Analytic expressions for time resolution of silicon pixel sensors

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Silicon sensors with high precision timing are used in present experiments, like the NA62 Gigatracker, or planned to be used for the LHC PhaseII upgrade, like the LGAD development. Trackers with 10um position and 10ps time resolution are quoted as a long term goal for these developments. This report will discuss analytic expressions for the time resolution of silicon sensors, with focus on the key contributions to the time resolution, namely Landau fluctuations, noise and variations of the weighting field. The impact of amplifier bandwidth is discussed as well.

Time resolution in Resistive Plate Chambers

Time response functions and avalanche fluctuations in resistive plate chambers

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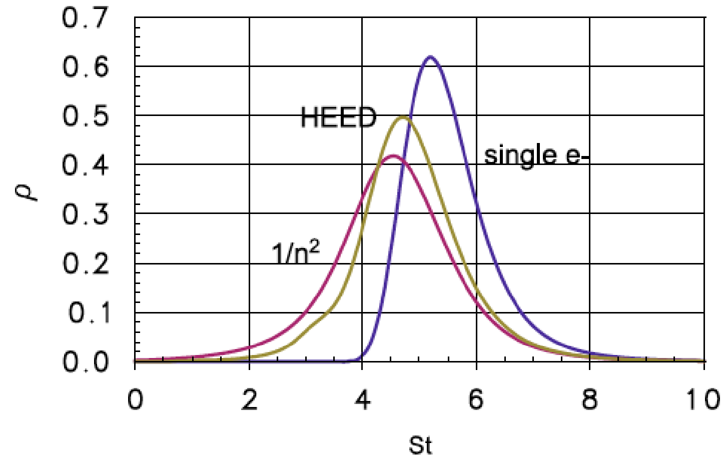
The time response function of RPCs is derived. First, primary electron distributions in the RPC gas gap are discussed. Then the exact expression for the fluctuations of an avalanche starting with a fixed number of primary electrons is derived, using Legler's model of avalanche multiplication in electronegative gases. By means of the Z-Transform formalism, the primary electron distributions and avalanche fluctuations are then combined and an analytic expression for the RPC time response function is derived. The solution is further used to discuss signal threshold and attachment effects. Finally, the time response function is evaluated for several primary ionization models.

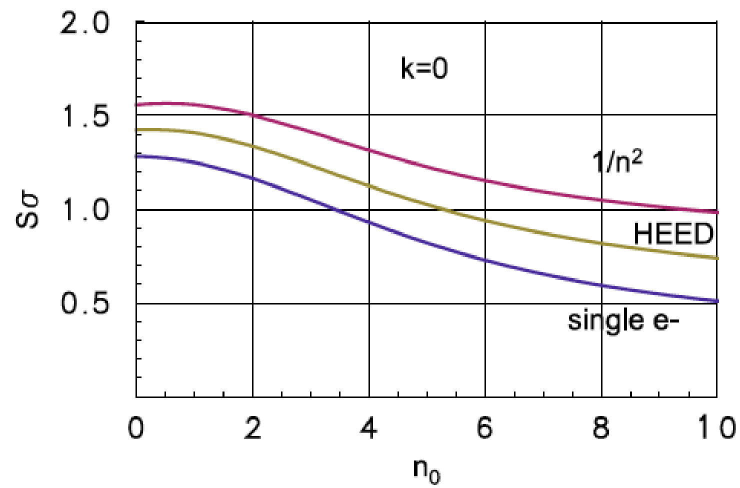
Time resolution in Resistive Plate Chambers

Before proceeding to the main body of the paper we shall state the principle result: the time response function for an RPC using a gas with drift-velocity v, Townsend coefficient α, attachment coefficient η, assuming (1) an average number of n_0 efficient clusters which fluctuate according to a Poisson distribution (2) a cluster size distribution $f(m)$ with Z-Transform $F(z)$ having a radius of convergence r_F (3) avalanche multiplication according to Legler's avalanche model and (4) a threshold of n electrons, is given by

$$\rho(n,t) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{e}^{n_0 F(z)} - 1}{\mathrm{e}^{n_0} - \mathrm{e}^{n_0 F(1/k)}} \frac{(1-k)^2 nS}{(1-kz)^2} \times \exp\left[-St - n\frac{(1-k)(1-z)}{1-kz}\mathrm{e}^{-St}\right] \mathrm{d}z \quad (1)$$

with $S = (\alpha - \eta)v$ and $k = \eta/\alpha$. The integration is over a circle with radius $r_F < r < 1/k$. Writing $z = r\exp(\mathrm{i}\phi)$ and $\mathrm{d}z = \mathrm{i}r\exp(\mathrm{i}\phi)\,\mathrm{d}\phi$ and





→ Interested in understanding how the basic contributions to the time resolution differ between RPCs and silicon sensors

Time resolution of silicon pixel sensors

Time resolution of silicon pixel sensors

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Abstract

We derive expressions for the time resolution of silicon detectors, using the Landau theory as a minimum model for describing the charge deposit of high energy particles. First we use the center of gravity time of the induced signal and derive analytic expressions for the three components contributing to the time resolution, namely charge deposit fluctuations, noise and fluctuations of the signal shape due to of the weighting field variations. Then we derive expressions for the time resolution using leading edge discrimination of the signal for various shaping times.

