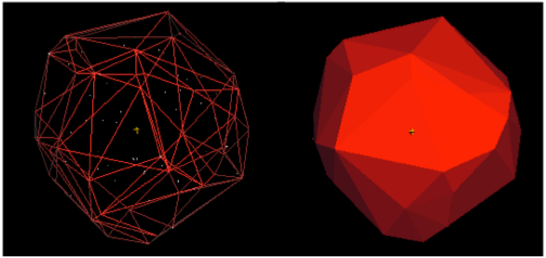
$$\frac{L_\alpha^o - L_\alpha^i}{L_\alpha^i} = \frac{2\epsilon_n^o F(\alpha) - 2\epsilon_n^i F(\alpha)}{2\epsilon_n^i F(\alpha)} = \boxed{\frac{\epsilon_n^o - \epsilon_n^i}{\epsilon_n^i}}. \tag{6}$$

$$\frac{e_\alpha^o - e_\alpha^i}{e_\alpha^i} = \frac{\langle A_\perp^o \rangle_T - \langle A_\perp^i \rangle_T}{\langle A_\perp^i \rangle_T} = \frac{\epsilon_n^o}{\epsilon_n^i}\frac{G(\alpha)}{G(\alpha)} - 1 = \boxed{\frac{\epsilon_n^o - \epsilon_n^i}{\epsilon_n^i}}. \tag{7}$$

Even if nonlinear, asymptotically we have $\Delta\epsilon_n/\epsilon_n^i = \lim\limits_{\alpha\to 0} \frac{\Delta e_\alpha}{e_\alpha^i}$

Fractional emittance calculation

An alternative method to using the subsample emittance e_α as an FOM is to use the volume occupied by the selected particles instead, i.e. the **fractional emittance**. This can be done at any dimension by calculating the volume of the **convex hull** of the subset of points.



The convex hull or convex envelope of a set X of points in the Euclidean space is the smallest convex set that contains X. For instance, when X is a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber membrane stretched around X.

Probability content of an n-RMS ellipsoid

An n-RMS ellipsoid is an ensemble of points $\mathcal{E}$ in n dimensions that satisfy $\mathbf{x}^T\Sigma^{-1}\mathbf{x} < 1$. The integral of the corresponding n-Gaussian over the entire ellipse $\mathcal{E}$ is, for an arbitrary scaled radius,

$$p(\mathbf{x} \in \mathcal{E}) = \int_0^{R^2} \chi_n^2(r)dr = \frac{\gamma(n/2, R^2/2)}{\Gamma(n/2)} \quad (8)$$

with $\gamma(n, r)$ the lower incomplete Euler Gamma function.

This yields radically different probability contents at different dimensions

n	1	2	3	4
V_n	$2(m\epsilon_n)^{1/2}R$	$\pi m\epsilon_n R^2$	$\frac{4}{3}\pi(m\epsilon_n)^{3/2}R^3$	$\frac{1}{2}\pi^2(m\epsilon_n)^2R^4$
$p(\mathbf{x} \in \mathcal{E})$	68.27 %	39.35 %	19.87 %	9.02 %
	$\mathrm{erf}(1/\sqrt{2})$	$1 - \frac{1}{\sqrt{e}}$	$\mathrm{erf}(1/\sqrt{2}) - \sqrt{\frac{2}{\pi e}}$	$1 - \frac{3}{2\sqrt{e}}$

Given a volume measurement, the RMS emittance of a Gaussian beam is

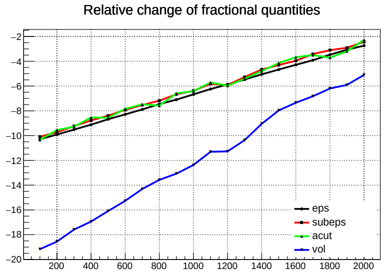
$$\epsilon_n = \frac{\left[\Gamma\left(\frac{n}{2}+1\right)V_n\right]^{\frac{2}{n}}}{m\pi R^2}, \qquad \epsilon_n^{4D} = \frac{\sqrt{2V_n}}{m\pi R^2} \quad (9)$$

