An extension of Ramo's theorem to include resistive elements

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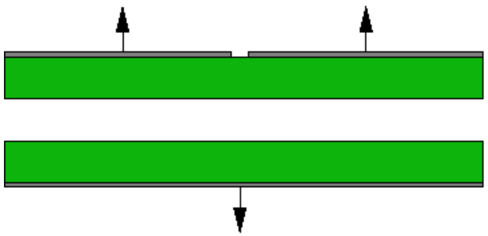
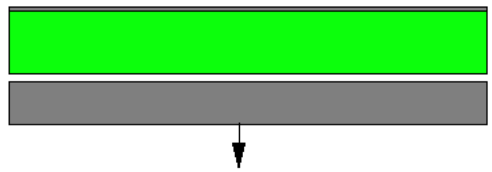
With the introduction of resistive elements in the detector volumes, like for Restistive Plate Chambers or resistive MICROMEGAs, the signal induced on the readout electrodes will not only be determined by the movement of the primary charges but also by the movement of charges inside these resistive elements. This report will present an extension of Ramo's theorem to include these effects, that might have an application on solid state detectors where resistive layers are used either to evacuate charge or to introduce discharge protection.

Detectors with resistive elements

Resistive Plate Chambers



2mm Bakelite, $\rho \approx 10^{10}\ \Omega cm$

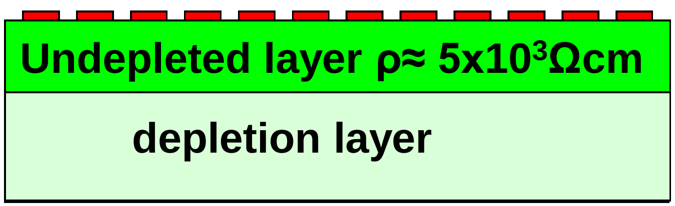


3mm glass, $\rho \approx 2 \times 10^{12}\ \Omega cm$

0.4mm glass, $\rho \approx 10^{13}\ \Omega cm$

Thin layers of ≈100kΩ/☐ to 10MΩ/☐ to apply HV.

Silicon Detectors



Resistive MICROMEGAS, TGCs

Thin layers of ≈100kΩ/☐ to 10MΩ/☐ for discharge protection and to apply HV.

Extended Theorems for Signal Induction in Particle Detectors VCI 2004

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Abstract

Most particle detectors are based on the principle that charged particles leave a trail of ionization in the detector and that the movement of these charges in an electric field induces signals on the detector electrodes. Assuming detector elements that are insulating and electrodes with infinite conductivity one can calculate the signals with an electrostatic approximation using the so called 'Ramo Theorem'. This is the standard way for calculation of signals e.g. in wire chambers and silicon detectors. In case the detectors contain resistive elements, which is e.g. the case in resistive plate chambers or underdepleted silicon detectors, the time dependence of the signals is not only given by the movement of the charges but also by the time dependent reaction of the detector materials. Using the quasi static approximation of Maxwell's equations we present an extended formalism that allows the calculation of induced signals for detectors with general materials by time dependent weighting fields. As examples we will discuss the signals in resistive plate chambers and underdepleted silicon detectors.

W. Riegler, *Extended theorems for signal induction in particle detectors VCI 2004*, *Nucl. Instrum. Meth.* A 535 (2004) 287.

Electric fields, weighting fields, signals and charge diffusion in detectors including resistive materials

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ABSTRACT: In this report we discuss static and time dependent electric fields in detector geometries with an arbitrary number of parallel layers of a given permittivity and weak conductivity. We derive the Green's functions i.e. the field of a point charge, as well as the weighting fields for readout pads and readout strips in these geometries. The effect of 'bulk' resistivity on electric fields and signals is investigated. The spreading of charge on thin resistive layers is also discussed in detail, and the conditions for allowing the effect to be described by the diffusion equation is discussed. We apply the results to derive fields and induced signals in Resistive Plate Chambers, MICROMEGAS detectors including resistive layers for charge spreading and discharge protection as well as detectors using resistive charge division readout like the MicroCAT detector. We also discuss in detail how resistive layers affect signal shapes and increase crosstalk between readout electrodes.

KEYWORDS: Charge induction; Detector modelling and simulations II (electric fields, charge transport, multiplication and induction, pulse formation, electron emission, etc); Micropattern gaseous detectors (MSGC, GEM, THGEM, RETHGEM, MHSP, MICROPIC, MICROMEGAS, InGrid, etc); Resistive-plate chambers

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Quasistatic Approximation of Maxwell's Equations

To include the frequency dependence of ε and σ we work in the Laplace domain i.e. we write

$$L[\vec{E}(\vec{x},t)] = \vec{\overline{E}}(\vec{x},s) \qquad L[\frac{\partial \vec{E}(\vec{x},t)}{\partial t}] = s\vec{\overline{E}}(\vec{x},s) \qquad \text{etc.} \tag{2}$$

where we have assumed that at $t=0$ all fields and charges are zero. Maxwell's equations for a linear isotropic medium with permittivity $\varepsilon(\vec{x},s)$ and conductivity $\sigma(\vec{x},s)$ then read as

$$\vec{\nabla}\vec{\overline{D}} = \overline{\rho} \quad \vec{\overline{D}} = \varepsilon\vec{\overline{E}} \qquad \vec{\nabla}\vec{\overline{B}} = 0 \quad \vec{\overline{B}} = \mu\vec{\overline{H}} \tag{3}$$

$$\vec{\nabla}\times\vec{\overline{E}} = -s\vec{\overline{B}} \qquad \vec{\nabla}\times\vec{\overline{H}} = \vec{\overline{j}}_e + \sigma\vec{\overline{E}} + s\vec{\overline{D}} \tag{4}$$

where $\vec{j}_e$ is an 'externally impressed' current that is connected with an 'external' charge density by $\vec{\nabla}\vec{\overline{j}}_e = -s\overline{\rho}_e$. Assuming weak conductivity σ we can set

$$\vec{\nabla}\times\vec{\overline{E}} = -s\vec{\overline{B}} = 0 \quad \Rightarrow \vec{\overline{E}} = -\vec{\nabla}\overline{\Phi} \tag{5}$$

and by taking the divergence of the second equation in (4) we find

$$\vec{\nabla}[\sigma(\vec{x},s)\vec{\nabla}]\overline{\Phi}(\vec{x},s) + \vec{\nabla}[\varepsilon(\vec{x},s)\vec{\nabla}]s\overline{\Phi}(\vec{x},s) = -s\rho_e(\vec{x},s) \tag{6}$$

which we can write as

$$\vec{\nabla}[\epsilon(\vec{x},s)\vec{\nabla}]\overline{\Phi}(\vec{x},s) = -\overline{\rho}_e(\vec{x},s) \quad \text{with} \quad \epsilon(\vec{x},s) = \varepsilon(\vec{x},s) + \frac{1}{s}\sigma(\vec{x},s) \tag{7}$$

Assuming 'slow' changes of the fields we can neglect Faraday's induction law …

… and we find back the Poisson equation with an effective permittivity in the Laplace domain.

j = σ E

σ=1/ρ

σ … conductivity

ρ … volume resistivity

Quasistatic Approximation

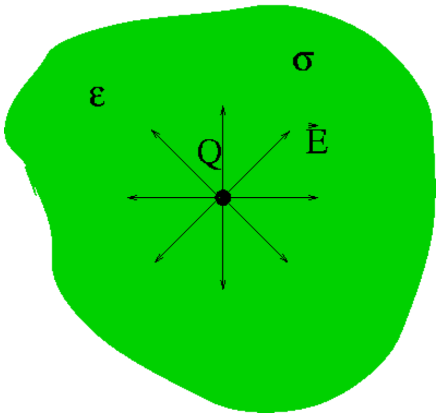
equations: Knowing the solution of the Poisson equation for a charge distribution $\rho(\vec{x})$ embedded in a geometry of a given permittivity $\varepsilon(\vec{x})$, we find the time dependent solution (in the Laplace domain with parameter s) for an 'externally impressed' charge density $\rho_e(\vec{x}, s)$ and a geometry that in addition includes a finite (weak) conductivity $\sigma(\vec{x})$ by replacing $\varepsilon(\vec{x})$ with $\varepsilon(\vec{x}) + \sigma(\vec{x})/s$ and $\rho(\vec{x})$ with $\rho_e(\vec{x}, s)$. For detector applications the volume resistivity $\rho(\vec{x}) = 1/\sigma(\vec{x})$ is traditionally used.

As an example we look at the potential of a point charge Q in a medium of constant permittivity ε, which is given by

$$\phi(r) = \frac{Q}{4\varepsilon\pi\, r} \tag{1}$$

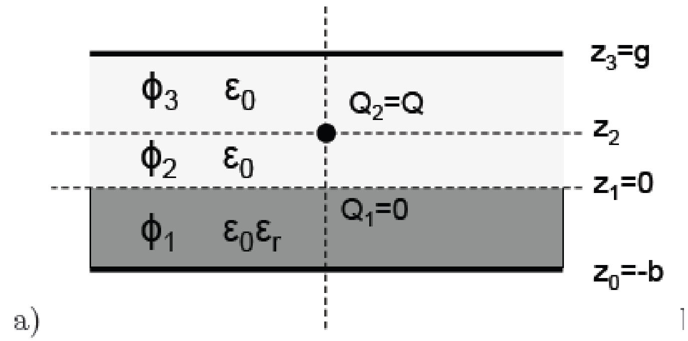
In case the medium has a conductivity σ and we place the 'external' charge Q at $t = 0$, i.e. $Q(t) = Q\Theta(t)$ and therefore $Q(s) = Q_0/s$, we replace ε by $\varepsilon + \sigma/s$ and Q by Q/s and perform the inverse Laplace transform, which gives

$$\phi(r, s) = \frac{Q}{4\pi(s\varepsilon + \sigma)r} \quad \rightarrow \quad \phi(r, t) = \frac{Q}{4\pi\varepsilon\, r}\, e^{\frac{-t}{\tau}} \qquad \tau = \varepsilon/\sigma = \rho\varepsilon \tag{2}$$



Examples

Single Gap RPC



$$\phi_2(r,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{r^2+(z_2-z)^2}} + \frac{1}{2\pi}\int_0^\infty J_0(kr)\left[f_2(k,z) - \frac{Q}{2\varepsilon_0}\,e^{-k(z_2-z)}\right]dk$$

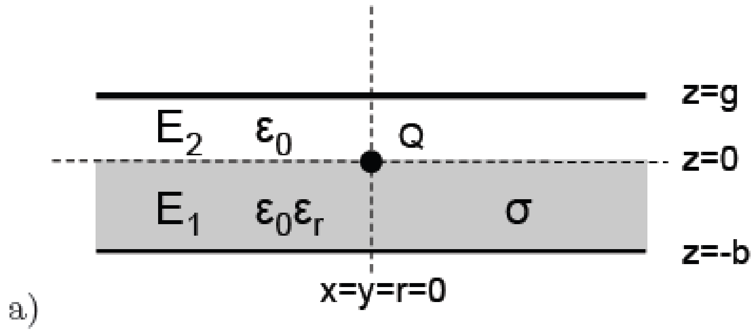
$$\phi_3(r,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{r^2+(z-z_2)^2}} + \frac{1}{2\pi}\int_0^\infty J_0(kr)\left[f_3(k,z) - \frac{Q}{2\varepsilon_0}\,e^{-k(z-z_2)}\right]dk$$

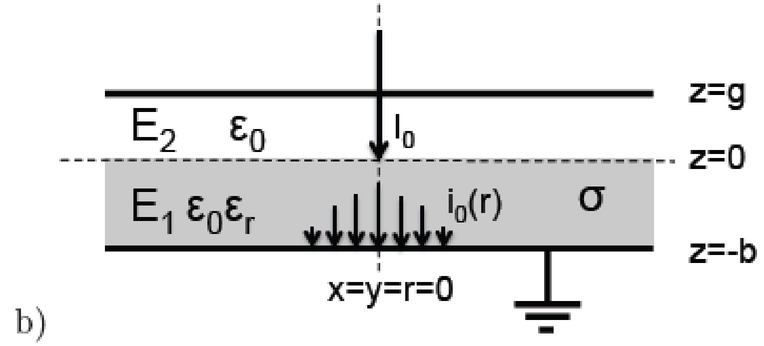
$$\begin{aligned}
f_1(k,z) &= Q\sinh(k(b+z))\sinh(k(g-z_2))/(\varepsilon_0 D(k)) \\
f_2(k,z) &= Q\sinh(k(g-z_2))[\sinh(bk)\cosh(kz)+\varepsilon_r\cosh(bk)\sinh(kz)]/(\varepsilon_0 D(k)) \\
f_3(k,z) &= Q\sinh(k(g-z))[\sinh(bk)\cosh(kz_2)+\varepsilon_r\cosh(bk)\sinh(kz_2)]/(\varepsilon_0 D(k))
\end{aligned}$$

with

$$D(k) = \sinh(bk)\cosh(gk) + \varepsilon_r\cosh(bk)\sinh(gk)$$

Single Gap RPC





$\varepsilon_1 = \varepsilon_0\varepsilon_r + \sigma/s \quad \varepsilon_2 = \varepsilon_0$ $\quad Q(t) = I_0\, t$ i.e. $Q(s) = I_0/s^2$ $\quad \lim_{t\to\infty} E(r,z,t) = \lim_{s\to 0} sE(r,z,s)$

$$i_0(r) = -\sigma E_1(r, z=-b) = \frac{I_0}{b^2\pi}\int_0^\infty \frac{1}{2} J_0\left(y\frac{r}{b}\right)\frac{y}{\cosh(y)}dy \qquad r_{50\%} \approx b \quad r_{90\%} \approx 2.3b \quad r_{99\%} \approx 3.9b$$