Keywords: silicon sensors, time resolution, weighting field
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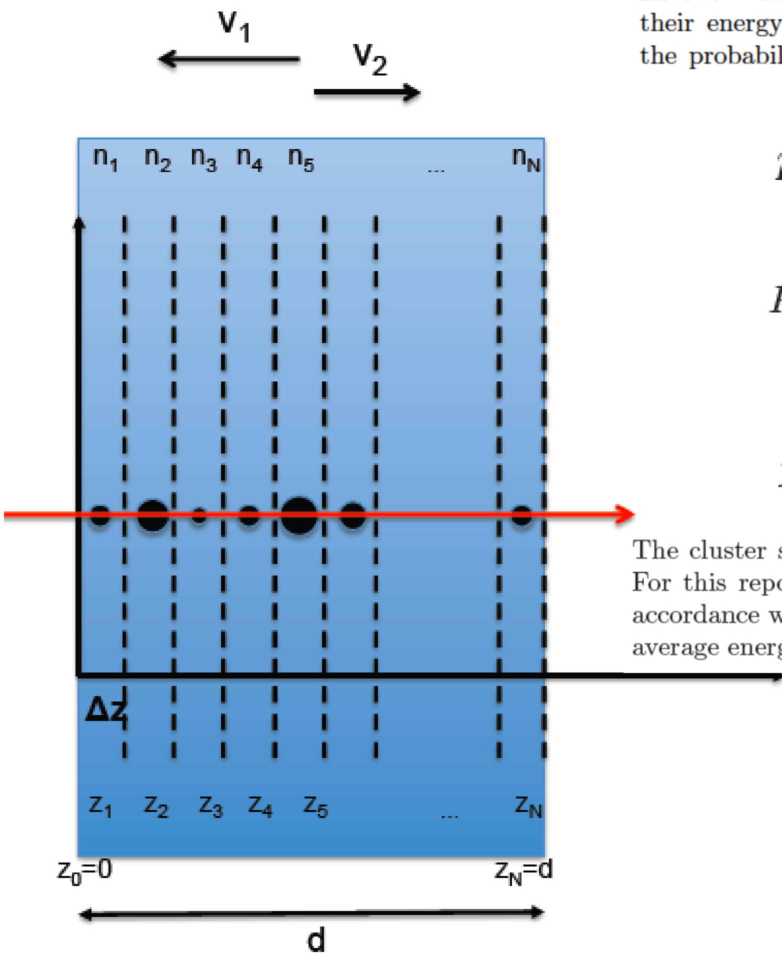
Energy deposit in silicon



A high energy particle passing a silicon sensor will experience a number of primary interactions with the material, with λ being the average distance between these primary interactions. For $\beta\gamma > 3$ we have $\lambda \approx 0.25\,\mu$m in silicon [6]. The electrons created in these primary interactions will typically lose their energy over very small distances and create a localised cluster of electron-hole pairs. We call the probability $p_{cl}(n)$ for creating n e-h pairs in a primary interaction the 'cluster-size distribution'.

$$p(n,\Delta z)dn = \left(1 - \frac{\Delta z}{\lambda}\right)\delta(n)dn + \frac{\Delta z}{\lambda} p_{clu}(n)dn$$

$$P(s,d) = \mathcal{L}[p(n,d)] = \mathcal{L}[p(n,\Delta z)]^N = \left(1 + \frac{d}{\lambda N}(P_{clu}(s) - 1)\right)^N$$

$$p(n,d) = \mathcal{L}^{-1}\left[e^{d/\lambda(P_{clu}(s)-1)}\right]$$

The cluster size distribution $p_{clu}(n)$ is typically calculated using some form of the the PAI model [7]. For this report we use Landau's approach to assume an $1/E^2$ distribution for the energy transfer in accordance with Rutherford scattering on free electrons and a lower cutoff energy ϵ chosen such that the average energy loss reproduces the Bethe-Bloch theory. The resulting cluster size distribution for a MIP