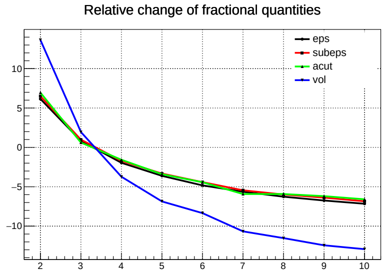
Toy analysis of fractional quantities

Toy

A thorough Toy analysis (Gaus) of the fractional quantities showed:

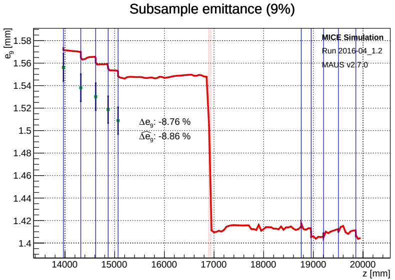
- The **same relative change** is seen in the RMS emittance and all of the fractional quantities, for any fraction
- The change in fractional quantities exhibit the **same relation** with $\beta_\perp$ and the input emittance, ϵ_i
- The fractional quantities are more robust against non-linearities as the tails do not influence their measurement

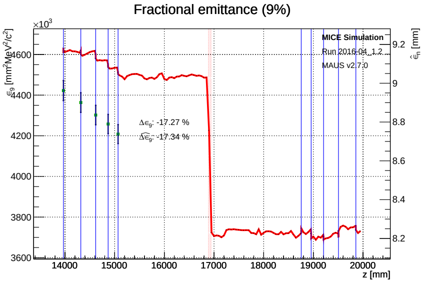




Fractional quantities in Step IV 2016/04 setting 1.2

Simulation





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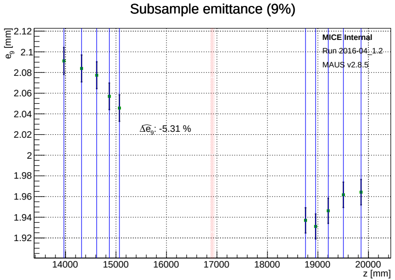
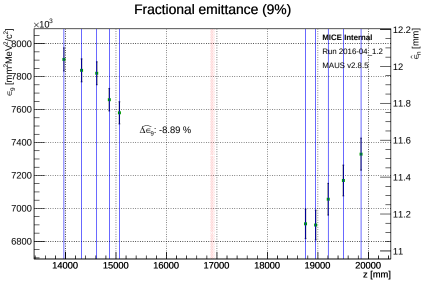




Phase-space density estimation

Amplitude methods work well for beams with a **Gaussian core**, so they are restricted to a small fraction of a non-linear beam (statistical limitation)

The volume of the phase-space is computed using the convex hull, which is a **bad approximation for concave sets**, i.e. non-Gaussian contours.

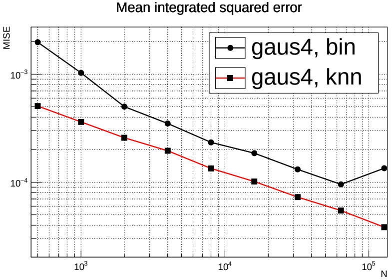
When using an estimator, there are three critical requirements

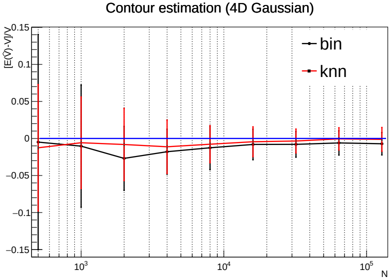
- **Consistency**: for large N, the estimator must converge to the true value that is estimated, $\lim_{n \to +\infty} \hat{\theta}_n = \theta$;
- **Unbiasedness**: the estimator must converge to the true value isotropically in the parameter space, $E(\hat{\theta}) = \theta$;
- **Robustness**: the estimator must return good results regardless of the true parent distribution it is estimating.

