Computing $\hat{q}$ on a quenched SU(3) lattice

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Outline

- Phenomenology based study of transport coefficient $\hat{q}$ in heavy-ion collisions

- Formulating $\hat{q}$ for hot QGP using Lattice gauge theory
  1) Previous study done on a quenched SU(2) lattice
  2) Extending calculations to a quenched SU(3) lattice

- Estimates of $\hat{q}$ on a quenched QGP plasma
Phenomenological extraction of $\hat{q}$

\[
\hat{q}(r, t) = \frac{\langle k_1^2 \rangle}{L}
\]

(1) \( q/T^3 \sim 3 - 5 \)

(2) Based on fit to experimental data
First principles calculation of $\hat{q}$

QGP is locally thermalized and highly non-perturbative

Thermal Lattice QCD to compute $\hat{q}$
Lattice formulation of $\hat{q}$

- Simplest process: A leading quark propagating through **hot plasma**

$$ q = \left( \frac{\mu^2}{2q^-}, q^-, 0 \right) = (\lambda^2, 1, 0)Q; \quad \text{Hard scale} = Q; \lambda \ll 1 $$

$$ k = (k^+, k^-, k_\perp 0) = (\lambda^2, \lambda^2, \lambda)Q; \quad \text{Transverse gluon} $$

$\mathcal{M}(k)$: Transition probability

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A. Majumder, PRC 87, 034905 (2013)
Lattice formulation of $\hat{q}$

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$$ k = (k^+, k^-, k_\perp 0) = (\lambda^2, \lambda^2, \lambda)Q; \quad \text{Transverse gluon} $$

$\mathcal{M}(k)$: Transition probability

$$ \hat{q}(\vec{r}, t) = \sum_k k_\perp^2 \frac{Disc[\mathcal{M}(k)]}{L} $$

$$ \hat{q} = \frac{16\sqrt{2}\pi^2\alpha_s}{N_c} \int \frac{dy^- d^2y_\perp}{(2\pi)^3} \; d^2k_\perp e^{-i\frac{k_\perp^2}{2q^-}y^- + i\vec{k}_\perp \cdot \vec{y}_\perp} $$

$$ \times \langle M | F^{+\mu}_\perp (y^-, y_\perp) F^+_{\mu \mu}(0) | M \rangle $$

Non-perturbative part (Lattice QCD)

A. Majumder, PRC 87, 034905 (2013)
Constructing a more general expression as $\hat{Q}$

- **Physical form of $\hat{q}$**

$$\hat{q} = \frac{16\sqrt{2}\pi^2 \alpha_s}{N_c} \int \frac{dy^- d^2y_\perp}{(2\pi)^3} \, d^2k_\perp e^{-i\frac{k_\perp^2}{2q^-}y^- + i\vec{k}_\perp \cdot \vec{y}_\perp} \langle M | F_{\perp}^{+\mu}(y^-, y_\perp) F_{\perp}^{+\mu}(0) | M \rangle$$

- **General form of $\hat{q}$: with $q^-$ is fixed; $q^+$ is variable**

$$\hat{Q}(q^+) = \frac{16\sqrt{2}\pi^2 \alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{i\vec{k} \cdot \vec{y}} q^- \frac{\langle M | F_{\perp}^{+\mu}(0) F_{\perp}^{+\mu}(y) | M \rangle}{(q + k)^2 + i\epsilon}$$

A. Majumder, PRC 87, 034905 (2013)
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- **Physical form of $\hat{q}$**

$$\hat{q} = \frac{16\sqrt{2}\pi^2\alpha_s}{N_c} \int \frac{dy^- d^2y_\perp}{(2\pi)^3} d^2k_\perp e^{-ik_\perp \cdot y^- + i\vec{k}_\perp \cdot \vec{y}_\perp} \langle M|F_\perp^{+\mu}(y^-,y_\perp)F_\perp^{+\mu}(0)|M\rangle$$

- **General form of $\hat{q}$: with $q^-$ is fixed; $q^+$ is variable**

$$\hat{Q}(q^+) = \frac{16\sqrt{2}\pi\alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{iky} q^- \frac{\langle M|F_\perp^{+\mu}(0) F_\perp^{+\mu}(y)|M\rangle}{(q + k)^2 + i\epsilon}$$

1) **When $q^+ \sim T$**

$$\text{Disc}[\hat{Q}(q^+)]_{at \, q^+ \sim T} = \hat{q}$$
Constructing a more general expression as \( \hat{Q} \)