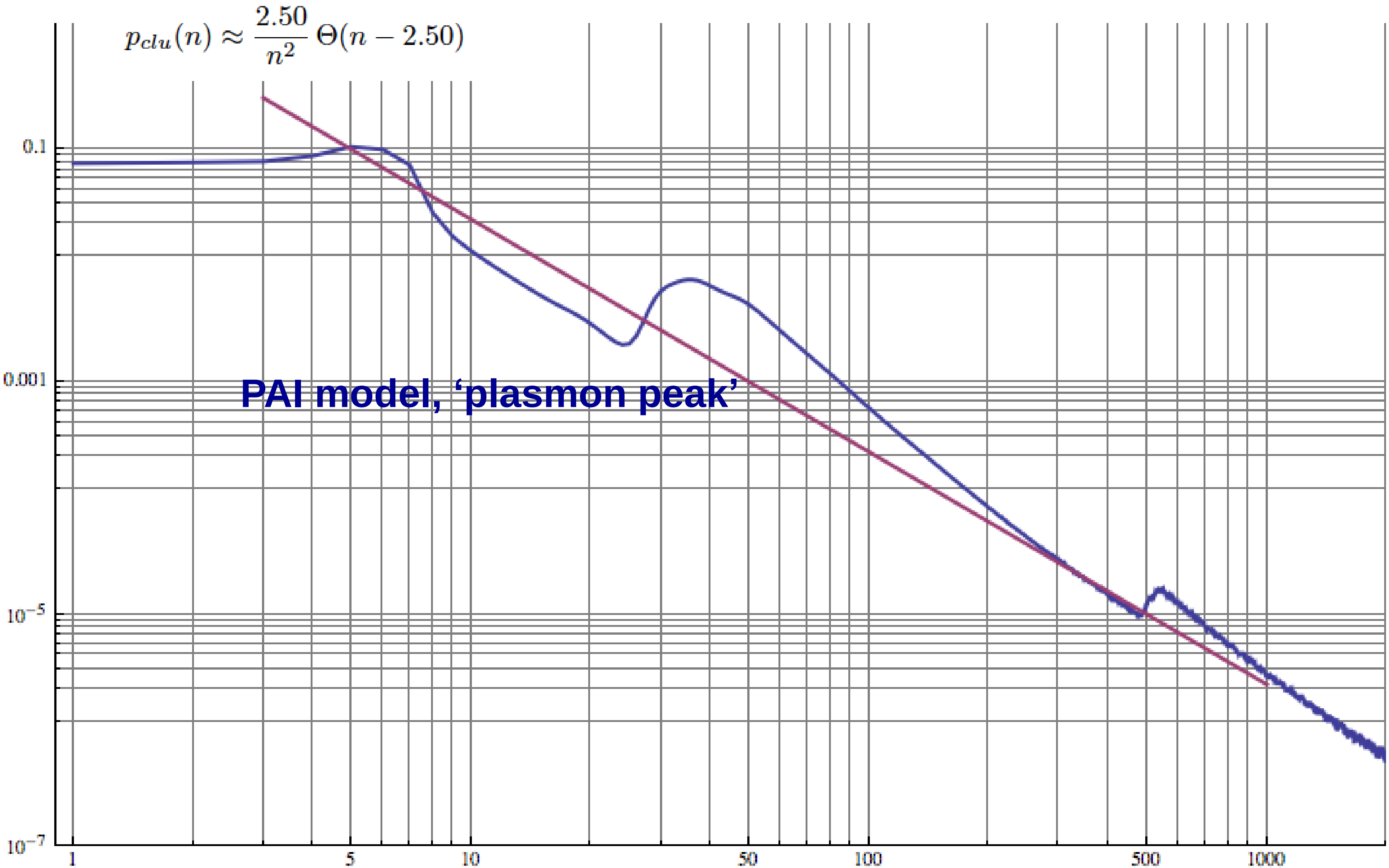
$$p_{clu}(n) \approx \frac{2.50}{n^2}\Theta(n - 2.50)$$

$$p(n,d)dn = \frac{\lambda}{2.50\,d} L\left(\frac{\lambda}{2.50\,d} n + C_\gamma - 1 - \log\frac{d}{\lambda}\right) dn$$

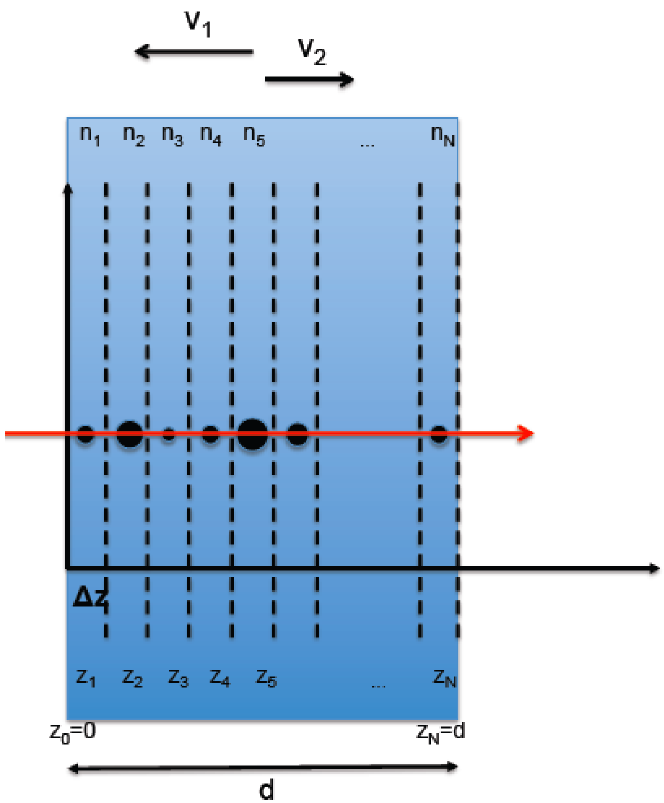
$$L(x) = \frac{1}{\pi}\int_0^\infty e^{-t\log t - x\,t}\sin(\pi t)dt$$

Figure 1: The silicon sensor is divided into slices of thickness Δz. The electrons and holes produced in one slice are assumed to move to the boundary of the sensor at constant velocity, which is correct in the limit of negligible depletion voltage.