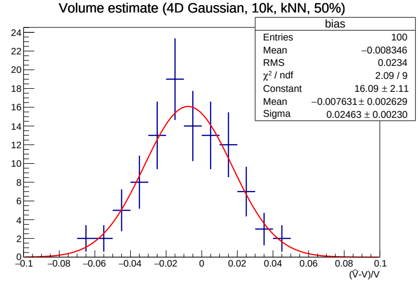
Deviation from the true estimated distribution can be quantified by computing the Mean Integrated Squared Error (**MISE**).

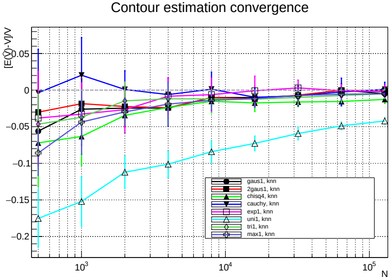
Systematic studies highlights

Toy







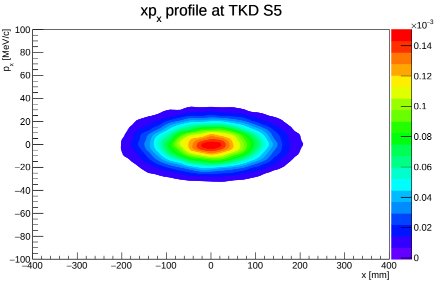


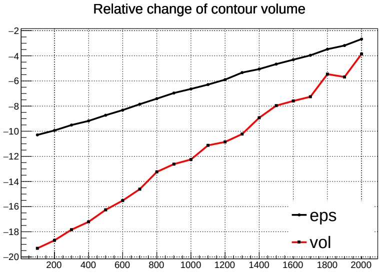
Phase-space evolution in a Gaussian world

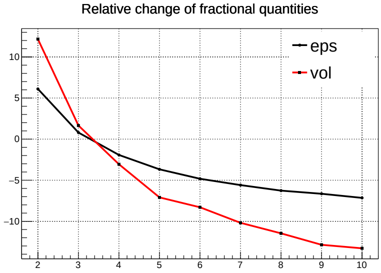
Toy

DE applied to toy MC:

- DE of a 4D Gaussian before and after absorber
- Find contour level, ρ_α, given α
- Measure the phase-space volume for which $\rho(x) > \rho_c$
- Equivalent to $A_\perp$ selection



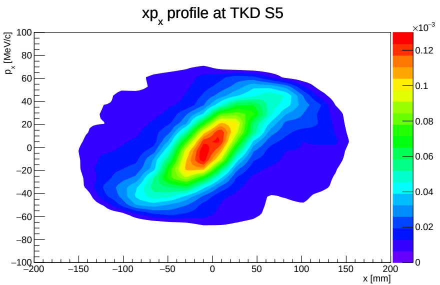


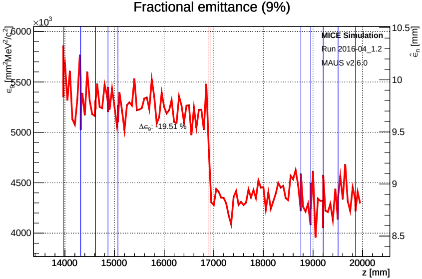


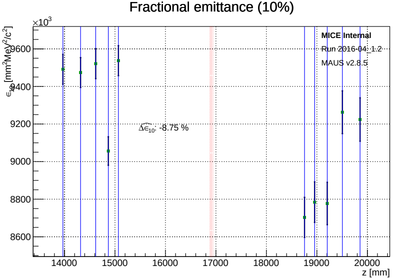
Phase-space evolution in Step IV 2016/04 setting 1.2

DE applied to MAUS MC and data:

- DE of a simulated phase-space in TKD station 5
- Find contour level, ρ_α, given α
- Measure the phase-space volume for which $\rho(x) > \rho_c$
- Use MC to compute volume