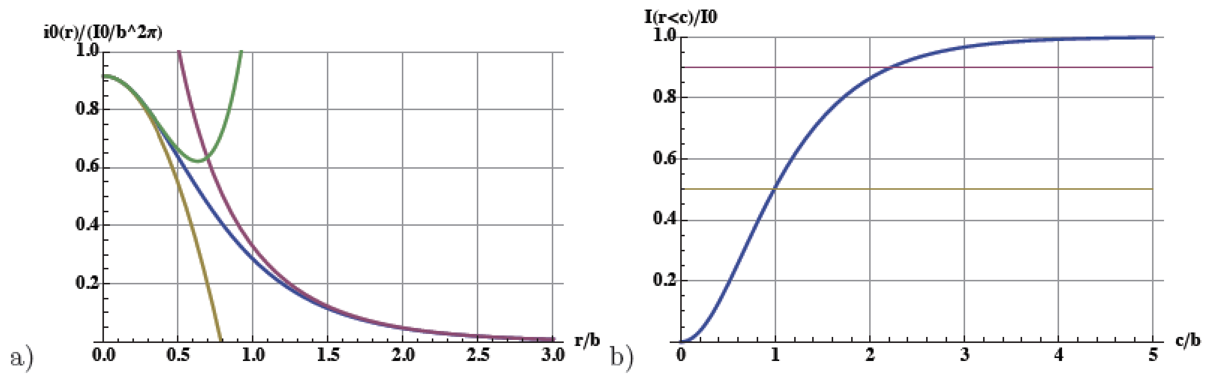
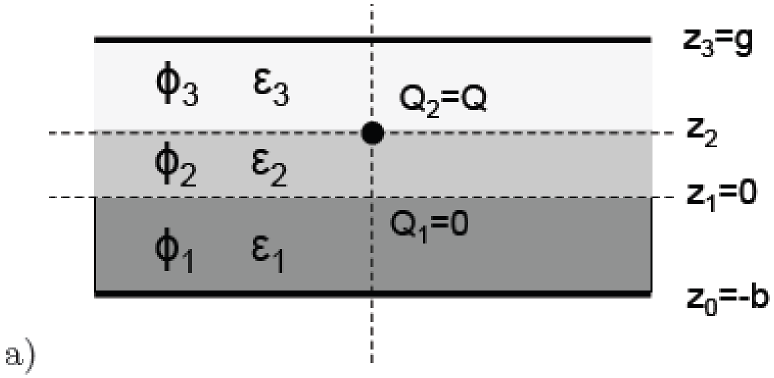
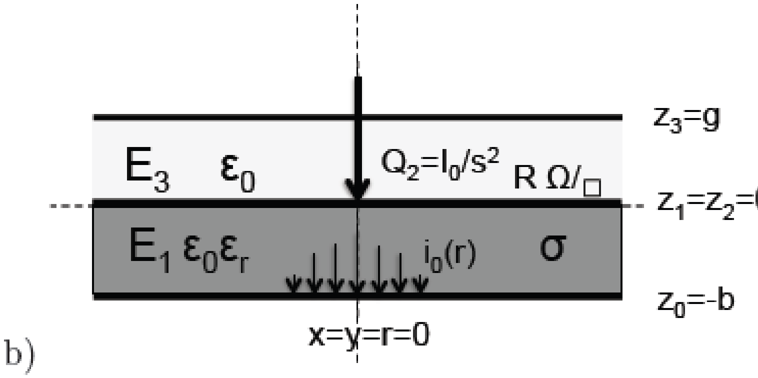


Figure 11: a) Current density $i_0(r)$ at $z=-b$. The exact curve together with the 2^{nd} order and 4^{th} order approximation from Eq. 94 and the exponential approximation from Eq. 96. b) Total current at $z=-b$ flowing inside a radius r from Eq. 97.

Single Gap RPC, increasing rate capability by a surface R



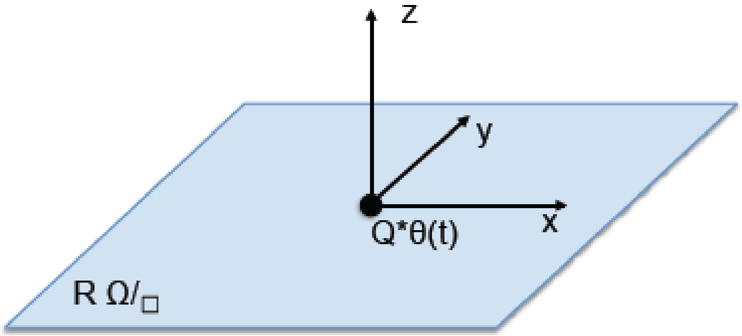


$$i_0(r) = -\sigma E_1(r, z=-b) = \frac{I_0}{b^2\pi}\int_0^\infty \frac{1}{2} J_0\left(y\frac{r}{b}\right)\frac{y}{\cosh(y)+\frac{y}{\beta^2}\sinh(y)}dy \qquad \beta^2 = R\sigma b$$

$$R < 1/(\sigma b) \rightarrow \beta^2 \ll 1$$

$$r_{50\%} \approx 1.26\sqrt{\frac{b}{R\sigma}} \quad r_{90\%} \approx 3.21\sqrt{\frac{b}{R\sigma}} \quad r_{99\%} \approx 5.77\sqrt{\frac{b}{R\sigma}}$$

Infinitely extended thin resistive layer



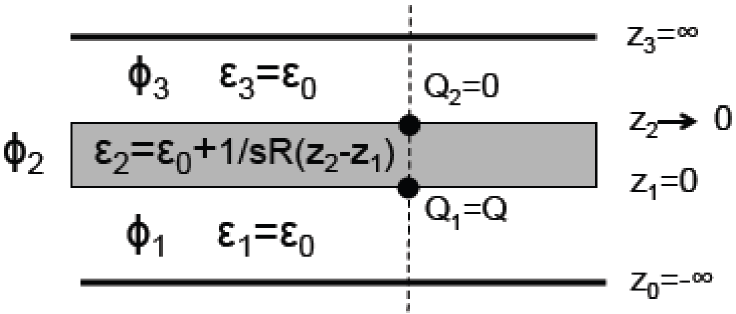


Figure 15: a) A resistive layer with surface resistance R [Ω/square]. b) The fields for this single layer can be calculated from the indicated 3-layer geometry by performing the indicated limits of the expressions for z_0, z_2, z_3.

Infinitely extended resistive layer

First we investigate an infinitely extended layer as shown in Fig. 12a. The charge Q will cause

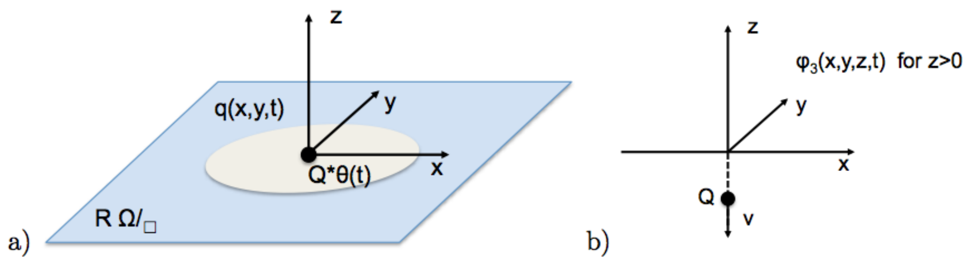


Figure 12: a) A point charge placed at an infinitely extended resistive layer at $t = 0$. b) The solution for the time dependent potential is equal to a point charge moving with velocity v along the z-axis.

$$\phi_1(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(-z+vt)^2}} \qquad \phi_3(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(z+vt)^2}} \tag{111}$$

We therefore conclude that the field due to a point charge placed on an infinite resistive layer at $t = 0$ is equal to the field of a charge Q that is moving with a velocity $\boxed{v = 1/2\varepsilon_0 R}$ away from the layer along the $z-$axis. As an example for a surface resistivity of $R = 1\,\mathrm{M}\Omega/\text{square}$ the velocity is $5.6\,\mathrm{cm}/\mu\mathrm{s}$.

The time dependent surface charge density on the resistive surface is given by

$$q(r,t) = \varepsilon_0\frac{\partial\phi_1}{\partial z}|_{z=0} - \varepsilon_0\frac{\partial\phi_3}{\partial z}|_{z=0} \tag{112}$$

which evaluates to

$$\boxed{q(r,t) = \frac{Q}{2\pi}\frac{vt}{\sqrt{(r^2+v^2t^2)^3}}} \tag{113}$$

The total charge on the resistive surface $Q_{tot} = \int_0^\infty 2r\pi q(r,t)dr$ is equal to Q at any time. The peak and the FWHM of the charge density are given by

$$q_{max} = \frac{Q}{2\pi}\frac{1}{v^2t^2} \qquad FWHM = 2(4^{1/3}-1)^{1/2} \approx 1.53vt \tag{114}$$

The charge is therefore 'diffusing' with a velocity v, and does not assume a gaussian shape as expected from a diffusion effect but has $1/r^3$ tails for large values of r. The radial current $I(r)$ at distance r are given by

$$I(r) = \frac{2r\pi}{R}E(r) = -\frac{2r\pi}{R}\frac{\partial\phi_1}{\partial r}|_{z=0} = \frac{Qvr^2}{(r^2+v^2t^2)^{3/2}} \tag{115}$$

It is easily verified that the rate of change of the total charge inside a radius r i.e. $dQ_r(t)/dt = d/dt\int_0^r 2r'\pi q(r',t),dr'$ is equal the the current $I(r)$.

A point charge Q is placed on an infinitely extended resistive layer with surface resistivity of R Ohms/square at t=0.

What is the charge distribution at time t>0 ?

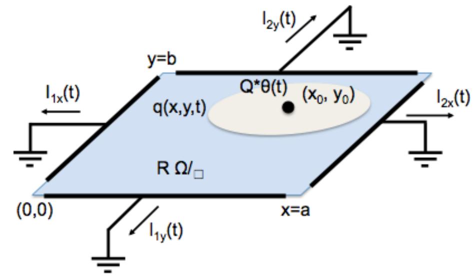
Note that this is not governed by any diffusion equation.

The solution is far from a Gaussian.