Cluster Size Distribution



Energy deposit in silicon



$$n_{MP} \approx \frac{2.50\, d}{\lambda}\left(0.2 + \log \frac{d}{\lambda}\right)$$

$$\frac{\Delta n_{FWHM}}{n_{MP}} \approx \frac{4.02}{0.2 + \log d/\lambda}$$

The most probable number of e-h pairs for a MIP in $50, 100, 200, 300\,\mu$m of silicon evaluate to $\approx$ $2750, 6190, 13770, 21870$, which is within 10% of the values shown in [8]. The relative width $\Delta n_{FWHM}/n_{MP}$ is 0.73, 0.65, 0.58, 0.55 for these values of thickness, which is 20-50% higher than the numbers from the PAI model and the actual values. It is well known that the Landau distribution overestimates the charge deposit fluctuations, so using this model we should have a slightly pessimistic estimate of the time resolution.

Timing of a detector signal, c.o.g. time

3. Center of gravity time of a signal

First we assume the measured time to be defined by the center of gravity (c.o.g.) time of the induced detector current signal $i(t)$. Assuming the Laplace Transform of the signal $I(s) = \mathcal{L}[i(t)]$, the c.o.g. time τ_{cur} of the signal is given by

$$\tau_{cur} = \frac{\int_0^\infty t\, i(t)dt}{\int_0^\infty i(t)dt} = \frac{\int_0^\infty t\, i(t)dt}{q} = -\frac{I'(0)}{I(0)} \tag{7}$$

where $q = \int_0^\infty i(t)dt$ is the total signal charge. We now consider the signal $i(t)$ to be processed by an amplifier having a delta response $f(t)$ with Laplace Transform $F(s)$, so the amplifier output signal $v(t)$ is given by

$$v(t) = \int_0^t f(t-t')i(t')dt' \qquad V(s) = F(s)I(s) \tag{8}$$

The c.o.g. time of the output signal is then given by

$$\tau_v = -\lim_{s\to 0}\frac{V'(s)}{V(s)} = -\frac{F'(0)I(0)+F(0)I'(0)}{F(0)I(0)} = -\frac{F'(0)}{F(0)} - \frac{I'(0)}{I(0)} = \tau_{amp} + \tau_{cur} \tag{9}$$

The represents the sum of the c.o.g. time of the delta response and the one from the current signal, and since the shape of the delta response does not vary in time, the c.o.g. time variation of the of the amplifier output signal is equal to the c.o.g. time variation of the original input signal and has no dependence on the amplifier characteristics.

Timing of a detector signal, c.o.g. time

as shown in the following. In case the duration T of the signal $i(t)$ is short compared to the 'peaking time' t_p of the amplifier ($i(t) = 0$ for $t > T \ll t_p$) we can approximate Eq. 8 for $t > T$ according to

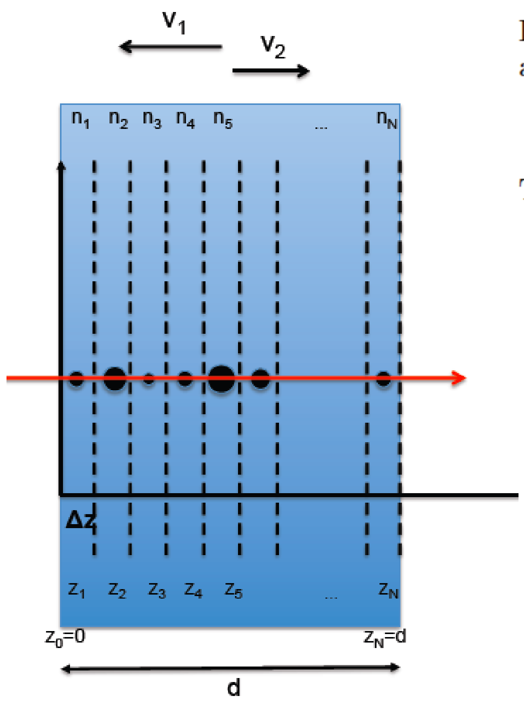
$$\begin{aligned} v(t) &= \int_0^T f(t-t')i(t')dt' \approx \int_0^T \left[f(t) - f'(t)t'\right] i(t')dt' \\ &= q\left[f(t) - f'(t)\frac{\int_0^T t'i(t')dt'}{q}\right] = q\left[f(t) - f'(t)\tau_{cur}\right] \\ &\approx q\, f(t-\tau_{cur}) \end{aligned} \qquad (10)$$

The amplifier output is simply equal to the amplifier delta response shifted by the c.o.g. time of the current signal and scaled by the total charge of the signal. Since the shape of the amplifier output signal is always equal to the amplifier delta response, we can determine the signal c.o.g. time either by the threshold crossing time at a given fraction of the signal or by sampling the signal and fitting the known signal shape to the samples.

→ An amplifier with a peaking time larger than the duration of the detector signal measures the c.o.g. time of the detector signal.