Alternate contour volume calculation: α-complices

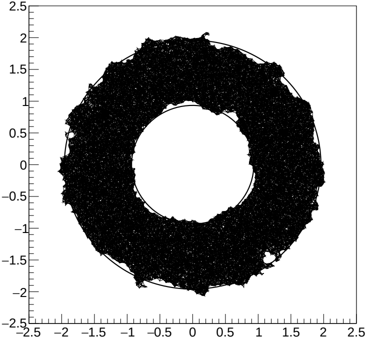
Convexe hulls work after amplitude selection because Σ has a convexe (L^2) symmetry around the mean.

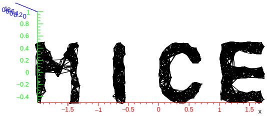
They cannot be used in the general case, as they drown underlying asymmetries of the distribution. Two options

- Compute the volume with MC
- Extend the concept to α-shapes

An α-complex is a set of n-simplices tessellating the points that verify $R < 1/\alpha$, R the circumradius of the simplex. One can **fix the scale of the features in the distribution** and get the right volume.

Using 1NN approximation, one expects $R \sim (N\rho_c)^{-1/n}$ for a contour level ρ_c.





Conclusions

Status of the amplitude-based analysis:

- Selecting the core amplitude-wise **gets rid of artificial cooling** due to scraping and **artificial growth** due to non-linearities
- The toy MC shows that the **exact same trends** are observed for the subsample and fractional emittance as for the RMS definition.
- Method shows a **clear cooling signal** in both MC and data, a proper MC is needed.

Status of the density estimation analysis:

- Systematic study under way, **kNN promising in 4D**, low MISE and no bias for large samples with the rule-of-thumb k selection.
- Method applied to the Toy to study its behaviour, must check for more input distribution shapes in 4D
- Method also shows cooling signal in simulation and data, **need more thorough analysis**, currently not as reliable as $A_\perp$ method

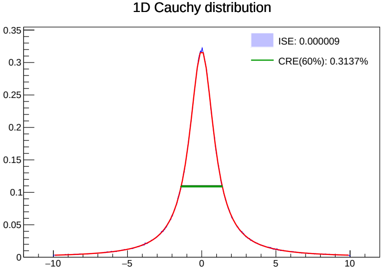
A. Density estimators

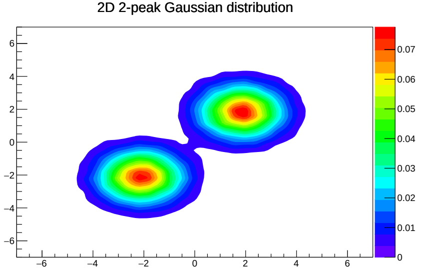
Optimal binning

Multiple ways to optimize the binning for density estimation

- Scott's rule, minimizes ISE for Gaussian distributions (bad)
- Minimize the ISE of a "leave-one-out" estimator for constant binning, i.e. minimize $J = \frac{2}{(M-1)\Delta} - \frac{M+1}{M^2(M-1)\Delta}\sum_k N_k^2$
- Maximize the "jackknife" likelihood, i.e. minimize the information $(-\ln L)$ with $\ln L = \sum_i N_i \ln\left(\frac{N_i}{\Delta\left[\sum_k (N_k+1)-1\right]}\right)$

An n-linear interpolation is used between the bin centres





k-Nearest Neighbours

For a given point $\boldsymbol{x}$, find the k closest points in the input cloud, find the distance R_k to the k^{th} point and compute the local density as

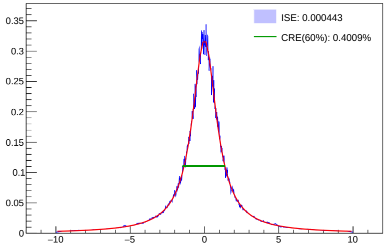
$$\rho(\boldsymbol{x}) = \frac{k}{\mathcal{V}_n} = \frac{k\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}} R_k^n}, \quad (10)$$

with $\mathcal{V}_n$ the volume of the n-sphere centred in $\vec{x}$ of radius R_k.

