Multi moment cancellation of participant fluctuations - MMCP method

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- The only experimentally controllable way to probe the **QCD phase diagram** is by studying interactions of **different system size** nuclei at various **energies**.
- **Participant fluctuations** is one of the **main background** effects in such study.
- It is the number of nucleons, N_P, that interacted inelastically and produced other particles during the collision.
- These fluctuations may hide the fluctuations from other sources.

There are several popular ways of reducing participant fluctuations:

(i) the selection of as **narrow centrality bins** as possible

(ii) the Centrality Bin Width Correction procedure (**CBWC**) (STAR, Luo (2011)),

(iii) the use of **strongly intensive** quantities (Gazdzicki, Mrowczynski (1992), Gorenstein, Gazdzicki (2011)),

We propose a different approach - to **cancel participant fluctuations** in a **combination** of several **high** fluctuation **moments** (V.B., Mackowiak-Pawlowska 1705.01110).

Fluctuation measures

A **multiplicity distribution**, $P(N)$, can be characterized by **central moments**, m_n,

$$m_n = \sum_N (N - \langle N \rangle)^n P(N) , \qquad \text{where} \quad \langle N^n \rangle = \sum_N N^n P(N) .$$

They are related to **cumulants**, κ_n,

$$\kappa_2 = m_2 , \qquad \kappa_3 = m_3 , \qquad \kappa_4 = m_4 - 3m_2^2 , \qquad \ldots ,$$

and **susceptibilities**, χ_n,

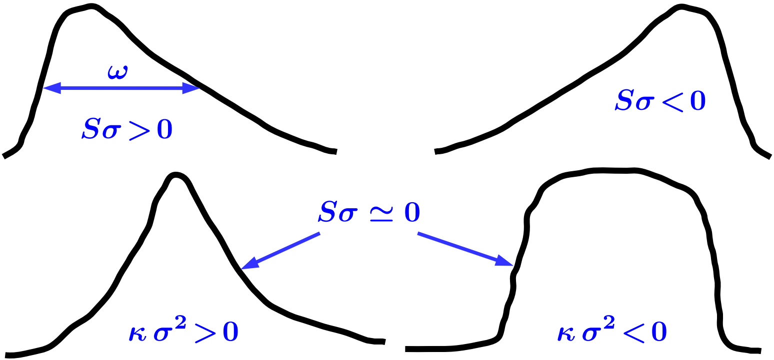
$$\chi_n = \frac{\partial^n (\mathcal{P}/T^4)}{\partial(\mu/T)^n} = \frac{\kappa_n}{V\, T^3} , \qquad \chi_{n,k} = \frac{\chi_n}{\chi_k} = \frac{\kappa_n}{\kappa_k} ,$$

where $\mathcal{P}$ is pressure, T - temperature, μ - chemical potential, and V - volume.

Frequently used cumulant ratios - **scaled variance**, normalized **skewness** and normalized **kurtosis** - are:

$$\omega = \frac{\kappa_2}{\langle N \rangle} = \frac{\sigma^2}{\langle N \rangle} , \qquad S\,\sigma = \frac{\kappa_3}{\kappa_2} , \qquad \kappa\,\sigma^2 = \frac{\kappa_4}{\kappa_2} ,$$

where $\sigma = \sqrt{\langle N^2 \rangle - \langle N \rangle^2} = \sqrt{\kappa_2}$ is **standard deviation**.



Poisson distribution: $\omega = S\sigma = \kappa\sigma^2 = 1$

Normal (Gauss) distribution: ω - **free** parameter, $S\sigma = \kappa\sigma^2 = 0$

Log-normal distribution: $\omega \sim \langle N \rangle,\ S\sigma \sim \langle N \rangle,\ \kappa\sigma^2 \sim \langle N \rangle^2$

- The approach `**just take negative binomial** (Poisson, Gauss...)' is **not working**, because it imposes a certain relation between moments, which might not exist.