The timescale is governed by the velocity $v=1/(2\varepsilon_0 R)$

Resistive layer grounded on a rectangle

Next we assume a rectangular grounded boundar
Q at position x_0, y_0 at $t = 0$ as indicated in Fig. 14ε



a)

Figure 14: a) A point charge placed on a resistive layer tha
resistive layer that is grounded on at $x = 0$ and $x = a$ but in:

expression Eq. 42. Assuming the currents pointing to the outside of the boundary, the currents flowing through the 4 boundaries are

$$I_{1x} = -\frac{1}{R}\int_0^b -\frac{\partial\phi_1}{\partial x}|_{x=0}dy \qquad I_{2x} = \frac{1}{R}\int_0^b -\frac{\partial\phi_1}{\partial x}|_{x=a}dy \tag{120}$$

$$I_{1y} = -\frac{1}{R}\int_0^a -\frac{\partial\phi_1}{\partial x}|_{y=0}dx \qquad I_{2y} = \frac{1}{R}\int_0^a -\frac{\partial\phi_1}{\partial x}|_{y=b}dx \tag{121}$$

which evaluates to

$$I_{1x}(t) = \frac{4Qv}{a^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{l}{m}\frac{1}{k_{lm}}\left[1-(-1)^m\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{122}$$

$$I_{2x}(t) = \frac{4Qv}{a^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{l}{m}\frac{1}{k_{lm}}(-1)^l\left[(-1)^m-1\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{123}$$

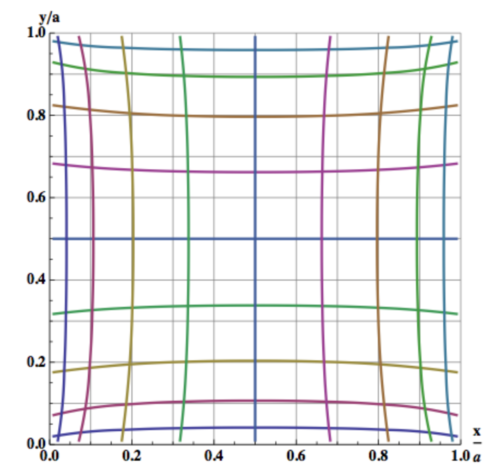
$$I_{1y}(t) = \frac{4Qv}{b^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{l}\frac{1}{k_{lm}}\left[1-(-1)^l\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{124}$$

$$I_{2y}(t) = \frac{4Qv}{b^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{l}\frac{1}{k_{lm}}(-1)^m\left[(-1)^l-1\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{125}$$

In case we want to know the total charge flowing through the grounded sides we have to integrate the above expressions from $t = 0$ to ∞ which results in the same expressions and just $e^{-k_{lm}vt}$ replaced by $1/(k_{lm}v)$. These measured currents can be used to find the position of the charge, a principle that is applied in the MicroCat detector. As an example, Fig. 15 shows the correction map that has to be applied in case one just uses linear interpolation of the measured charges.

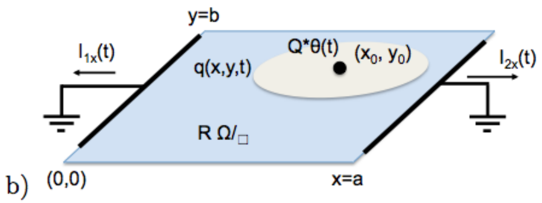
A point charge Q is placed on a resistive layer with surface resistivity of R Ohms/square that is grounded on 4 edges

What are the currents induced on these grounded edges for time t>0 ?



› for the case where the position of the charge is determined by linear i
boundaries of the geometry in Fig. 14a.

Resistive layer grounded on two sides and i



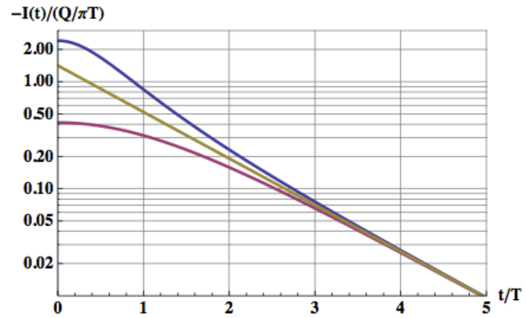


Figure 16: Currents for the geometry of Fig. 14b for $x_0 = a/4$.

5.4. Resistive layer grounded at $\pm a$ and insulated at $\pm b$.

In case the resistive layer is grounded at $x = 0, x = a$ and insulated at $y = 0, y = b$, as shown in Fig. 14, the currents are only flowing into the grounded elements at $x = 0$ and $x = a$. We use Eq. 43 and with some effort the summation can be achieved and evaluates to

$$I_{1x}(t) = -\frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=0}dy = -\frac{Q}{\pi T}\frac{\sin(\pi\frac{x_0}{a})}{\cosh(\frac{t}{T}) - \cos(\pi\frac{x_0}{a})} \tag{126}$$

$$I_{2x}(t) = \frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=a}dy = -\frac{Q}{\pi T}\frac{\sin(\pi\frac{x_0}{a})}{\cosh(\frac{t}{T}) + \cos(\pi\frac{x_0}{a})} \tag{127}$$

with $T = \frac{2a\varepsilon_0 R}{\pi} = \frac{a}{\pi v}$. For large times both expressions tend to

$$I_{1x}(t) = I_{2x}(t) \approx -\frac{2Q}{\pi T}\cos\left(\pi\frac{x_0}{a}\right) e^{-t/T} \tag{128}$$

Fig. 16 shows the two currents for a charge deposit at position $x_0 = a/4$ together with the asymptotic expression from Eq. 128. The total charge that is flowing through the grounded ends is given by

$$q_1 = \int_0^\infty I_{1x}(t)dt = Q\frac{a - x_0}{a} \qquad q_2 = \int_0^\infty I_{2x}(t)dt = Q\frac{x_0}{a} \tag{129}$$

so we learn that the charges are just shared in proportion to the distance from the grounded boundary, equal to the resistive charge division.

Possibility of position measurement in RPC and Micromegas

A point charge Q is placed on a resistive layer with surface resistivity of R Ohms/square that is grounded on 2 edges and insulated on the other two.

What are the currents induced on these grounded edges for time t>0 ?

The currents are monotonic.

Both of the currents approach exponential shape with a time constant T.

The measured total charges satisfy the simple resistive charge division formulas.

Uniform currents on resistive layers

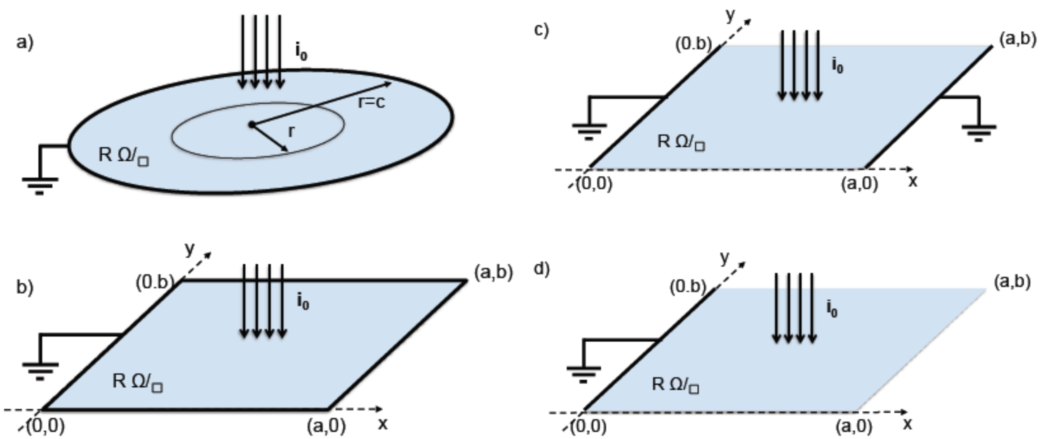


Figure 25: A uniform current 'impressed' on the resistive layer will result in a potential distribution that depends strongly on the boundary conditions. The 4 geometries shown in this figure are discussed.

Uniform illumination of the resistive layers results in 'chargeup' and related potentials.

In this section we want to discuss the potentials that are created on thin resistive layers for uniform charge deposition. In detectors like RPCs and Resistive Micromegas such resistive layers are used for application of the high voltage and for spark protection. The resistivity must be chosen small enough to ensure that potentials that are established on these layers due to charge-up are not influencing the applied electric fields responsible for the proper detector operation. If such detectors are in an environment of uniform particle irradiation the situation can be formulated by placing a uniform 'externally impressed' current per unit area i_0 [A/cm^2] on the resistive layer. For illustration we use the example of a resistive layer an absence of any grounded planes from Section 5. First we want to investigate the geometry shown in Fig. 25a) where the layer is grounded on a circle at $r = c$. The charge dq placed on an infinitesimal area at position r_0, ϕ_0 after time t is given by $dq(t) = i_0 r_0 dr_0 d\phi_0 t$, or in the Laplace domain $dq(s) = i_0 r_0 dr_0 d\phi_0 / s^2$. We therefore have to replace Q/s in Eq. 119 by $q(s)$, which results in

$$f_1(k,z,s) = \frac{i_0}{s}\frac{R r_0 dr_0 d\phi_0}{k + 2\varepsilon_0 R s}\, e^{kz} \qquad f_2(k,z,s) = \frac{i_0}{s}\frac{R r_0 dr_0 d\phi_0}{k + 2\varepsilon_0 R s}\, e^{-kz} \tag{160}$$

Since we want to know the steady situation for long times i.e. for $t \to \infty$ we $f(k,z,t \to \infty) = \lim_{s\to 0} s f(k,z,s)$ and have

$$f_1(k,z) = \frac{R i_0 r_0 dr_0 d\phi_0}{k}\, e^{kz} \qquad f_2(k,z) = \frac{R i_0 r_0 dr_0 d\phi_0}{k}\, e^{-kz} \tag{161}$$

$$\phi_1(r,z) = \phi_3(r,-z) = 2c^2 R i_0 \sum_{l=1}^{\infty} \frac{J_0(j_{0l} r/c)}{j_{0l}^3 J_1(j_{0l})} \, e^{j_{0l} z/c} \tag{162}$$

For $z = 0$ i.e. on the surface of the resistive layer, the expression can be summed and we have

$$\phi_1(r, z=0) = \phi_3(r, z=0) = \frac{1}{4} R i_0 (c^2 - r^2) \tag{163}$$

This expression can also be derived in an elementary way: the total current on a disc of radius r i.e. $r^2 \pi i_0$, is equal to the total radial current flowing at radius r i.e. $2 r \pi E_r / R$. This defines the radial field inside the layer to $E_r = R i_0 r/2$. With the boundary condition $\phi(c) = \int_0^c E_r(r) dr = 0$ we find back the above expression. The maximum potential is therefore in the centre of the disc and is equal to

$$\phi(r=0) = \frac{c^2 \pi R i_0}{4\pi} = \frac{1}{4\pi} R I_{tot} \approx 0.08 \, R I_{tot} \tag{164}$$

To find the potentials in the rectangular geometry of Fig. 25b we again have f_1, f_2 from Eq. 161 we just have to replace $r_0 dr_0 d\phi_0$ by $dx_0 dy_0$ and perform the integration $\int_0^a dx_0 \int_0^b dy_0$ of Eq. 47, which results in

$$\phi_1(x,y,z) = \phi_3(x,y,-z) = ab R i_0 \frac{4}{\pi^4} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{[1-(-1)^l][1-(-1)^m] \sin(l\pi x/a) \sin(m \pi y/b)}{l^3 m b/a + m^3 l a/b} \, e^{k_{lm} z} \tag{165}$$

The expression cannot be written in closed form but converges quickly, so numerical evaluation is straight forward. The peak of the potential can be found by setting $d\phi_1/dx = 0, d\phi_1/dy = 0$ and is found at $x = a/2, y = b/2$, which is also evident by the symmetry of the geometry. The maximum potential on the resistive layer is then

$$\phi_{max} = \phi(a/2, b/2, z=0) = \frac{1}{8} R i_0 a^2 b^2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{128}{\pi^4} \frac{(-1)^{l+m}}{b^2 (2l-1)^3 (2m-1) + a^2 (2m-1)^3 (2l-1)} \tag{166}$$

For a square geometry $(b = a)$ the sum evaluates to ≈ 0.59 so the peak voltage in the center is

$$\phi_{max} \approx 0.074 R i_0 a^2 = 0.074 \, R I_{tot} \tag{167}$$

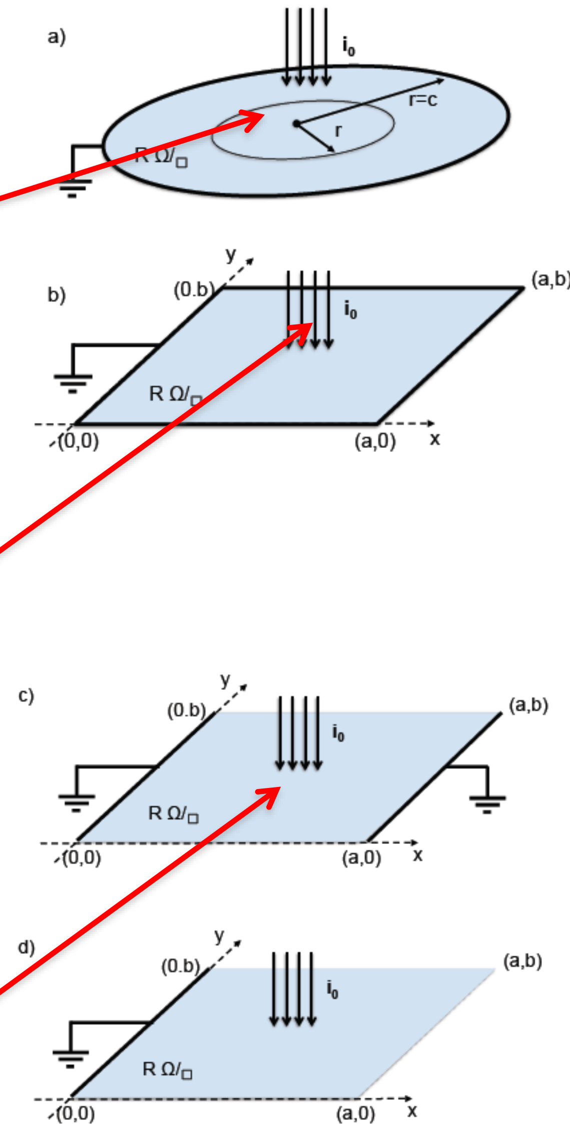
We see that the value is only less than 10 % different from the peak voltage for the circular boundary in Eq. 164.