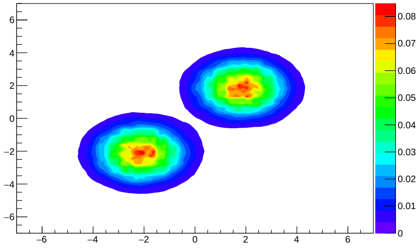
Several ways to optimize k

- Rule of thumb, simply fix $k = \sqrt{N}$
- Minimize the information criterion (AIC, BIC)

1D Cauchy distribution



2D 2-peak Gaussian distribution



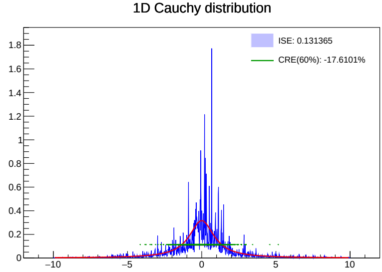
Tessellation Density Estimation (TDE)

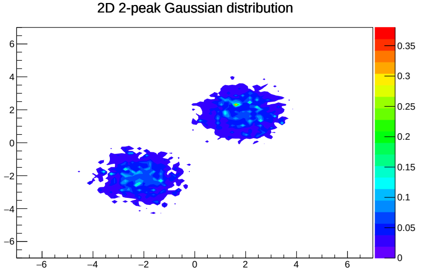
For a given point set, find the Voronoi tessellation of the space. The density in each of the Voronoi cell is simply

$$\rho_j = V_j^{-1} \tag{11}$$

with $\mathcal{V}_j$ the volume of the j^th cell. It is analogous to 1NN. One can either find the closest cell and use ρ_j or use Delaunay interpolation.

- $\rightarrow$ Unbiased estimator but extremely large variance
- $\rightarrow$ Must find ways to reduce variance to make it practical





Penalised Bootstrap Aggregate TDE (PBATDE)

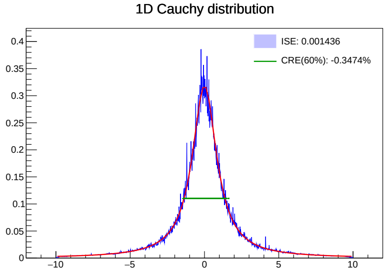
One way to reduce variance is to resample the N points into M (~ 1000) tessellated sets of J points (bootstrapping) and compute

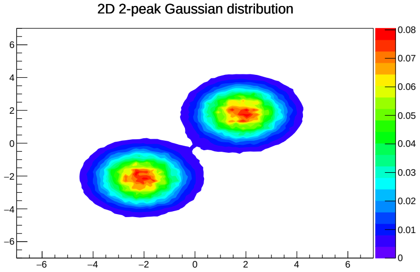
$$\rho(\boldsymbol{x}) = \frac{1}{J}\sum_{j=1}^{J} V_{jk}^{-1} \tag{12}$$

with $\mathcal{V}_j$ the volume of the k^{th} cell of the subset j.

→ Still unbiased, variance reduced by bootstrapping

→ Minimize the information criterion (AIC, BIC), penalises complexity J





B. Theoretical functions

Uniform distribution

Probability density function:

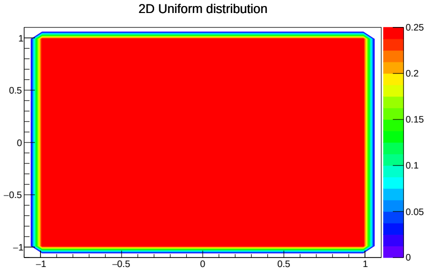
$$X \sim \mathcal{U}(\boldsymbol{\mu}, \boldsymbol{L}) = \frac{1}{\prod_{i=1}^{n} L_i} \prod_{i=1}^{n} \chi_{\left[-\frac{L_i}{2}, \frac{L_i}{2}\right]}(x_i - \mu_i) \tag{13}$$