The only assumptions are that participants are **identical** and **independent**. Then mean multiplicity N is the sum of contributions from N_P participants,

$$N = n_1 + n_2 + \ldots + n_{N_P}, \qquad \text{and} \quad \langle N \rangle = \langle N_P \rangle \langle n_A \rangle,$$

where ***identical*** and ***independent*** means that, $\langle n_i \rangle = \langle n_j \rangle = \langle n_1 \rangle = \langle n_A \rangle$, and $\langle n_i\, n_j \ldots n_k \rangle = \langle n_A \rangle^k$. Using multinomial theorem

$$N^k = \left(n_1 + n_2 + \ldots + n_{N_P}\right)^k = \sum_{k_1, k_2, \ldots k_{N_P}} \frac{k!}{k_1! k_2! \ldots k_{N_P}!}\, n_1^{k_1} n_2^{k_2} \ldots n_{N_P}^{k_{N_P}}\; \delta\left(k - \sum_{i=1}^{N_P} k_i\right),$$

where δ is the Kronecker delta, one can obtain arbitrarily high moments, e.g.

$$\omega = \omega_A + \langle n_A \rangle\, \omega_P,$$

$$S\sigma = \frac{\omega_A S_A \sigma_A + \langle n_A \rangle \omega_P \left[3\,\omega_A + \langle n_A \rangle S_P \sigma_P \right]}{\omega_A + \langle n_A \rangle \omega_P},$$

$$\kappa\sigma^2 = \frac{\omega_A \kappa_A \sigma_A^2 + \langle n_A \rangle \omega_P \left[\langle n_A \rangle^2 \kappa_P \sigma_P^2 + \omega_A \left(3\,\omega_A + 4\, S_A \sigma_A + 6 \langle n_A \rangle S_P \sigma_P \right) \right]}{\omega_A + \langle n_A \rangle \omega_P},$$

red - what we would like to measure, **black** - what we measure, **blue** - participant fluctuations (V.B. 1606.05358, Skokov, Friman, Redlich (2013), Braun-Munzinger, Rustamov, Stachel (2017))

The problem

- A moment of a rank n is a function of all lower moments for both participants, $\langle N_{\mathbf{P}}^n\rangle$, and a source, $\langle n_{\mathbf{A}}^n\rangle$,

$$\langle N^n\rangle = \mathcal{F}\left(\langle n_{\mathbf{A}}^1\rangle, \langle n_{\mathbf{A}}^2\rangle, \ldots \langle n_{\mathbf{A}}^n\rangle,\ \langle N_{\mathbf{P}}^1\rangle, \langle N_{\mathbf{P}}^2\rangle, \ldots \langle N_{\mathbf{P}}^n\rangle\right) .$$

  Therefore, one has only n **measures**, but $2n$ **unknowns** for their description.

- **Strongly intensive** measures **require** two types of values, e.g. pions and kaons,

$$\langle N_{\mathbf{A}}^n\rangle = \mathcal{F}\left(\langle n_{\mathbf{A}}^1\rangle, \langle n_{\mathbf{A}}^2\rangle, \ldots \langle n_{\mathbf{A}}^n\rangle,\ \langle N_{\mathbf{P_A}}^1\rangle, \langle N_{\mathbf{P_A}}^2\rangle, \ldots \langle N_{\mathbf{P_A}}^n\rangle\right) ,$$

$$\langle N_{\mathbf{B}}^n\rangle = \mathcal{F}\left(\langle n_{\mathbf{B}}^1\rangle, \langle n_{\mathbf{B}}^2\rangle, \ldots \langle n_{\mathbf{B}}^n\rangle,\ \langle N_{\mathbf{P_B}}^1\rangle, \langle N_{\mathbf{P_B}}^2\rangle, \ldots \langle N_{\mathbf{P_B}}^n\rangle\right) .$$

  and the **assumption** that all corresponding participant fluctuations moments are the same $\langle N_{\mathbf{P_A}}^n\rangle = \langle N_{\mathbf{P_B}}^n\rangle = \langle N_{\mathbf{P}}^n\rangle$, which gives $3n$ **unknowns** for $2n$ **measured** values.

- **Wounded nucleon** model gives all $\langle N_{\mathbf{P}}^n\rangle$, but it is **not working** - **participants are not protons**. The new SPS data of the **NA49** and **NA61/SHINE** show that

  $\omega_{\mathbf{p+p}} > \omega_{\mathbf{Ar+Sc}}\ ,\ \omega_{\mathbf{Pb+Pb}}$ at **SPS** (Rybczynski (2013), Aduszkiewicz (2015), Seryakov (2017))

  i.e. $\omega_{\mathbf{P}}$ can be negative, which is forbidden by definition. At higher energies the wounded nucleon model clearly contradicts the data, because

  $\omega_{\mathbf{p+p}} \gg \omega_{\mathbf{Pb+Pb}}$ at **LHC** (V.B. 1606.05358)