→ This does not mean that the c.o.g. time gives the best possible time resolution, but it allows the characterization of time resolution independently of the amplifier and might still be of practical importance …

C.o.g. time of a silicon detector signal



For later use we remark that for the sum of two current signals $i(t) = i_1(t) + i_2(t)$ with c.o.g. times τ_1 and τ_2 we have

$$\tau = \frac{\int t i(t) dt}{\int i(t) dt} = \frac{\tau_1 \int i_1(t) dt + \tau_2 \int i_2(t) dt}{\int i_1(t) dt + \int i_2(t) dt} = \frac{\tau_1 q_1 + \tau_2 q_2}{q_1 + q_2} \qquad (11)$$

The c.o.g. time for the sum of N signals $i_n(t)$ is therefore given by

$$\tau = \frac{1}{\sum_{k=1}^{N} q_k} \sum_{k=1}^{N} q_k \tau_k \qquad (12)$$

We assume a silicon sensor operated at large over-depletion i.e. at a voltage that is large compared to the depletion voltage and the electric field can therefore be assumed to be constant throughout the sensor. Consequently the velocities of electrons and holes are constant and the signal from a single electron or single hole has a box shape. We assume a parallel plate geometry with one plate a $z = 0$ and one at $z = d$, where a pair of charges $+q, -q$ is produced at position z and $-q$ moves with velocity v_1 to the electrode at $z = 0$ while q moves with velocity v_2 to the electrode at $z = d$. The weighting field of the electrode at $z = 0$ is $E_w = 1/d$ and the induced current is therefore

$$i(t) = -\frac{q v_1}{d} \Theta(z/v_1 - t) - \frac{q v_2}{d} \Theta((d - z)/v_2 - t) \qquad (13)$$

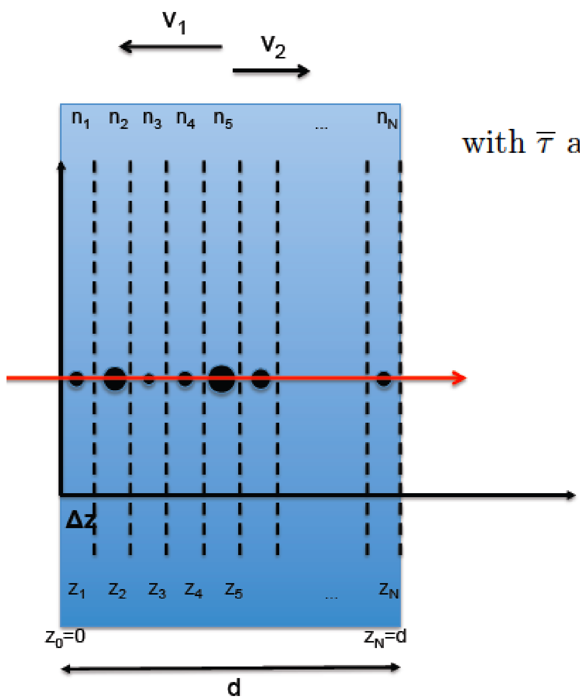
with $\Theta(t)$ being the Heaviside step function. We have $\int i(t) dt = -q$ and according to Eq. 7 the c.o.g. time of this signal is then

$$\tau = \frac{1}{2d} \left[\frac{z^2}{v_1} + \frac{(d - z)^2}{v_2} \right] \qquad (14)$$

If $n_1, n_2, ..., n_N$ charges are produced at positions $z_1, z_2, ..., z_N$ and are moving to the electrodes with v_1 and v_2, the resulting c.o.g. time of the signal is

$$\tau(n_1, n_2, ..., n_N) = \frac{1}{2d \left(\sum_{k=1}^{N} n_k\right)} \sum_{k=1}^{N} n_k \left[\frac{z_k^2}{v_1} + \frac{(d - z_k)^2}{v_2} \right] \qquad (15)$$

C.o.g. time of a silicon detector signal



$$\Delta_\tau^2 = \overline{\tau^2} - \overline{\tau}^2$$

with $\overline{\tau}$ and $\overline{\tau^2}$ being the average and the second moment of τ. The average $\overline{\tau}$ is given by

$$\overline{\tau} = \int_0^\infty \int_0^\infty ... \int_0^\infty \tau(n_1, n_2, ..., n_N) p(n_1, \Delta z) p(n_2, \Delta z) ... p(n_N, \Delta z)\, dn_1\, dn_2 ... dn_N$$

$$\tau(n_1, n_2, ..., n_N) = \frac{1}{2d\left(\sum_{k=1}^N n_k\right)} \sum_{k=1}^N n_k \left[\frac{z_k^2}{v_1} + \frac{(d-z_k)^2}{v_2}\right]$$

$$\overline{\tau^2} = \int_0^\infty \int_0^\infty ... \int_0^\infty \tau^2(n_1, n_2, ..., n_N) p(n_1, \Delta z) p(n_2, \Delta z) ... p(n_N, \Delta z)\, dn_1\, dn_2 ... dn_N$$

$$\tau^2(n_1, n_2, ..., n_N) = \frac{1}{4d^2\left(\sum_{k=1}^N n_k\right)^2} \sum_{k=1}^N \sum_{r=1}^N n_k n_r \left[\frac{z_k^2}{v_1} + \frac{(d-z_k)^2}{v_2}\right] \left[\frac{z_r^2}{v_1} + \frac{(d-z_r)^2}{v_2}\right]$$