For uniform illumination of the geometry Fig. 25c that is grounded at $x = 0, a$ and insulated at $y = 0, b$ we use expression Eq. 48 and proceed as before and find

$$\phi_1(x,z) = \phi_3(x,-z) = 2 R i_0 a^2 \sum_{l=1}^{\infty} \frac{(1-(-1)^l) \sin(l\pi x/a)}{l^3 \pi^3} \, e^{l\pi z/a} \tag{168}$$

The potential is is independent of y and for $z = 0$ the sum can be written inclosed form

$$\phi_1(x, z=0) = \frac{1}{2} R i_0 (ax - x^2) \qquad \phi_{max} = \frac{1}{8} a^2 R i_0 \tag{169}$$



Infinitely extended resistive layer with parallel ground plane

Assuming an infinitely extended geometry, the time dependent charge density evaluates to

$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{2}\int_0^\infty \kappa J_0(\kappa\frac{r}{b})\,\exp\left[-\kappa(1-e^{-2\kappa})\frac{t}{T}\right]d\kappa \qquad T=\frac{b}{v}=2b\varepsilon_0 R \tag{134}$$

It can be verified that $\int_0^\infty 2r\pi q(r,t)dr = Q$ at any time. For long times i.e. large values of t/T we can approximate the exponent of the above expression by

$$-\kappa(1-e^{-2\kappa})\frac{t}{T} \approx -2\kappa^2\frac{t}{T} \tag{135}$$

and the integral evaluates to

$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{8t/T}e^{-\frac{r^2}{8b^2t/T}} \tag{136}$$

In analogy to the one dimensional transmission line, the discussed geometry is often assumed to be defined by the two dimensional diffusion equation

$$\frac{\partial q}{\partial t} = h\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right) \qquad h=1/RC \quad C=\frac{\varepsilon_0}{b} \tag{137}$$

where C is the capacitance per unit area between the resistive layer and the grounded plate. The solution of this equation for a point charge Q put at $r=0, t=0$ evaluates exactly to the above Gaussian expression In Fig. 19 the charge distribution from Eq. 134 is compared to the above Gaussian as well as Eq. 113 for the geometry without a ground plane. Although the order of magnitude is similar, the solution of the diffusion equation does not work very well. The reason for the discrepancy can be understood when investigating how Eq. 135 is derived: the current $\vec{j}(x,y,t)$ flowing inside the resistive layer is related to the electric field $\vec{E}(x,y,t)$ in the resistive layer by $\vec{j}=\vec{E}/R$. The relation between the current and the charge density $q(x,y,t)$ is $\vec{\nabla}\vec{j} = -\partial q/\partial t$. With $\vec{E}=-\vec{\nabla}\phi$ we then get

$$\frac{\partial q}{\partial t} = \frac{1}{R}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right) \tag{138}$$

If we set $q=C\phi$ we have the diffusion equation Eq. 135. This relation between voltage and charge($Q=CU$) is however only a good approximation if the charge distribution does not have a significant gradient over distances of the order of b. For small times when the charge distribution is very peaked around zero this is certainly not a good approximation. It means that for long times when the distribution if very broad when compared to the distance b the two solutions should approach each other. Indeed this can be seen if we calculate the current that is induced on the grounded plate, which we do next. The presence of the charge on the resistive layer induces a charge on the grounded metal plane. If we assume that the metal plane is segmented into strips, as shown in Fig. 20b, we can calculate the induced charge through the electric field on the surface of the plane. Assuming a strip centred at $x=x_p$ with a width of w and infinite extension in y direction, we find the induced charge to

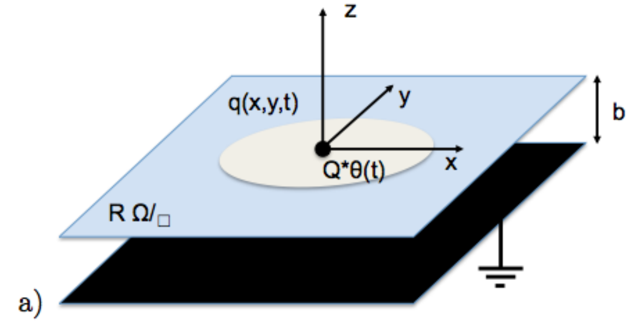


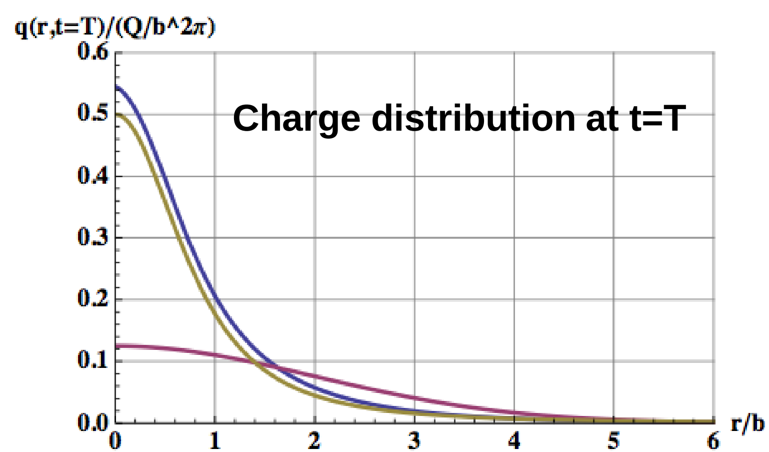
Figure 18: a) An infinitely extended resistive layer in presenc
radius $r=c$.

A point charge Q is placed on an infinitely extended resistive layer with surface resistivity of R Ohms/square and a parallel ground plane at t=0.

What is the charge distribution at time t>0 ?

This process is in principle NOT governed by the diffusion equation.

In practice it is governed by the diffusion equation for long times.



Infinitely extended resistive layer with parallel ground plane

$$Q_{ind}(t) = \int_{x_p-w/2}^{x_p+w/2} \int_{\infty}^{\infty} -\varepsilon_0 \frac{\partial \phi_1}{\partial z}|_{z=-b} dy dx \tag{139}$$

which evaluates to

$$Q_{ind}(t) = \frac{2Q}{\pi} \int_0^{\infty} \frac{1}{\kappa} \cos(\kappa \frac{x_p}{b}) \sin(\kappa \frac{w}{2b}) \exp\left[-\kappa - \kappa(1-e^{-2\kappa})\frac{t}{T}\right] d\kappa \tag{140}$$

The solution of the diffusion equation assumes the relation of a capacitor where the ground plate should just carry the charge density $-q(x, y, t)$, so the total charge on the strip is

$$Q^g_{ind}(t) = \int_{x_p-w/2}^{x_p+w/2} \int_{\infty}^{\infty} q_g(x,y) dx dy = \frac{Q}{2} \left[\mathrm{erf}\left(\frac{2x_p + w}{4b\sqrt{2t/T}}\right) - \mathrm{erf}\left(\frac{2x_p - w}{4b\sqrt{2t/T}}\right)\right] \tag{141}$$

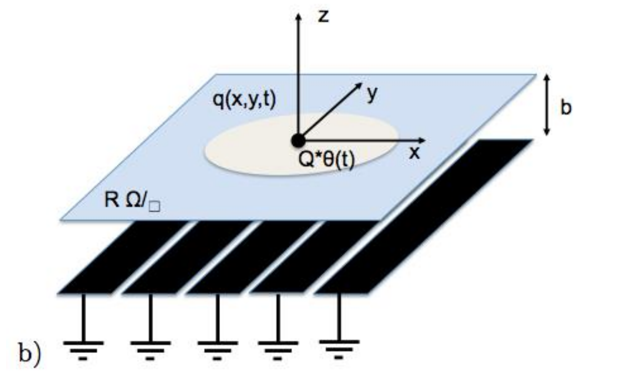
Both expression are shown in Fig. 19b. Although there are significant differences at small times the curves approach each other for longer times when the charge distribution becomes broad. Indeed, if take Eq. 139 we see that for large values of t/T only small values of κ contribute to the integral, so if we expand the exponent as

$$-\kappa - \kappa(1-e^{-2\kappa})\frac{t}{T} \approx -2\kappa^2 \frac{t}{T} \tag{142}$$

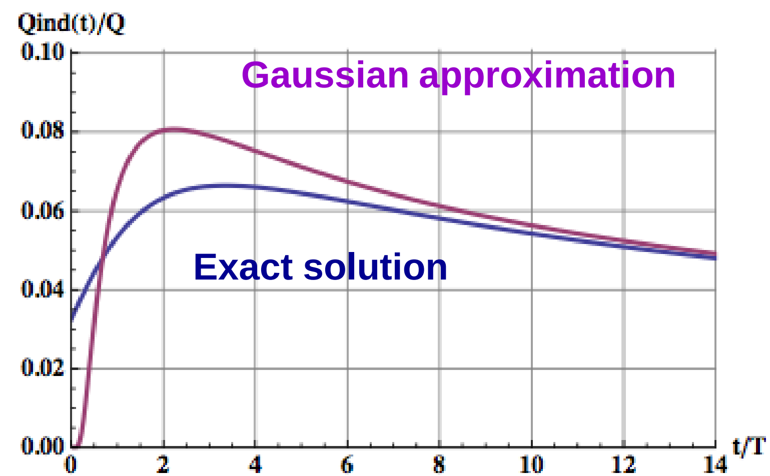
the integral evaluates precisely to expression Eq. 140.

broad. The solutions still do not represent a detector signal due to the unphysical assumption that the charge is created 'out of nowhere' at $t = 0$. The correct signal on a strip due to a pair of charges $\pm Q$ moving in a detector will be discussed in Section 8.

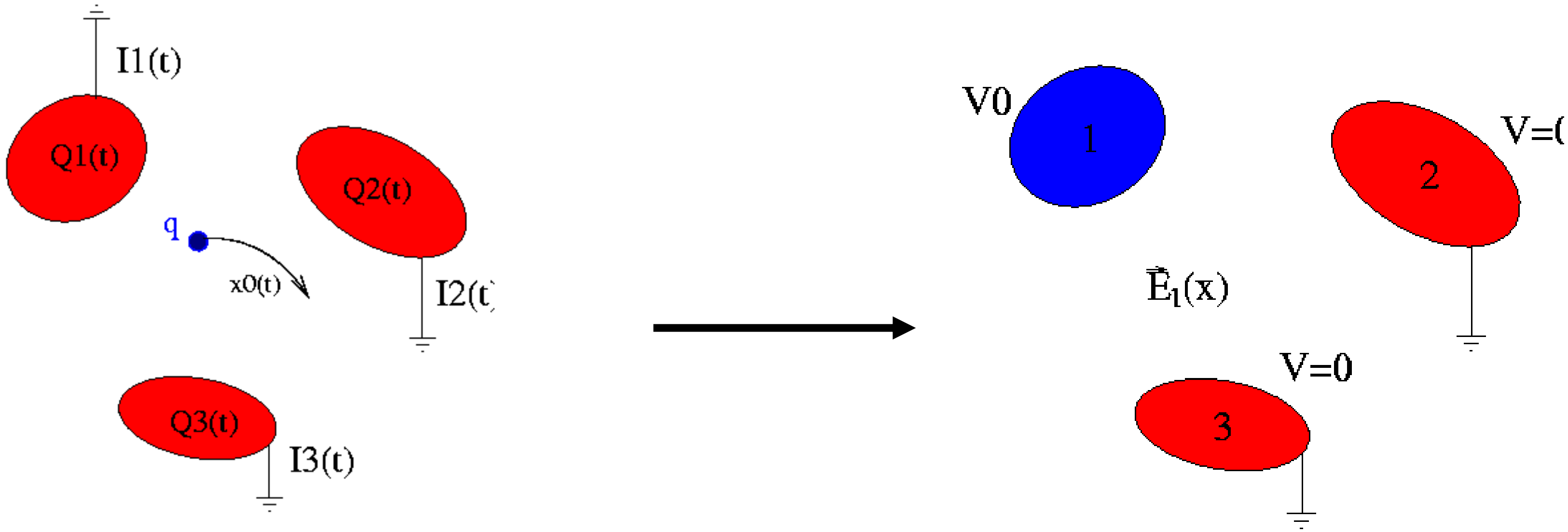
What are the charges induced metallic readout electrodes by this charge distribution?