- **Suppose** that **current** experimental **methods** are **effective enough** to make scaled variance for the fluctuations from a source close to the measured fluctuations,

$$\omega \simeq \omega_{\mathbf{A}}\,, \qquad \text{then} \quad \alpha = \frac{\omega - \omega_{\mathbf{A}}}{\omega_{\mathbf{A}}} = \langle n_{\mathbf{A}}\rangle \frac{\omega_{\mathbf{P}}}{\omega_{\mathbf{A}}} \ll 1 \quad \text{is a small parameter}$$

- $\omega_{\mathbf{A}}$ competes with $\langle n_{\mathbf{A}}\rangle\, S_{\mathbf{P}}\, \sigma_{\mathbf{P}}$ and $\langle n_{\mathbf{A}}\rangle^2\, \kappa_{\mathbf{P}}\, \sigma^2_{\mathbf{P}}$, **assume** that their ratio is also small,

$$|\beta| = \langle n_{\mathbf{A}}\rangle \frac{|S_{\mathbf{P}}\, \sigma_{\mathbf{P}}|}{\omega_{\mathbf{A}}} \ll 1\,, \qquad |\gamma| = \langle n_{\mathbf{A}}\rangle^2 \frac{|\kappa_{\mathbf{P}}\, \sigma^2_{\mathbf{P}}|}{\omega^2_{\mathbf{A}}} \ll 1\,,$$

The $\alpha,\ |\beta|,\ |\gamma| \ll 1$ is the **mathematical meaning** of the **'small participant fluctuations'**. Then

$$\omega_{\mathbf{A}} \simeq \omega - \alpha\, \omega\,,$$

$$S_{\mathbf{A}}\, \sigma_{\mathbf{A}} \simeq S\, \sigma + \alpha\, (S\, \sigma - 3\, \omega_{\mathbf{A}})\,,$$

$$\kappa_{\mathbf{A}}\, \sigma^2_{\mathbf{A}} \simeq \kappa\, \sigma^2 + \alpha \left(\kappa\, \sigma^2 - 3\, \omega^2_{\mathbf{A}} - 4\, \omega_{\mathbf{A}}\, S_{\mathbf{A}}\, \sigma_{\mathbf{A}}\right)\,, \qquad \alpha,\ |\beta|,\ |\gamma| \ll 1\,.$$

The skewness $S_{\mathbf{A}}\, \sigma_{\mathbf{A}}$ **has a contribution** $\omega_{\mathbf{A}}$, while **kurtosis** $\kappa_{\mathbf{A}}\, \sigma^2_{\mathbf{A}}$ **includes** $\omega^2_{\mathbf{A}}$. Therefore, it is reasonable to introduce two more notations:

$$\delta = \frac{S_{\mathbf{A}}\, \sigma_{\mathbf{A}}}{\omega_{\mathbf{A}}} \quad \text{and} \quad \varepsilon = \frac{\kappa_{\mathbf{A}}\, \sigma^2_{\mathbf{A}}}{\omega^2_{\mathbf{A}}}$$

Then, without any approximation

$$\omega = \omega_{A}(1+\alpha)\,,$$
$$S\,\sigma = \omega_{A}\,\frac{3\alpha+\delta+\alpha\,\beta}{1+\alpha}\,,$$
$$\kappa\,\sigma^2 = \omega_{A}^2\,\frac{3\alpha+\varepsilon+\alpha\left[\,6\beta+\gamma+4\delta\,\right]}{1+\alpha}\,.$$

One can make α, $|\beta|$, $|\gamma|$ $\ll$ 1 small by**decreasing bin width**, or by decreasing $\langle n_{A}\rangle$, choosing **rare particles**, or **net charges** for analysis (Braun-Munzinger, Rustamov, Stachel (2017)).

Close to critical point $\kappa_n \sim \xi^{\frac{5(n-1)-1}{2}}$, $\xi \to \infty$ (Stephanov (2008), Mukherjeea, Venugopalan, Yin (QM2017)), then

$$\alpha \sim \langle n_{A}\rangle^2\,\frac{\omega_{P}}{\xi^2} \to 0\,, \qquad |\beta| \sim \langle n_{A}\rangle^2\,\frac{|S_{P}\,\sigma_{P}|}{\xi^2} \to 0\,, \qquad |\gamma| \sim \langle n_{A}\rangle^4\,\frac{|\kappa_{P}\,\sigma_{P}^2|}{\xi^4} \to 0\,,$$

while $\delta \sim \langle n_{A}\rangle\,\xi^{0.5} \to \infty\,,$ and $\varepsilon \sim \langle n_{A}\rangle^2\,\xi \to \infty\,.$

Away from critical point, e.g. in hadron gas, (Karsch, Redlich (2011))

$$\delta = \varepsilon = \frac{1}{\omega_{A}^2} = \tanh^2(\mu_B/T) \to 0\,, \quad \mu_B/T \to 0$$

$\delta = \varepsilon = 0$ for Gauss, and $\delta = \varepsilon = 1$ for Poisson distribution.