Volume occupied by a fraction α of the population:

$$\mathcal{V} = \alpha \prod_{i=1}^{n} L_i \tag{14}$$

The α-probability contour:

- $\rightarrow$ n-orthotope centred in $\boldsymbol{\mu}$
- $\rightarrow$ Side lengths are the αL_i



Gaussian distribution

Probability density function:

$$X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left[-(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})/2\right] \quad (15)$$

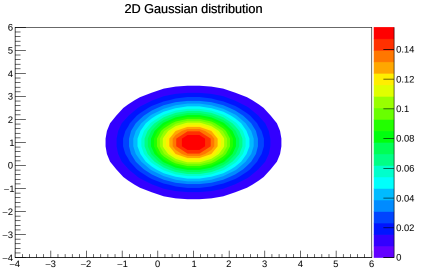
Volume occupied by a fraction α of the population:

$\rightarrow$ First find R^2 so that $P\left(\frac{n}{2}, \frac{R^2}{2}\right) = \alpha$, P the NLI Γ function

$$\mathcal{V} = \frac{\pi^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}}{\Gamma(\frac{n}{2}+1)} R^n \quad (16)$$

The α-probability contour:

- $\rightarrow$ n-ellipsoid centred in $\boldsymbol{\mu}$
- $\rightarrow$ Diagonalize $\Sigma = U^T \Lambda U$, axes radii are the $\sqrt{\lambda_i} R$
- $\rightarrow$ Angles are ratios between the eigenvector components, u_{ij}



Exponential distribution

Probability density function:

$$X \sim \mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \frac{1}{2^n} \prod_{i=1}^{n} \lambda_i \exp\left[-\sum_{i=1}^{n} \lambda_i |x_i - \mu_i|\right], \tag{17}$$

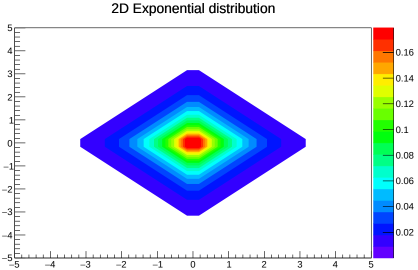
it is the L^1 extension of the 1D exponential distribution.
Volume occupied by a fraction α of the population:

$\rightarrow$ First find R so that $\gamma(n, R) = \alpha$, γ the LI Γ function

$$\mathcal{V} = \frac{2}{\prod_{i=1}^{n} \lambda_i} R^n \tag{18}$$

The α-probability contour:

- $\rightarrow$ n-rhombus centred in $\boldsymbol{\mu}$
- $\rightarrow$ Axes half-lengths are the R/λ_i
- $\rightarrow$ L^1 symmetry



Triangular distribution

Probability density function:

$$X \sim \mathcal{T}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \frac{n(n+1)}{2^n} \prod_{i=1}^{n} \lambda_i \left(1 - \sum_{i=1}^{n} \lambda_i \left| x_i - \mu_i \right| \right), \qquad (19)$$

it is the L^1 extension of the 1D triangular distribution.
Volume occupied by a fraction α of the population:

→ First find R so that $R^n[(n+1) - nR] = \alpha$, then

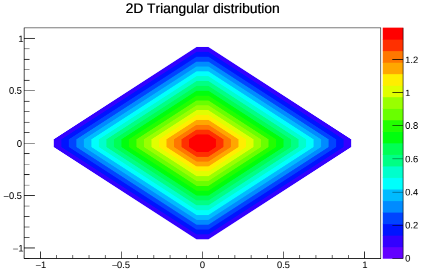
$$\mathcal{V} = \frac{2}{\prod_{i=1}^{n} \lambda_i} R^n \qquad (20)$$

The α-probability contour:

→ n-rhombus centred in $\boldsymbol{\mu}$

→ Axes half-lengths are R/λ_i

→ L^1 symmetry



Maxwell-Boltzmann distribution

Probability density function:

$$X \sim \mathcal{M}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{n(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{3}{2}}} \boldsymbol{z}^T\boldsymbol{z} \exp\left[-\boldsymbol{z}^T\boldsymbol{z}/2\right], \quad (21)$$

with $\boldsymbol{z}^T\boldsymbol{z} = (\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$. It is the L^2 extension.