→

$$\Delta_\tau^2 = \overline{\tau^2} - \overline{\tau}^2 = \frac{b_N}{\Delta z} \frac{d^3(4v_1^2 - 7v_1v_2 + 4v_2^2)}{180 v_1^2 v_2^2}$$

→

$$b_N = \int_0^\infty \left[\frac{\Delta z}{\lambda} \int_0^\infty \frac{n_1^2 p_{clu}(n_1)}{(n_1+n)^2} dn_1\right] p(n, d) dn$$

C.o.g. time of a silicon detector signal

Up to this point the expression for b_N is still completely general for any kind of cluster size distributions $p_{clu}(n)$ and resulting $p(n,d)$. Using the Landau theory we use $p_{clu}(n)$ from Eq. 4 and have

$$\int_0^\infty \frac{n_1^2 p_{clu}(n_1)}{(n_1+n)^2} dn_1 = \int_{2.50}^\infty \frac{2.50}{(n_1+n)^2} dn_1 = \frac{2.50}{n+2.50} \approx \frac{2.50}{n} \tag{35}$$

(for $n \gg 1$) and with Eq. 5 we get

$$b_N \approx 2.50 \frac{\Delta z}{\lambda} \int_0^\infty \frac{p(n,d)}{n} dn = \frac{\Delta z}{d} \int_0^\infty \frac{L(z+\gamma-1-\log d/\lambda)}{z} dz \approx \frac{\Delta z}{d} \frac{1}{(1+1.155 \log d/\lambda)} \tag{36}$$

The last expression is an approximation of better than 1% in the interval $4 < \log d/\lambda < 10$, which corresponds to a range of the silicon sensor thickness of $15 < d < 5000\,\mu$m for a value of $\lambda = 0.25\,\mu$m. For the standard deviation we therefore finally have

$$\Delta_\tau \approx \frac{d}{\sqrt{1+1.155 \log d/\lambda}} \sqrt{\frac{4}{180 v_1^2} - \frac{7}{180 v_1 v_2} + \frac{4}{180 v_2^2}} \tag{37}$$

This is the time resolution of s silicon sensor when measuring the c.o.g. time.
At constant electric field i.e. at constant drift velocity v_1 and v_2, the time resolution scales with d, which represents the trivial fact that the duration of the signal and therefore also Δ_τ scales with d. For a given voltage V, the electric fields in the thinner sensors, and therefore the velocities of electrons and holes are of course larger, so the time resolution improves significantly beyond the $1/d$ scaling for thin sensors.
If we associate v_1 and v_2 with the electron and hole velocity, $T_1 = d/v_1$ and $T_2 = d/v_2$ are the total drift times of electrons and holes, and $T_{12} = d/\sqrt{v_1 v_2}$ is the total drift time assuming the geometric mean of the electron and hole velocity. The expression $1/\sqrt{1+1.155 \log d/\lambda}$ varies only from 0.37 to 0.33 for d from $50\,\mu$m to $300\,\mu$m for $\lambda = 0.25\,\mu$m, which means that the effect of the Landau fluctuations does not vary significantly in this range of sensor thickness. So by approximating it with the value of 0.35 we have

$$\Delta_\tau \approx \frac{1}{20} \sqrt{T_1^2 - 1.75\, T_{12} + T_2^2} \qquad 50\,\mu\text{m} < d < 300\,\mu\text{m} \tag{38}$$

Driftvelocities of electrons and holes

$$v_e(E) = \frac{\mu_e E}{\left[1 + \left(\frac{\mu_e E}{v_{sat}^e}\right)^{\beta_e}\right]^{1/\beta_e}} \qquad v_h(E) = \frac{\mu_h E}{\left[1 + \left(\frac{\mu_h E}{v_{sat}^h}\right)^{\beta_h}\right]^{1/\beta_h}} \tag{39}$$

where we chose $\mu_e = 1417\,\mathrm{cm}^2/\mathrm{Vs}$, $\mu_h = 471\,\mathrm{cm}^2/\mathrm{Vs}$, $\beta_e = 1.109$, $\beta_h = 1.213$ and $v_{sat}^e = 1.07 \times 10^7\,\mathrm{cm/s}$ and $v_{sat}^h = 0.837 \times 10^7\,\mathrm{cm/s}$ at 300 K in accordance with the default models in Sentaurus Device [9]. The resulting driftvelocity together with the time that the electrons and holes need to traverse the sensor (assuming $V_{dep} = 0$) are given in Fig. 2. For a 50 μm sensor at 250 V the electrons take 0.5 ns and the holes take 0.7 ns to traverse the sensor, so the total signal duration is < 0.7 ns.
The values for the time resolution according to Eq. 37 are given in Fig. 3. For an applied voltage of 250 V the values are 305, 152, 52, 21 ps for 300, 200, 100, 50 μm sensors.