For $\alpha,\ |\beta|,\ |\gamma| \ll 1$, and neglecting **either** δ or ε, one can solve the system

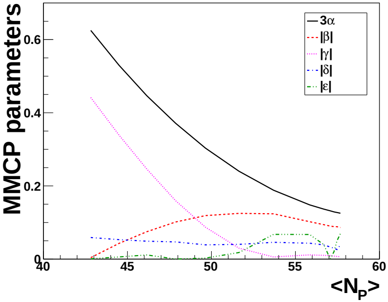
$$\begin{aligned}
\omega &= \omega_{\mathbf{A}}(1+\alpha)\,,\\
S\,\sigma &= \omega_{\mathbf{A}}\,\frac{3\alpha+\delta+\alpha\,\beta}{1+\alpha}\,,\\
\kappa\,\sigma^2 &= \omega_{\mathbf{A}}^2\,\frac{3\alpha+\varepsilon+\alpha\left[\,6\beta+\gamma+4\delta\,\right]}{1+\alpha}\,.
\end{aligned}$$

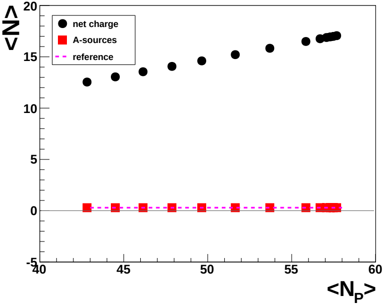
In case of $\delta \ll \varepsilon$ there is a simple **analytic solution**:

$$\begin{aligned}
\alpha &\simeq \frac{S\,\sigma}{3\,\omega - S\,\sigma}\,,\\
\omega_{\mathbf{A}} &\simeq \omega - \frac{S\,\sigma}{3}\,,\\
\kappa_{\mathbf{A}}\,\sigma_{\mathbf{A}}^2 &\simeq \kappa\,\sigma^2 - \omega_{\mathbf{A}}\,S\,\sigma\,, \qquad \alpha,\ |\beta|,\ |\gamma| \ll 1,\quad \delta \ll \varepsilon\,.
\end{aligned}$$

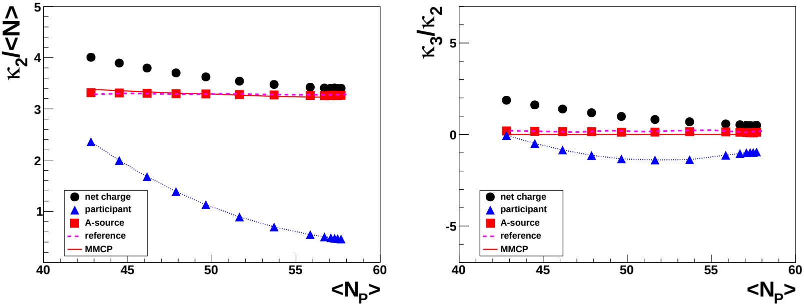
These approximate equations **remove** fluctuations of **participants** and **obtain** the fluctuation of **sources** through ***measured*** values. This is the meaning of the **MMCP method**.

Net electric **charge** in $^{40}_{18}Ar + ^{45}_{21}Sc$ at $p_{lab} = 150$ **GeV/c** and with $y^{CMS} > 0$ in centrality, **left** to **right**: **20%**, 17.5%, 15%, 12.5%, 10%, 7.5%, 5%, 2.5%, 1.5%, 1%, 0.75%, 0.5% and **0.2%**.