ıce of a grounded layer. b) The same geometry grounded at a



Ramo's Theorem

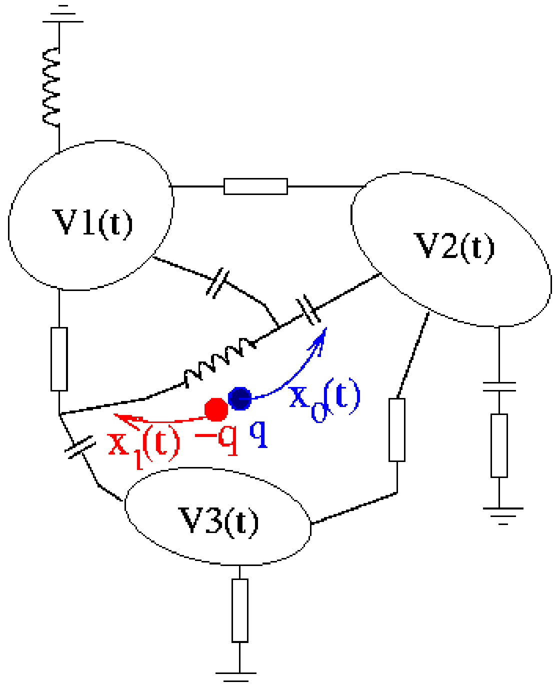


The current induced on a grounded electrode by a charge q moving along a trajectory $x_0(t)$ can be calculated the following way:

- **One removes the charge, sets the electrode in question to voltage V_0 and grounds all other electrodes.**
- **This defines an electric field E(x), the so called weighting field of the electrode.**
- **The induced current is the given by $I_1(t) = -q/V_0\, E_1(x_0(t))\, d/dt\, x_0(t)$**

Extensions of the Theorem

An extension of the theorem, where the electrodes are connected with arbitrary discrete impedance elements, has been given by Gatti et al., NIMA 193 (1982) 651, details in Blum, Riegler, Rolandi, Particle Detection with Drift Chambers, Springer



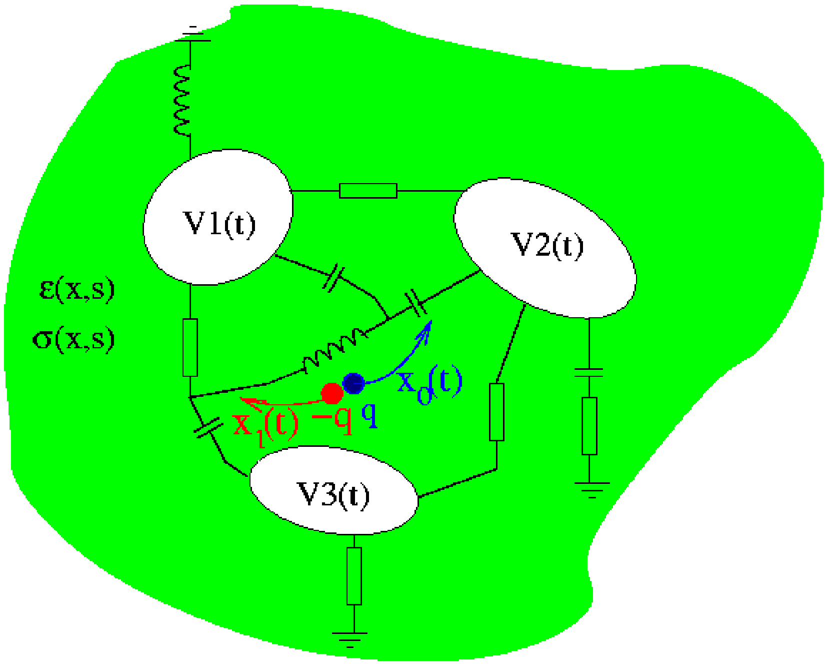
However this still doesn't include the scenario where a conductive medium is present in between the electrodes, as for example in Resistive Plate Chambers or undepleted Silicon Detectors.

Formulation of the Problem

At t=0, a pair of charges +q,-q is produced at some position in between the electrodes.

From there they move along trajectories $x_0(t)$ and $x_1(t)$.

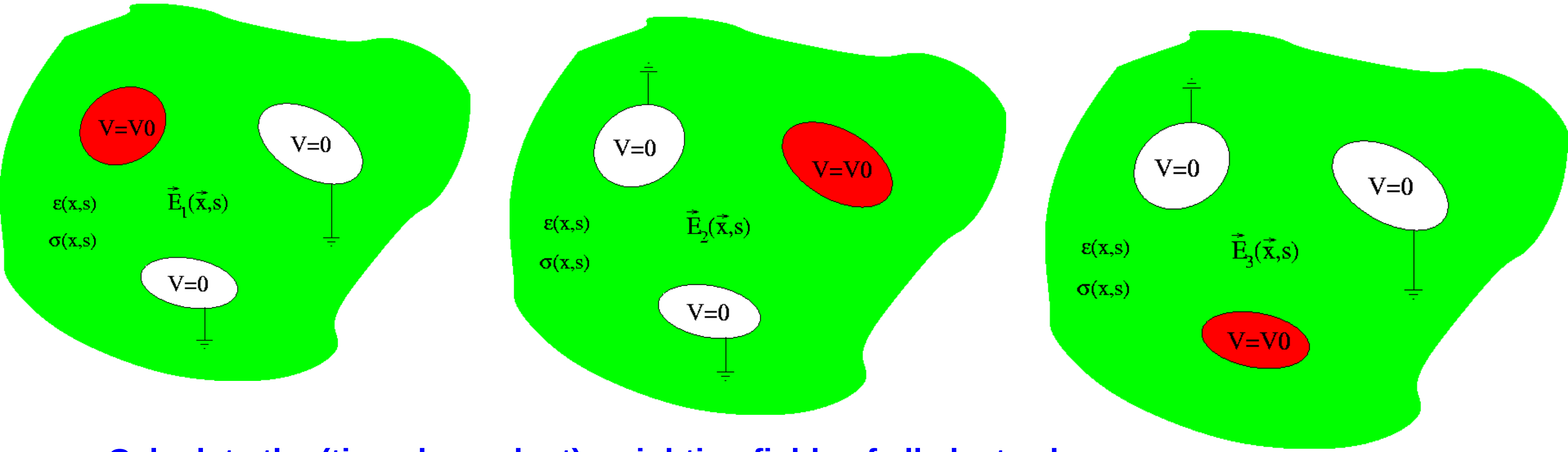
What are the voltages induced on electrodes that are embedded in a medium with position and frequency dependent permittivity and conductivity, and that are connected with arbitrary discrete elements ?



Theorem (1,4)

Remove the charges and the discrete elements and calculate the weighting fields of all electrodes by putting a voltage $V_0\delta(t)$ on the electrode in question and grounding all others.

In the Laplace domain this corresponds to a constant voltage V_0 on the electrode.

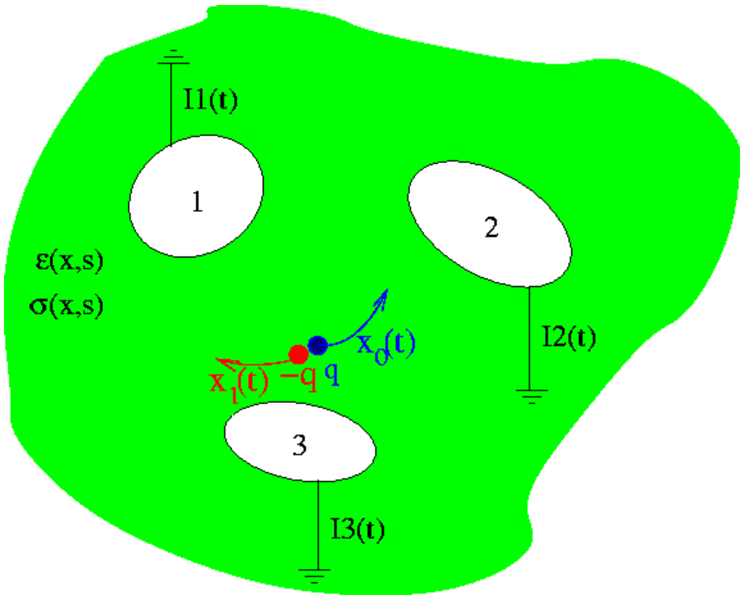


Calculate the (time dependent) weighting fields of all electrodes

$$\vec{\nabla}\left[\varepsilon_{eff}(\vec{x},s)\vec{\nabla}\right]\phi(\vec{x},s)=0 \qquad \phi_n(\vec{x},s)\big|_{\vec{x}=\vec{A}_m}=V_0\delta_{nm}$$

$$\vec{E}_n(\vec{x},s)=-\vec{\nabla}\phi_n(\vec{x},s) \qquad \vec{E}_n(\vec{x},t)=\mathcal{L}^{-1}\left[\vec{E}_n(\vec{x},s)\right]$$

Theorem (2,4)



Calculate induced currents in case the electrodes are grounded

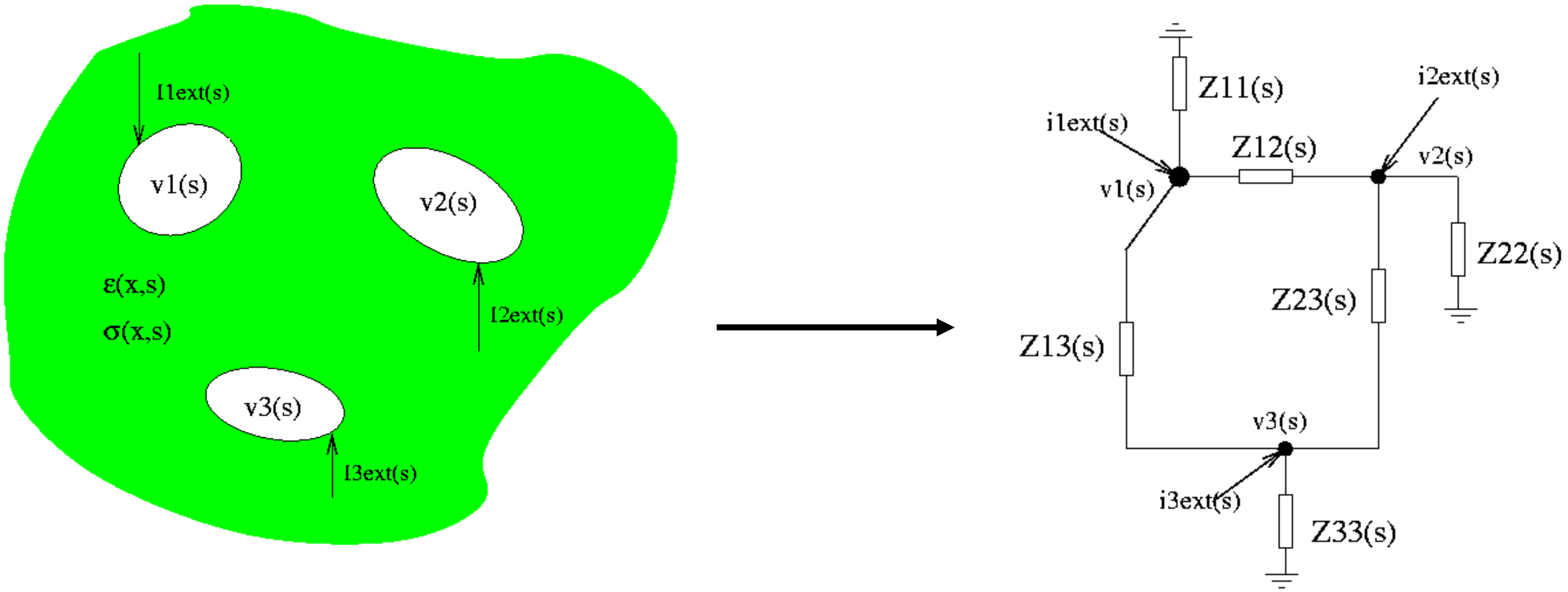
$$I_n(t) = \frac{q}{V_0}\int_0^t \vec{E}_n\left[\vec{x}_0(t'), t-t'\right]\dot{\vec{x}}_0(t')dt'$$
$$-\frac{q}{V_0}\int_0^t \vec{E}_n\left[\vec{x}_1(t'), t-t'\right]\dot{\vec{x}}_1(t')dt'$$

Theorem (3,4)

Calculate the admittance matrix and equivalent impedance elements from the weighting fields.