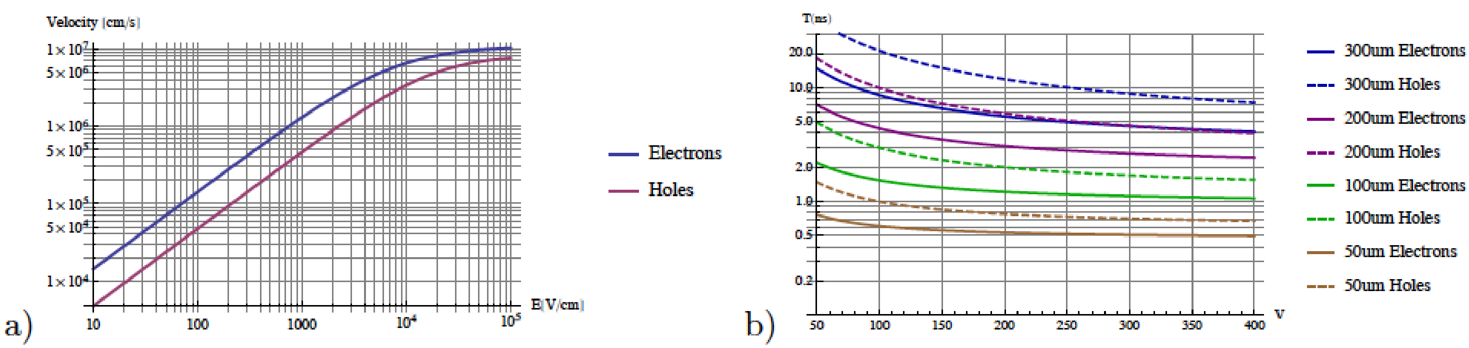


Figure 2: a) Velocity of electrons and holes as a function of electric field. b) Time for electrons an holes to transit the full thickness of the sensor assuming $V_{dep} = 0$.

Standard deviation of the c.o.g. time

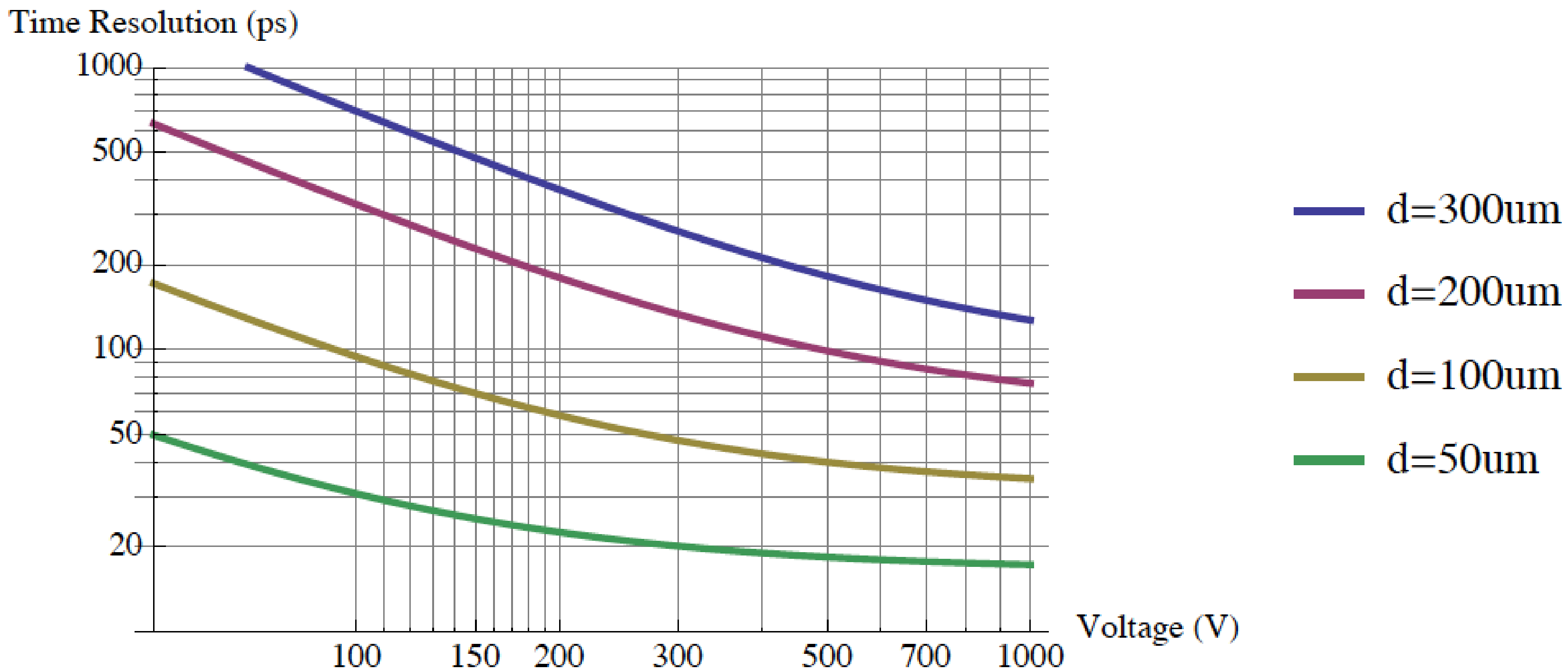


Figure 3: Standard deviation of the c.o.g. time from Eq. 37 for different values of silicon sensor thickness as a function of applied voltage V, assuming $\lambda = 0.25\,\mu$m, the Landau theory and $V_{dep} \approx 0$.