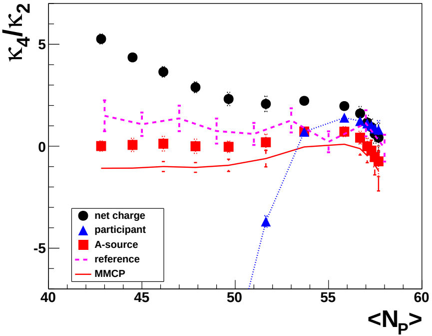
- The condition α, $|\beta|$, $|\gamma| \ll 1$ is valid for the system that we study
- The $|\delta| \ll |\varepsilon|$ is not always satisfied, however, $|\delta|$, $|\varepsilon| < 3\alpha$.
- $\langle N_{\mathbf{net\ charge}} \rangle / \langle N_{\mathbf{P}}^{max} \rangle = (18 + 21)/(40 + 45) \simeq 0.5$, condition $y^{CMS} > 0$ corresponds to $1/2$ of created system, therefore, $\langle n_A \rangle \simeq 0.5 * 0.5 = 0.25 \simeq 0.3$.
- $\langle n_A \rangle$ is **independent** of **centrality**. (V.B., Mackowiak-Pawlowska 1705.01110)



red - what we would like to measure, **black** - what we measure, **blue** - participants, **magenta** - the 'reference' is obtained selecting $\mathbf{N_P = const}$.

- ω_A is **independent** of **centrality** window (V.B., Mackowiak-Pawlowska 1705.01110)
- $\langle n_A \rangle$ **is smal, but** it's **fluctuations** give the **main contribution** $\omega \sim \omega_A \simeq 3.3 \gg \omega_P \gtrsim 0.5$
- ω_P is small, but it **exists**, even if **bin width** goes to **zero**
- The **skewness of a source**, $S_A\,\sigma_A$, is also **independent** on centrality and is close to **zero**.
- The large values of the net charge $S\,\sigma$ are due to the ***second*** moment fluctuations of **participants**, $S_A\,\sigma_A \simeq 0.1 \ll S\,\sigma \simeq 3\,\langle n_A \rangle\,\omega_P > 0$, while $S_P\,\sigma_P \leq 0$, for all δc

Left to **right**: **20%**, 17.5%, 15%, 12.5%, 10%, 7.5%, 5%, 2.5%, 1.5%, 1%, 0.75%, 0.5% and **0.2%**.



red - what we would like to measure, **black** - what we measure, **blue** - participants, **magenta** - the 'reference' is obtained selecting $N_P = const$.

- The **higher** the **order**, the **stronger** is the **dependence** on the **bin width** δc
- In very central collision in a fixed target experiment $N_P \simeq const$, while in collider mode and in peripheral collisions $N_P \neq const$. Therefore, **STAR**, **NA61** and **STAR fixed target** data may be **hard to compare** (V.B., Mackowiak-Pawlowska 1705.01110).

The **CBWC** procedure used by **STAR** means that a value X is measured in r sub-samples, and then summed up with the relative weights w_r of the sub-samples r,

$$X = \sum_r w_r X_r , \qquad w_r = n_r / \sum_r n_r ,$$

where n_r is the number of events in the bin r.

bin width	n_r	w_r	$\langle N \rangle$	$\sigma^2/\langle N \rangle$	$S\sigma$	$\kappa\sigma^2$
0-1%	624827	$\simeq 0.2$	16.88(1)	3.405(3)	0.503(4)	1.2(3)
1-2%	626043	$\simeq 0.2$	16.36(1)	3.417(3)	0.603(4)	2.4(3)
2-3%	611242	$\simeq 0.2$	**15.83(1)**	**3.447(3)**	**0.609(4)**	**2.2(3)**
3-4%	665988	$\simeq 0.2$	15.32(1)	3.467(3)	0.660(4)	2.9(3)
4-5%	623110	$\simeq 0.2$	14.81(1)	3.486(3)	0.741(3)	2.3(3)
0-5%	3151210	1.0	**15.834(4)**	**3.442(1)**	**0.625(2)**	**2.2(1)**

The sub-bin values for the collected number of events n_r, the weight of the sub-bin w_r, the average net charge $\langle N \rangle$, scaled variance $\omega = \kappa_2/\langle N \rangle = \sigma^2/\langle N \rangle$, normalized skewness $S\sigma = \kappa_3/\kappa_2$, and normalized kurtosis $\kappa\sigma^2 = \kappa_4/\kappa_2$ (V.B., Mackowiak-Pawlowska 1705.01110).

- The **CBWC reduces** statistical **uncertainty**.
- The **CBWC gives average measured** fluctuations over the selected sub-bins.