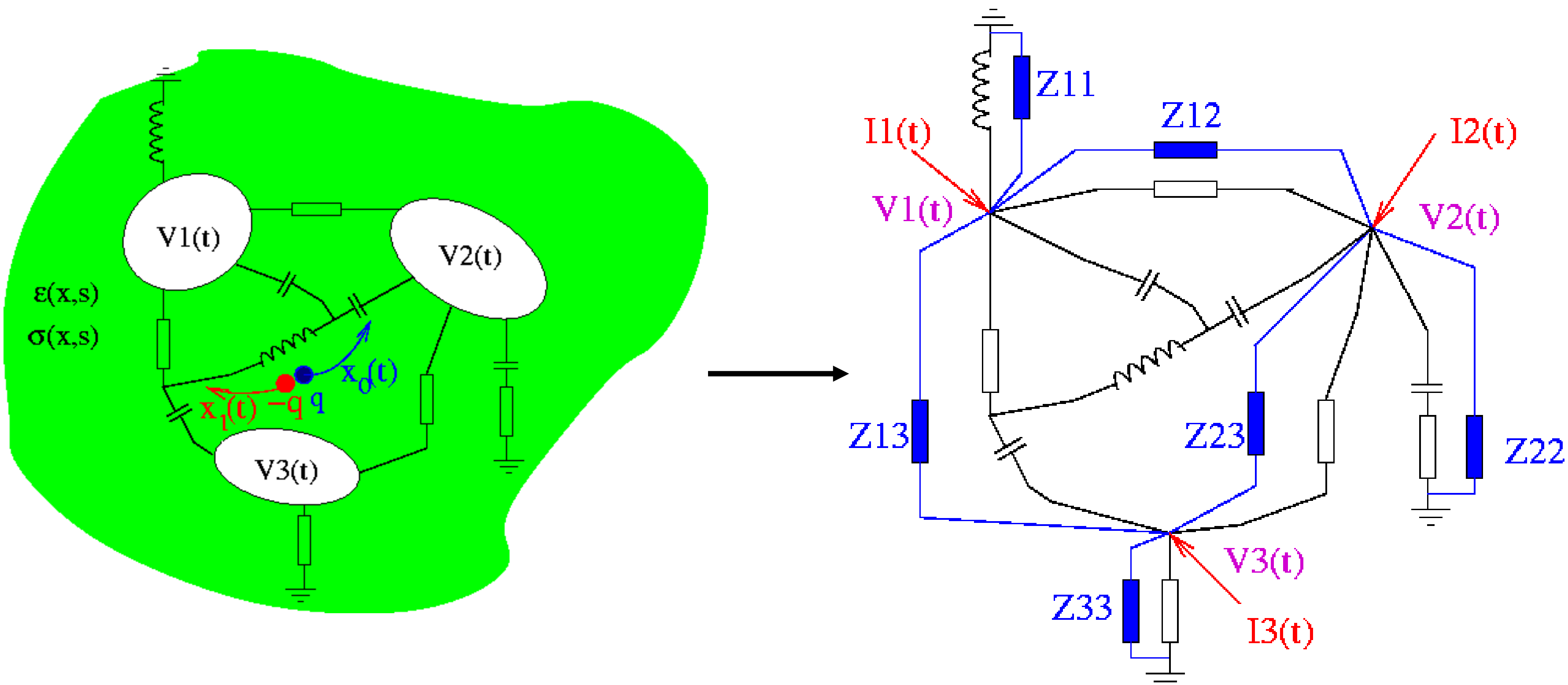
$$y_{nm}(s) = \int_{A_n} \varepsilon_{eff}(\vec{x}, s)\vec{E}_m(\vec{x}, s)d\vec{A} \qquad y_{nm}(s) = y_{mn}(s)$$

$$Z_{nm}(s) = -\frac{1}{y_{nm}(s)} \qquad Z_{nn}(s) = \frac{1}{\sum_{m=1}^{N} y_{nm}(s)}$$

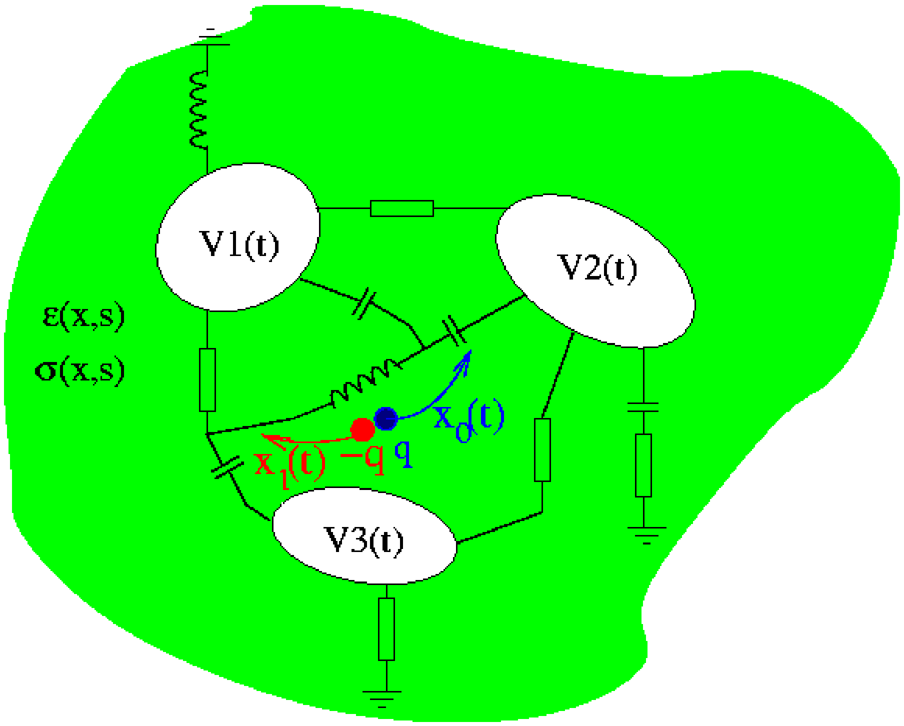


Theorem (4,4)

Add the impedance elements to the original circuit and put the calculated currents On the nodes 1,2,3. This gives the induced voltages.

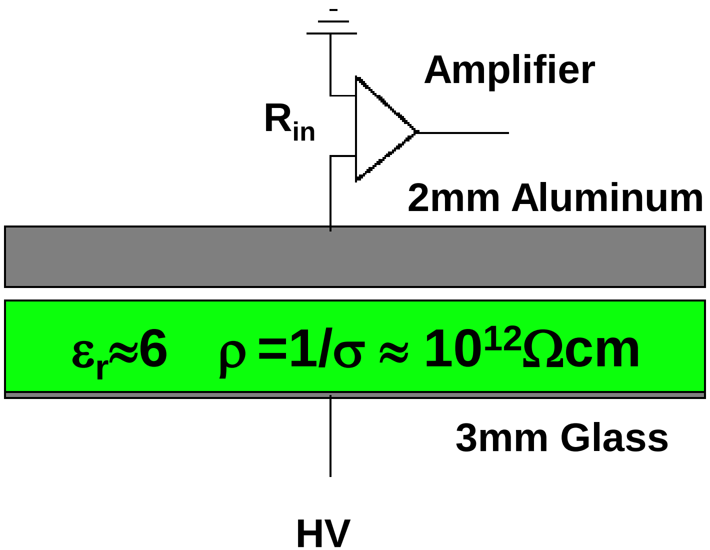


This theorem is applies to situations where the influence of the moving charges on the system is through their electric field by j=σE and the movement of the charges is not affected.



Examples

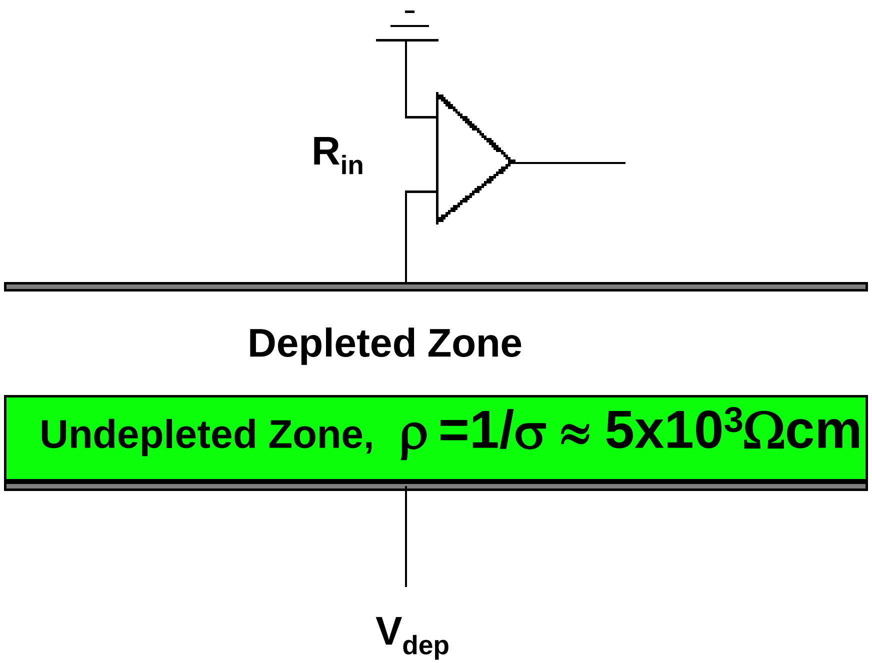
RPC



300µm Gas Gap

$\tau \approx \varepsilon_0 / \sigma \approx$ 100msec

Silicon Detector

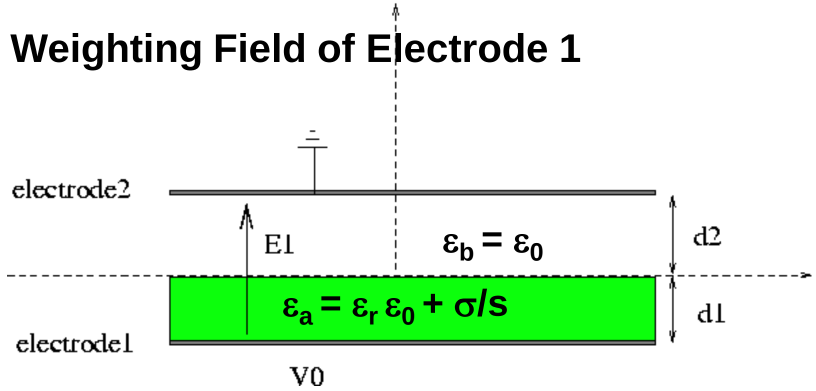


$\tau \approx \varepsilon_0 / \sigma \approx$ 1ns

heavily irradiated silicon has larger resistivity that can give time constants of a few hundreds of ns,

Example, Weighting Fields (1,4)

Weighting Field of Electrode 1

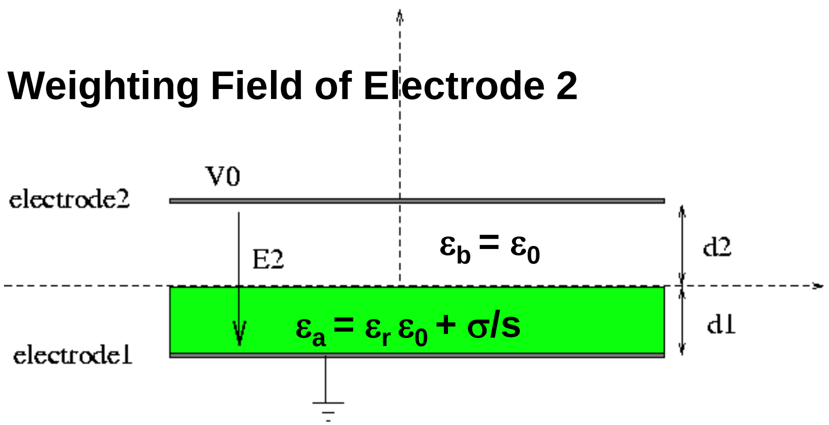


$$E_{1z}(s) = \frac{\varepsilon_a V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0 \varepsilon_r}{(d_1 + d_2\varepsilon_r)} \frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\tau_2}} \qquad z > 0$$

$$= \frac{\varepsilon_b V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0}{(d_1 + d_2\varepsilon_r)} \frac{s}{s + \frac{1}{\tau_2}} \qquad z < 0$$

$$\tau_1 = \frac{\varepsilon_r \varepsilon_0}{\sigma} \qquad \tau_2 = \frac{\varepsilon_0}{\sigma}\left(\frac{d_1 + d_2\varepsilon_r}{d_2}\right)$$