- **Physical form of \( \hat{q} \)**

\[
\hat{q} = \frac{16\sqrt{2}\pi^2 \alpha_s}{N_c} \int d\gamma d^2y_\perp \frac{d^2k_\perp}{(2\pi)^3} e^{-\frac{|k_\perp|^2}{2q^2} - ik_\perp \cdot \gamma_\perp} \langle M|F_{\perp \mu}^+(y^-, y_\perp)F_{\perp \mu}^+(0)|M \rangle
\]

- **General form of \( \hat{Q} \): with \( q^- \) is fixed; \( q^+ \) is variable**

\[
\hat{Q}(q^+) = \frac{16\sqrt{2}\pi^2 \alpha_s}{N_c} \int d^4y d^4k \frac{e^{iky}}{(2\pi)^4} \frac{\langle M|F_{\perp \mu}^+(0)F_{\perp \mu}^+(y)|M \rangle}{(q+k)^2+i\epsilon}
\]

1) **When \( q^+ \sim T \)**

\[
\text{Disc}[\hat{Q}(q^+)]_{q^+ \sim T} = \hat{q}
\]

2) **When \( q^+ = -q^- \)**

\[
\frac{1}{(q+k)^2} \approx \frac{1}{-2q^-q^- + 2q^- (k^+ - k^-)} = -\frac{1}{2(q^-)^2} \left[ 1 - \left( \frac{k^+ - k^-}{q^-} \right) \right]^{-1} = -\frac{1}{2(q^-)^2} \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}k_z}{q^-} \right)^n \right]
\]
Constructing a more general expression as $\hat{\mathcal{Q}}$

- **Physical form of $\hat{\mathcal{Q}}$**

$$\hat{\mathcal{Q}} = \frac{16\sqrt{2}\pi^2\alpha_s}{N_c} \int \frac{dy^-d^2y_\perp}{(2\pi)^3} \, d^2k_\perp e^{-\frac{k_\perp^2}{2q^-}y^- + i\vec{k}_\perp \cdot \vec{y}_\perp} \langle M|F^{+\mu}_\perp(y^-, y_\perp)F^{+\mu}_\perp(0)|M \rangle$$

- **General form of $\hat{\mathcal{Q}}$: with $q^-$ is fixed; $q^+$ is variable**

$$\hat{\mathcal{Q}}(q^+) = \frac{16\sqrt{2}\pi\alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{iky^-} q^- \frac{\langle M|F^{+\mu}_\perp(0) F^{+\mu}_\perp(y)|M \rangle}{(q + k)^2 + i\varepsilon}$$

1) When $q^+ \sim T$

$$\text{Disc}[\hat{\mathcal{Q}}(q^+)]_{at \, q^+ \sim T} = \hat{\mathcal{Q}}$$

2) When $q^+ = -q^-$

$$\frac{1}{(q + k)^2} \simeq -2q^-q^- + 2q^-(k^+ - k^-) = -\frac{1}{2(q^-)^2} \left[ 1 - \left( \frac{k^+ - k^-}{q^-} \right) \right]^{-1} = -\frac{1}{2(q^-)^2} \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{2k_z}}{q^-} \right)^n \right]$$

$$\hat{\mathcal{Q}}(q^+ = -q^-) = \frac{8\sqrt{2}\pi\alpha_s}{N_c \, q^-} \langle M|F^{+\mu}_\perp(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2D_z}}{q^-} \right)^n F^{+\mu}_\perp(0) |M \rangle$$
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

\[ \hat{Q}(q^+) = \frac{16\sqrt{2}\pi\alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{iky} q^- \frac{\langle M|^F^{+\mu}(0) F^{+\mu}(y)|M\rangle}{(q+k)^2+i\epsilon} \]

1) When $q^+ \in [-#T, #T]$

\[ q^2 \approx 0 \text{ (in-medium scattering)} \]
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

$$\tilde{\mathcal{Q}}(q^+) = \frac{16\sqrt{2}\pi\alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{iky} q^- \frac{\langle \mathcal{M}(\mu) \rangle}{(q+k)^2+i\epsilon}$$

1) When $q^+ \in [-\#T, \#T]$  
$q^2 \approx 0$ (in-medium scattering)

2) When $q^+ \in [\#T, +\infty)$  
$q^2 \gg 0$ (Bremsstrahlung radiation)
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

$$\hat{Q}(q^+)=\frac{16\sqrt{2}\pi\alpha_s}{N_c} \int \frac{d^4y d^4k}{(2\pi)^4} e^{iky} q^- \frac{\langle M F_{\perp\mu}(0) F_{\perp\mu}^+(y) | M \rangle}{(q+k)^2+i\epsilon}$$