Comparison of different methods for 0%–5% and 0%–20% centrality

0 – 5%	CBWC	net charge	reference	**A-source**	**MMCP**
$\kappa_2/\langle N\rangle$	3.4	3.5	3.3	**3.3**	**3.3**
κ_3/κ_2	0.6	0.7	0.2	0.2	0
κ_4/κ_2	2.2(1)	2.2(2)	1.3(5)	0.7(2)	0.0(2)

0 – 20%	net charge	**A-source**	**MMCP**
$\kappa_2/\langle N\rangle$	4.0	**3.3**	**3.4**
κ_3/κ_2	1.9	0.2(1)	0
κ_4/κ_2	5.3(2)	0.0(2)	-1.1(2)

- The **MMCP coincides** within the uncertainty with the **A-source** for $\omega = \kappa_2/\langle N\rangle$ in the **0 – 5%** centrality, and deviates only for **2%** from A-source in the **0 – 20%** centrality.
- The **CBWC overestimates** $S\sigma = \kappa_3/\kappa_2$ and $\kappa\sigma^2 = \kappa_4/\kappa_2$ **three times**.
- The κ_3/κ_2 of sources is zero by definition in the **MMCP**. It **agrees** within three standard deviations with the **A-source** generated by EPOS.
- The κ_4/κ_2, in MMCP for 0-5% bin underestimates the A-source. This is the result of neglecting skewness of the participants $S_P\,\sigma_P$ (V.B., Mackowiak-Pawlowska 1705.01110).

- The average number of particles produced by a source, $\langle n_A \rangle$, and it's fluctuations of the second, ω_A, and the third order, $S_A\,\sigma_A$, **do not depend on centrality**.
- However, the fourth order fluctuations of a source, $\kappa_A\,\sigma_A^2$, **change non-monotonously** for the bin width smaller than **5%** in the range from **−1** to **+1**. This effect **should be studied**, especially **for higher moments**.
- $S\,\sigma$ and $\kappa\,\sigma^2$ depend on the lower order fluctuations, which give the largest contribution to their values. The fluctuations from a source that one would like to access, $S_A\,\sigma_A$ and $\kappa_A\,\sigma_A^2$, **are almost zero** in the considered example:

$$S\,\sigma \;\simeq\; 3\,\langle n_A \rangle\,\omega_P \;>\; 0\,, \quad \kappa\,\sigma^2 \;\simeq\; 3\,\langle n_A \rangle\,\omega_A\,\omega_P \;>\; 0\,, \quad S_A\,\sigma_A \;\simeq\; 0\,, \quad \kappa_A\,\sigma_A^2 \;\simeq\; 0\,.$$

- The **CBWC** is **unable to remove participant fluctuations**. It gives the average of the 'total', i.e. non-processed fluctuations, which are dominated by participant fluctuations. Its results **depend** on the **width**, **weight**, and the **position** of the sub-bins, i.e. may give arbitrary result.
- **The same is true if one** does not mix the bins, but **selects a particular centrality bin** and increases statistics.

Conclusions II

- A **fluctuation of participants** and a contribution from lower moments may **persist** in higher moments, even if the bin width approaches zero.
- Summary of the methods reducing participant fluctuations:

  (i) Narrowing centrality bin gives $\omega = \omega_A + X$, while ω_A **is needed**,

  (ii) CBWC gives $\langle\omega\rangle = \langle\omega_A\rangle + \langle X\rangle$, while ω_A **is needed** ,

  (iii) Strongly Intensive quantities give $\langle n_B\rangle\, \omega_A \pm \langle n_A\rangle\, \omega_B$, while ω_A or ω_B **is needed**,

  (iv) MMCP gives ω_A (!), if X/ω_A is a small parameter.

- We introduce the way how to **quantify smallness** and **largeness** of fluctuations:

$$\alpha = \langle n_A\rangle \frac{\omega_P}{\omega_A}\,, \quad \beta = \langle n_A\rangle \frac{S_P\,\sigma_P}{\omega_A}\,, \quad \gamma = \langle n_A\rangle^2 \frac{\kappa_P\,\sigma_P^2}{\omega_A^2}\,, \quad \delta = \frac{S_A\,\sigma_A}{\omega_A}\,, \quad \varepsilon = \frac{\kappa_A\,\sigma_A^2}{\omega_A^2}\,.$$

- The **MMCP works well for** the **scaled variance**. Is it possible to obtain $S_A\sigma_A$ and $\kappa_A\sigma_A^2$ measuring κ_5 and κ_6, and **further developing MMCP?**

Thank you!

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