Weighting Field of Electrode 2



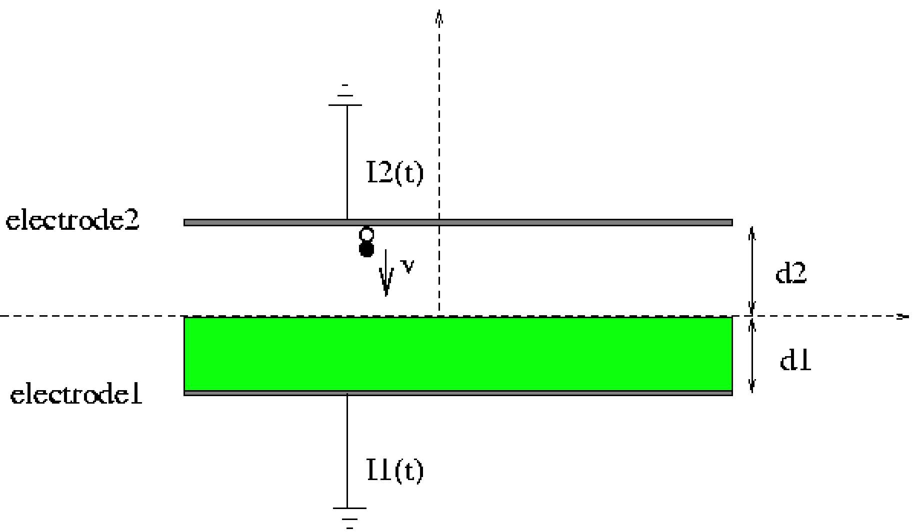
$$E_{2z}(s) = -E_{1z}(s)$$

Example, Induced Currents (2,4)

At t=0 a pair of charges q, -q is created at z=d_2.
One charge is moving with velocity v to z=0
Until it hits the resistive layer at T=d_2/v.

$$x_0(t) = d_2 - vt \qquad t < T$$
$$= 0 \qquad t > T$$

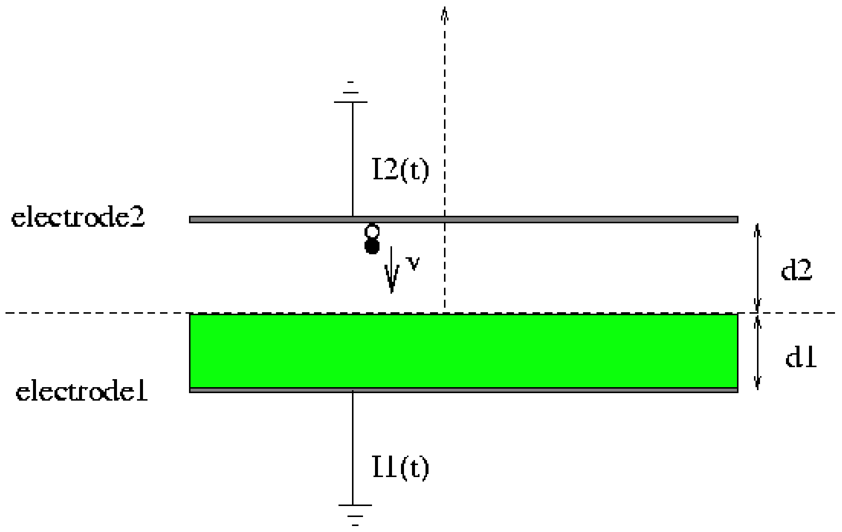
$$\dot{x}_0(t) = -v \qquad t < T$$
$$= 0 \qquad t > T$$

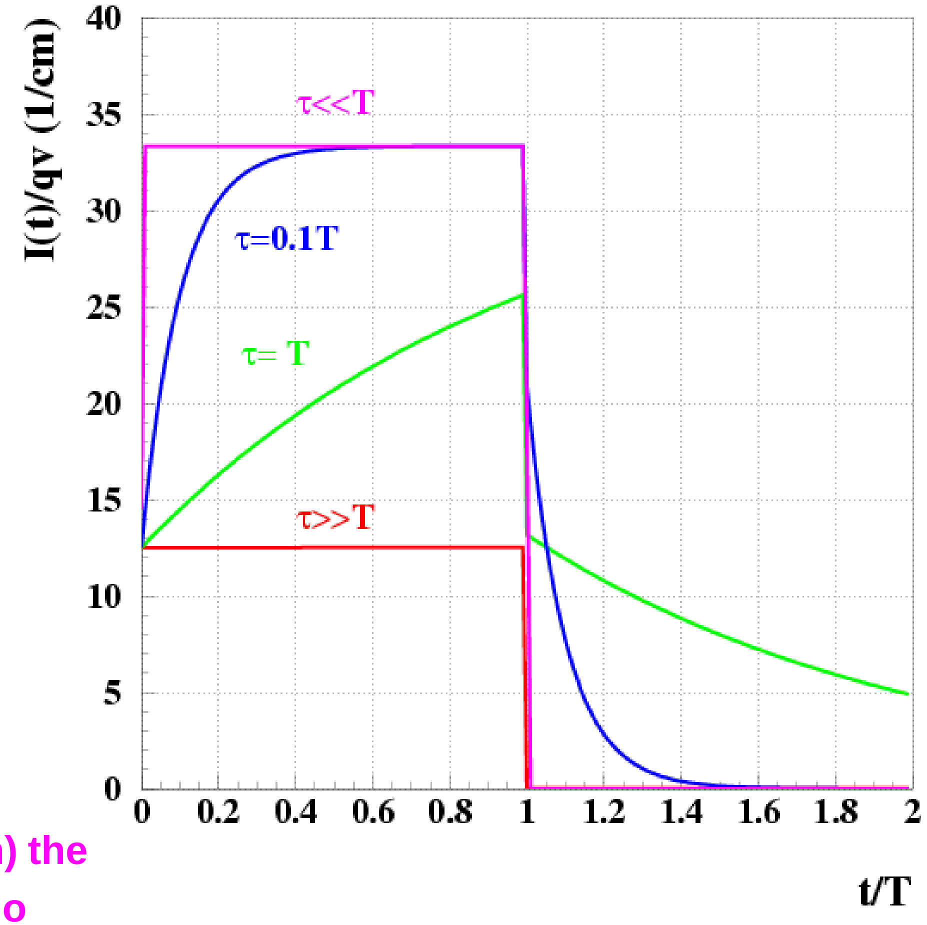


$$E_{1z}(\vec{x}, t) = \frac{\varepsilon_r V_0}{d_1 + \varepsilon_r d_2}\left[\delta(t) + \frac{\tau_2 - \tau_1}{\tau_1 \tau_2}\, e^{-\frac{t}{\tau_2}}\right] \qquad z > 0$$

$$I_1(t) = qv\frac{\varepsilon_r}{d_1 + \varepsilon_r d_2}\left[1 + \frac{d_1}{d_2 \varepsilon_r}(1 - e^{-\frac{t}{\tau_2}})\right] \qquad t < T$$
$$= qv\frac{1}{d_1 + \varepsilon_r d_2}\frac{d_1}{d_2}\left(e^{\frac{T}{\tau_2}} - 1\right) e^{-\frac{t}{\tau_2}} \qquad t > T$$

Example, Induced Currents (2,4)

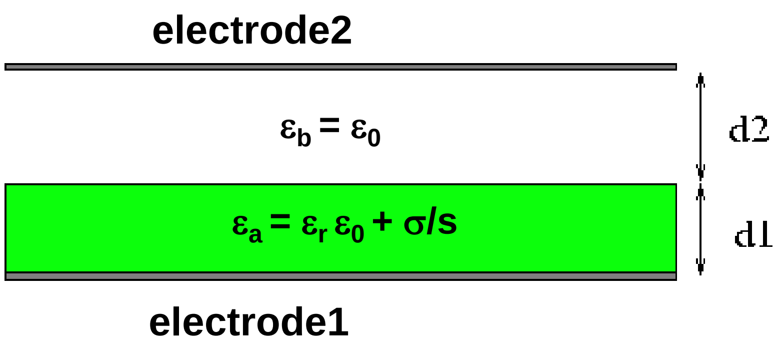


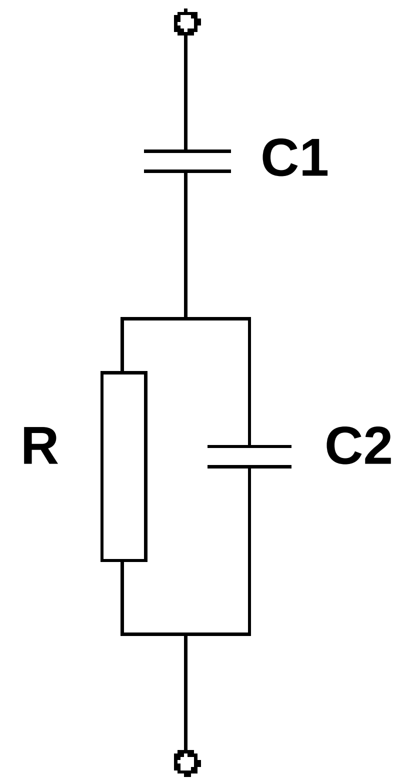


In case of high resistivity (τ>>T, RPCs, irradiated silicon) the layer is an insulator.

In case of very low resistivity (τ <<T, silicon) the layer acts like a metal plate and the scenario is equal to a parallel plate geometry with plate separation d_2.

Example, Admittance Matrix (3,4)





$$y_{nm}(s) = \frac{A\varepsilon_0 s(\sigma + \varepsilon_0 s)}{\sigma d_2 + (d1_1 + d_2)\varepsilon_0 s} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

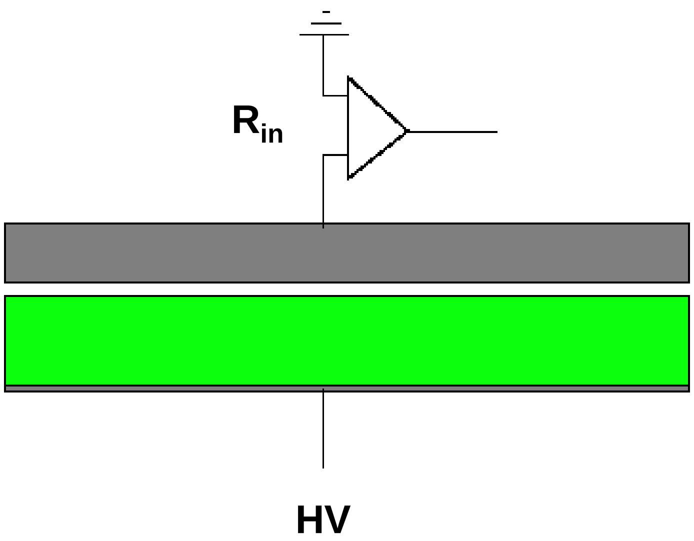
$$Z_{11}(s) = \infty$$

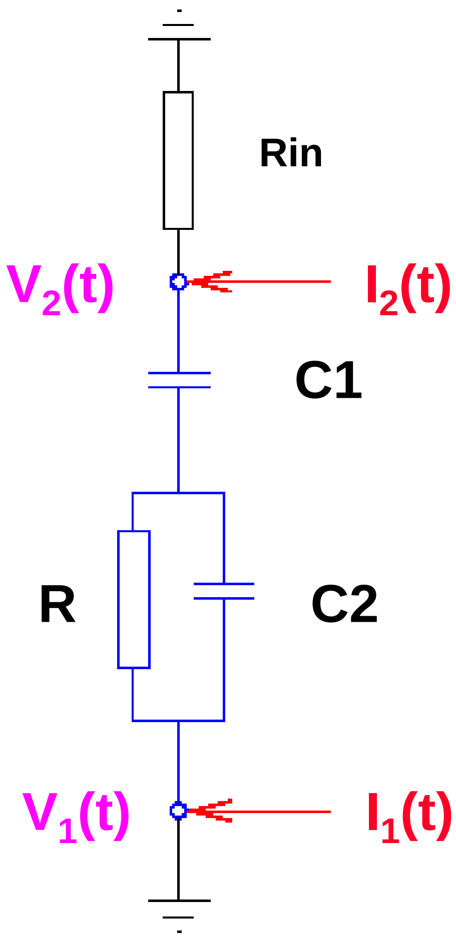
$$Z_{22}(s) = \infty$$

$$Z_{12}(s) = \frac{1}{sC_1} + \frac{R/sC_2}{R + 1/sC_2}$$

$$C_1 = \varepsilon_0 \frac{A}{d_2} \qquad C_2 = \varepsilon_r \varepsilon_0 \frac{A}{d_1} \qquad R = \frac{1}{\sigma}\frac{d_1}{A}$$