1) **When** $q^+ \in [-\#T, \#T]$ \[ q^2 \approx 0 \text{ (in-medium scattering)} \]

2) **When** $q^+ \in [\#T, +\infty)$ \[ q^2 \gg 0 \text{ (Bremsstrahlung radiation)} \]

3) **When** $q^+ \in (-\infty, -\#T]$ \[ q^2 \ll 0 \text{ (Space-like)}; \lim_{q^+\rightarrow-q^-} \text{Disc}[\hat{Q}(q^+)]=0 \]
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

Contour C1:

$$I_1 = \int dq^+ \frac{\hat{Q}(q^+)}{2\pi i (q^+ + q^-)} = \hat{Q}(q^+ = -q^-)$$
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

Contour C1:

$$I_1 = \oint \frac{dq^+}{2\pi i} \frac{\hat{Q}(q^+)}{(q^+ + q^-)} = \hat{Q}(q^+) = -q^-$$

Contour C2: On extending it to infinity

$q^+ = -q^-$
Extract $\hat{q}$ using analytic continuation of $\hat{Q}(q^+)$

Contour C1:

$\int dq^+ \frac{\hat{Q}(q^+)}{2\pi i (q^+ + q^-)} = \hat{Q}(q^+ = -q^-)$

Contour C2: On extending it to infinity

$I_1 = \int_{-\#T}^{\#T} dq^+ \frac{\hat{q}(q^+)}{q^+ + q^-} + \int_0^\infty dq^+ \frac{Disc[\hat{Q}(q^+)]}{q^+ + q^-}$

Pure thermal part
Pure Vacuum part

Width of thermal discontinuity $2T$ or $4T$ (HTL analysis)
⇒ Source of systematic error

$\hat{q} = \frac{8\sqrt{2}\pi\alpha_s}{N_c(T_1 + T_2)} \langle M | F_{\perp \mu}^+(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2} \mathcal{D}_z}{q^-} \right)^n F_{\perp \mu}^+(0) | M \rangle_{\text{Thermal-Vacuum}}$
$\hat{q}$ as a series of local operators

- Physical form of $\hat{q}$ at LO:

$$\hat{q} = \frac{8\sqrt{2}\pi\alpha_s}{N_c(T_1 + T_2)} \langle M | F^+_{\perp \mu} (0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F^+_{\perp \mu} (0) | M \rangle_{(\text{Thermal-Vacuum})}$$
\( \hat{q} \) as a series of local operators

- Physical form of \( \hat{q} \) at LO:

\[
\hat{q} = \frac{8\sqrt{2}\pi\alpha_s}{N_c(T_1 + T_2)} \langle M | F_{\perp}^{+\mu}(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F_{\perp}^{+\mu}(0) | M \rangle_{\text{Vacuum}}
\]

Xiangdong Ji, PRL 110, 262002 (2013)
Parton PDF operator product expansion with \( D_z \) derivatives
\( \hat{q} \) as a series of local operators

- Physical form of \( \hat{q} \) at LO:
  \[
  \hat{q} = \frac{8\sqrt{2}\pi \alpha_s}{N_c(T_1 + T_2)} \langle M|F^+_{\perp \mu}(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F^+_{\perp \mu}(0) |M \rangle_{\text{Thermal-Vacuum}}
  \]

Rotating to Euclidean space:
\[
\begin{align*}
  x^0 &\rightarrow -ix^4; \quad A^0 \rightarrow iA^4 \\
  \Rightarrow \quad F^{0i} &\rightarrow iF^{4i}
\end{align*}
\]

LO operators:
\[
\sum_{i=1}^{2} \text{Trace}[F^{3i}F^{3i} - F^{4i}F^{4i}] + 2i \sum_{i=1}^{2} \text{Trace}[F^{3i}F^{4i}]
\]

- Uncrossed operator
- Crossed operator

LO operators with \( D_z \) derivative:
\[
\sum_{i=1}^{2} \text{Trace}[F^{3i}D_zF^{3i} - F^{4i}D_zF^{4i}] + i \sum_{i=1}^{2} \text{Trace}[F^{3i}D_zF^{4i} + F^{4i}D_zF^{3i}]
\]

where, \( D_z \) is covariant derivative along leading parton direction (z-dir)

Xiangdong Ji, PRL 110, 262002 (2013)
Parton PDF operator product expansion with \( D_z \) derivatives

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Operators in quenched SU(2) plasma