Volume occupied by a fraction α of the population:

→ Numerically optimize inner (r) and outer (R) radii of content α, then

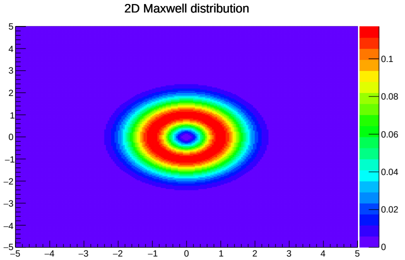
$$\mathcal{V} = \frac{\pi^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}(R^n - r^n) \quad (22)$$

The α-probability contour:

→ Two concentric n-ellipsoid centred in $\boldsymbol{\mu}$

→ Inner and outer radii are r and R

→ L^2 symmetry



Cauchy distribution

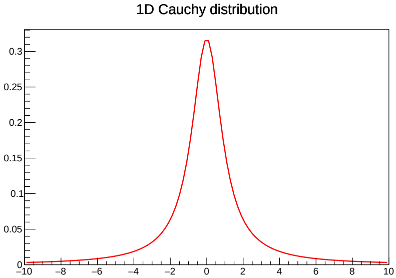
Probability density function:

$$X \sim \mathcal{C}(\mu, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2} \tag{23}$$

this definition cannot be extended to higher dimensions as it wouldn't be normalisable on $\mathbb{R}^2$ and above.

Volume occupied by a fraction α of the population:

$$\mathcal{V} = 2\sigma \tan\left(\frac{\pi\alpha}{2}\right) \tag{24}$$



Chi-squared distribution

Probability density function:

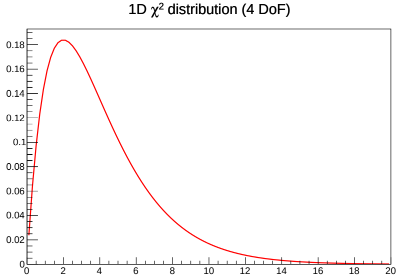
$$X \sim \chi_n^2 = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-x/2}, \tag{25}$$

it has a mean and variance of n and a maximum at $x = n - 2$ if $n \geq 2$.

Volume occupied by a fraction α of the population:

$\rightarrow$ Find r, R so that $P\left(\frac{n}{2}, \frac{R}{2}\right) - P\left(\frac{n}{2}, \frac{r}{2}\right) = \alpha$, P the NLI Γ function:

$$\mathcal{V} = R - r \tag{26}$$



C. Linear transport

Drift space

At first order, in a drift space, the 4 transverse coordinates evolve as

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 1 & L/p_z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L/p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{27}$$

The covariance matrix transforms as

$$\Sigma \rightarrow M \Sigma M^{-1} \tag{28}$$

and, as the transfer matrix has $\det M = 1$, the emittance is left unchanged after a drift.

Solenoid field

At first order, in a solenoid field, the 4 transverse coordinates evolve as

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\theta) & \frac{1}{K}\sin(\theta) & 0 & 0 \\ -K\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \frac{1}{K}\sin(\theta) \\ 0 & 0 & -K\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \quad (29)$$

with $\theta = KL$ and $K = \frac{B_0}{2B\rho_0}$, where B_0 is the longitudinal field inside the solenoid and $(B\rho_0)$ is the magnetic rigidity of the central trajectory.
The covariance matrix transforms as

$$\Sigma \rightarrow M\Sigma M^{-1} \quad (30)$$

and, as the transfer matrix has $\det M = 1$, the emittance is left unchanged after going through a solenoid.