Example, Voltage (4,4)





Charge spread in e.g. a Micromega with bulk or surface resistivity

Micromega Mesh

Avalanche region

Bulk

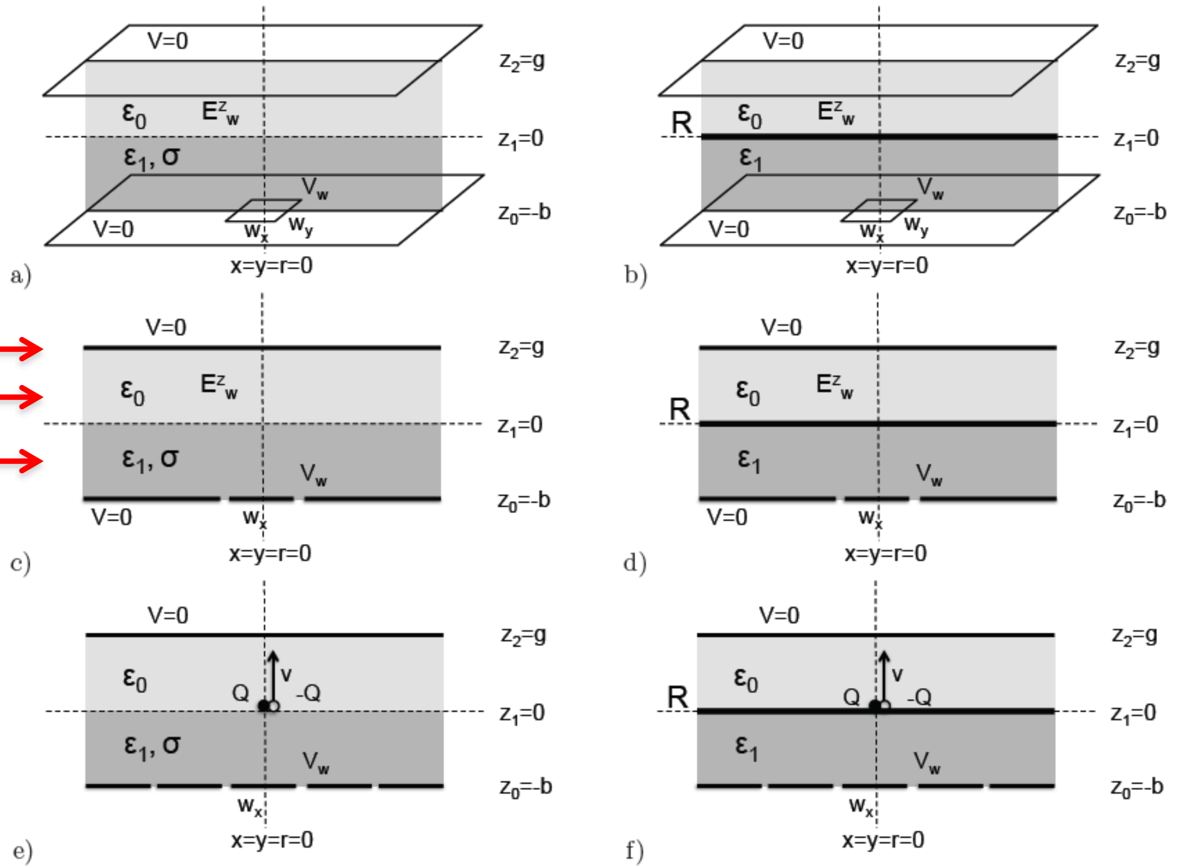
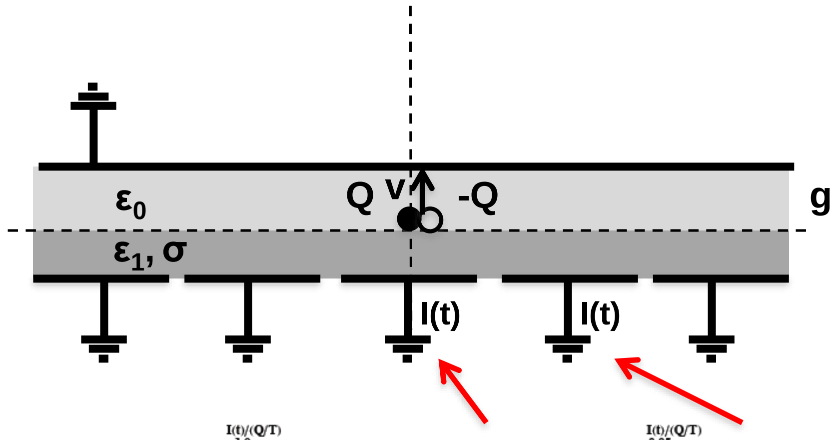


Figure 27: Weighting field for a geometry with a resistive layer having a bulk resistivity of $\rho = 1/\sigma[\Omega\text{cm}]$ (left) and a geometry with a thin resistive layer of value $R\,[\Omega/\text{square}]$ (right).

$$I(t) = -\frac{q}{V_w}\int_0^t \vec{E}_w(\vec{x}_1(t'), t-t')\dot{\vec{x}}_1(t')dt' + \frac{q}{V_w}\int_0^t \vec{E}_w(\vec{x}_2(t'), t-t')\dot{\vec{x}}_2(t')dt'$$

Charge spread in e.g. a Micromega with bulk resistivity



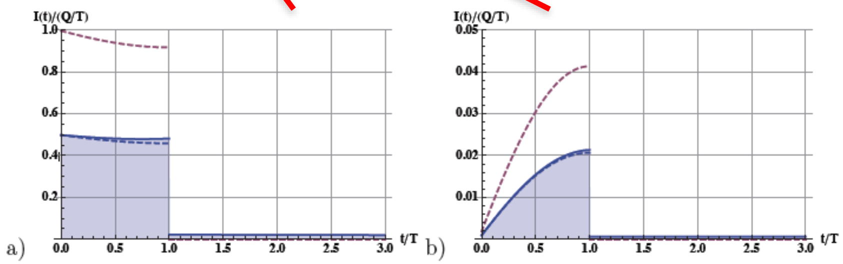


Figure 28: Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = 10T$ for a) $x = 0$ and b) $x = 4g$.

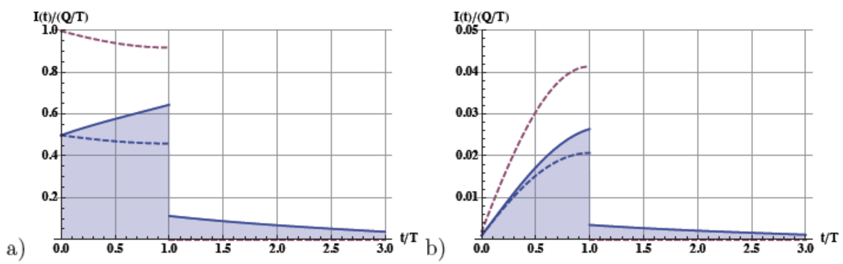


Figure 29: Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = T$ for a) $x = 0$ and b) $x = 4g$.

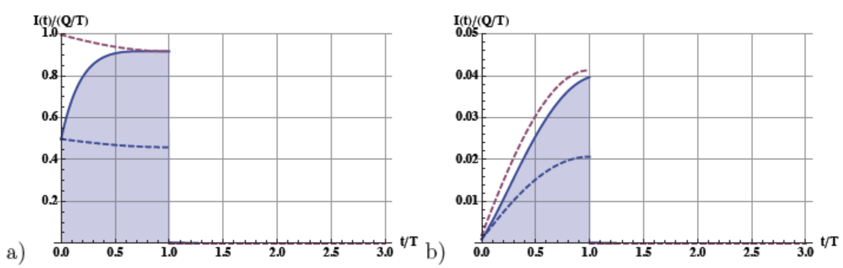


Figure 30: Uniform charge movement from $z = 0$ to $z = g$, with $\varepsilon_r = 1, w_x = 4g, b = g, \tau_0 = 0.1T$ for a) $x = 0$ and b) $x = 4g$.

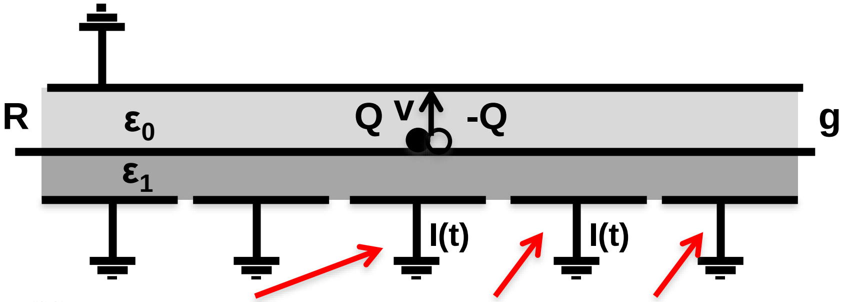
$$\tau_0 = \varepsilon_0/\sigma = \varepsilon_0\rho.$$

$$T = g/v$$

Zero Resistivity

Infinite Resistivity (insulator)

All signals are unipolar since the charge that compensates Q sitting on the surface is flowing from all the strips.



Charge spread in e.g. a Micromega with surface resistivity

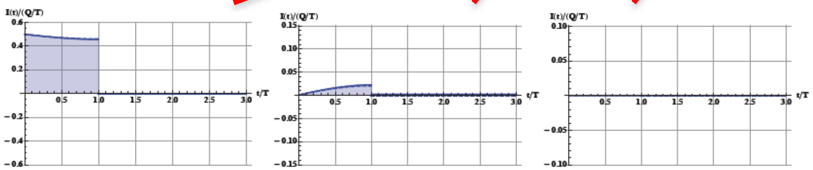


Figure 31: $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 10T$ for $x = 0, x = 4g, x = 8g$

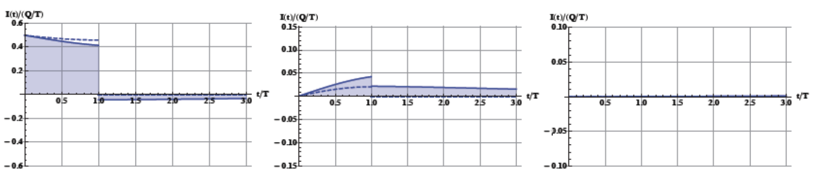


Figure 32: $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = T$ for $x = 0, x = 4g, x = 8g$

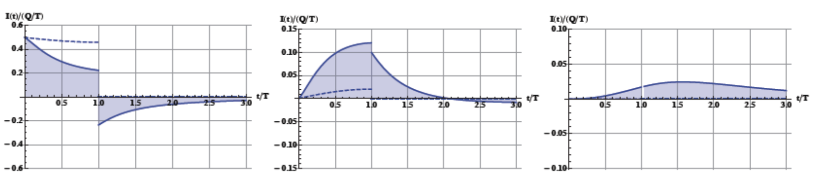


Figure 33: $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.1T$ for $x = 0, x = 4g, x = 8g$

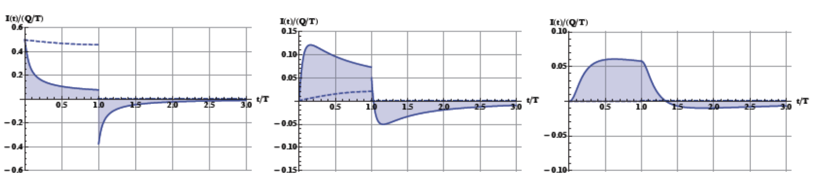


Figure 34: $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.01T$ for $x = 0, x = 4g, x = 8g$

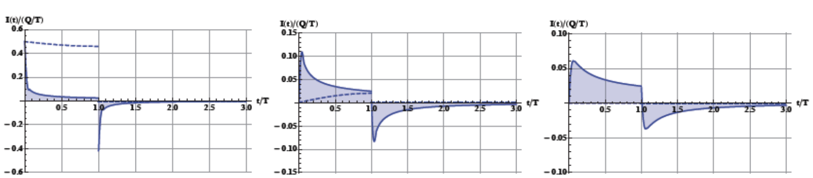


Figure 35: $\varepsilon_r = 1, w_x = 4g, b = g, T_0 = 0.001T$ for $x = 0, x = 4g, x = 8g$

$$T_0 = \varepsilon_0 R g$$

$$T = g/v$$

Zero Resistivity

Infinite Resistivity (insulator)

All signals are bipolar since the charge that compensates Q sitting on the surface is not flowing from the strips.

Summary

An extension of Ramo's theorem for detectors containing elements of finite resistivity has been presented.

Using the quasistatic approximation of Maxwell's equations the time dependent weighting fields can be derived from electrostatic solutions.