- Average over 5000 configuration
- Transition temperature $T_c \in [170, 350]$ MeV
- Crossed correlator is small for $T \sim 400$ MeV

A. Majumder, PRC 87, 034905 (2013)
Operators in quenched SU(3) plasma

Thermal + Vacuum part

- $100g^2a^4\langle F^3F^3i - F^4F^4i \rangle$
- $g^2a^4\langle F^3F^4i + F^4iF^3 \rangle$
- $100g^2a^5\langle F^3DzF^3i - F^4iDzF^4i \rangle$
- $100g^2a^5\langle F^3DzF^4i + F^4iDzF^3 \rangle$

(N$_t=4, N_s=16$)

Vacuum part

- $100g^2a^4\langle F^3F^3i - F^4F^4i \rangle$
- $g^2a^4\langle F^3F^4i + F^4iF^3 \rangle$
- $100g^2a^5\langle F^3DzF^3i - F^4iDzF^4i \rangle$
- $100g^2a^5\langle F^3DzF^4i + F^4iDzF^3 \rangle$

(N$_t=16, N_s=16$)

$\beta_0 = 6/g^2$
Operators in quenched SU(3) plasma

- Operators looks similar to SU(2)
- Need to set the lattice spacing ($\alpha_L$) i.e. relation between $g$ and $\alpha_L$
Scale setting using Polyakov loop

- Expectation value of Polyakov loop:
  \[ P = \frac{1}{n_x n_y n_z} \text{tr} \left[ \sum_{r=0}^{n_t-1} \prod_{n=0}^{r} U_4(na, \vec{r}) \right] \]

- Nonperturbative correction to two loop beta function
  \[ a_L = \frac{f}{\Lambda_L} \left( \frac{11}{16\pi^2 g^2} \right)^{-\frac{51}{12\pi}} \exp \left( -\frac{8\pi^2}{11g^2} \right) \]

Temperature, \( T = \frac{1}{n_t a_L} \)

Critical Temperature, \( T_c = 265 \text{ MeV} \) (Pure SU(3))

Tune \( f \) such that \( \frac{T_c}{\Lambda_L} \) is independent of \( g \)
Real part of FF correlator in quenched SU(3)

- Uncrossed correlator is dominant at high temperature
- Crossed correlator goes to zero at high temperature
- Correlator with Dz derivative are suppressed
Uncrossed correlator is dominant at high temperature
Crossed correlator goes to zero at high temperature
Correlators with $Dz$ derivative are suppressed
\( \hat{q} \) in quenched SU(3) plasma

\[
\hat{q} = \frac{8\sqrt{2}\pi\alpha_s}{N_c(T_1 + T_2)} \langle M | F_{\perp}^+(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F_{\perp\mu}^+(0) | M \rangle_{(\text{Thermal–Vacuum})}; \text{ Width of thermal discontinuity 2T or 4T (HTL analysis)}
\]

At high temperature:
\[
\frac{\hat{q}}{T^3} \sim 0.5 - 1
\]

At high temperature:
\[
\text{Im} \left[ \frac{\hat{q}}{T^3} \right] \text{ goes to zero}
\]
Pure gluon plasma vs QGP plasma

Going from pure gluon to quark-gluon system
1) Critical temperature shifts to lower temperature
2) Magnitude amplifies by factor of 3-4

- JHEP 1011 (2010) 07
Pure gluon plasma vs QGP plasma

Going from pure gluon to quark-gluon system
1) Critical temperature shifts to lower temperature
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\( \hat{q} \propto S \) (Entropy density of the medium)

- JHEP 1011 (2010) 07

Amit Kumar, Hard Probes 2018
Going from pure gluon to quark-gluon system

1) Critical temperature shifts to lower temperature
2) Magnitude amplifies by factor of 3-4

\( \hat{q} \propto S \) (Entropy density of the medium)
First calculation of $\hat{q}$ on SU(3) quenched plasma

- Lattice extraction of $\hat{q}$ is consistent with JET collaboration plot
- Analytic continuation to deep Euclidean space and expressed as local operators
- $\hat{q}$ shows scaling behavior (Nt=4, 6, 8 and 10)
- Scale setting using perturbative loop beta function with non-perturbative correction using Polyakov loop.

Future work

- Extend calculation to unquenched plasma (QGP)
- Extend calculation using Improved Action and bigger lattice sizes
- Include radiation diagram contributions
Imaginary part of FF correlator in quenched SU(3)

- Imaginary part of FF correlator does not contribute
Operators in quenched SU(3) plasma

Preliminary (in collaboration with Chihoko Nonaka)

10/4/2018

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