$$\Delta_\tau \approx \frac{d}{\sqrt{1 + 1.155\,\log d/\lambda}} \sqrt{\frac{4}{180 v_1^2} - \frac{7}{180 v_1 v_2} + \frac{4}{180 v_2^2}} \qquad (37)$$

Noise

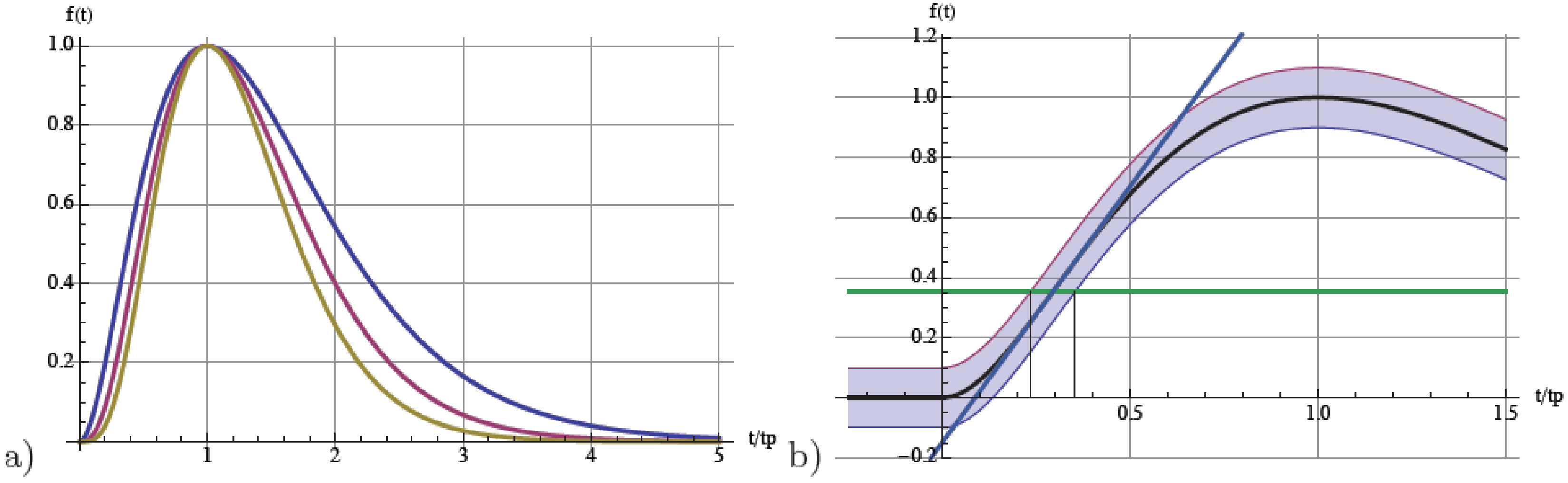


Figure 4: a) Amplifer response for $n = 2, 3, 4$ from Eq. 40. b) Contribution to the time resolution from the noise.

$$f(t) = \left(\frac{t}{t_p}\right)^n e^{n(1-t/t_p)}\,\Theta(t) \qquad |W(i2\pi f)| = \frac{1}{\sqrt{[1+(2\pi f)^2 t_p^2/n^2]^{n+1}}} \qquad f_{bw} = \frac{1}{2\pi\, t_p}\, n\sqrt{2^{1/(n+1)}-1}$$

$$\sigma_t = \frac{\sigma_{noise}}{A}\frac{1}{f'(t_s)} = \frac{\sigma_{noise}}{A}\frac{t_p}{e^{\sqrt{n}} n^{(3/2-n)}(n-\sqrt{n})^{n-1}} = \frac{\sigma_{noise}}{A}\frac{1}{2\pi f_{bw}}\frac{\sqrt{2^{1/(n+1)}-1}}{e^{\sqrt{n}} n^{(1/2-n)}(n-\sqrt{n})^{n-1}}$$

$$= \frac{\sigma_{noise}}{A}\, t_p \times (0.59, 0.57, 0.54, 0.51) \qquad \text{for} \qquad n = 2, 3, 4, 5$$

$$= \frac{\sigma_{noise}}{A}\frac{1}{f_{bw}} \times (0.10, 0.12, 0.13, 0.14) \qquad \text{for} \qquad n = 2, 3, 4, 5$$

Noise

So for an amplifier with a peaking time of t_p=1 ns and $n = 2$, the time resolution is 60 ps for a signal to noise ration of 10 and 20 ps for a signal to noise ratio of 30.
The pulse-height of the sensor signal is given by the total number n of deposited e-h pairs, so if we write the noise σ_{noise} in units of electrons the signal to noise ratio is σ_{noise}/n. Since n is varying according to the Landau distribution $p(n,d)$ from Eq. 5, using Eq. 36 we can calculate the average signal to noise ratio and the average resolution to

$$\overline{\sigma}_t = \frac{\sigma_{noise}}{f'(t_s)} \int_0^\infty \frac{p(n,d)}{n} dn \approx \frac{\sigma_{noise}}{f'(t_s)} \frac{0.4\lambda}{d} \frac{1}{1+1.155 \log\frac{d}{\lambda}} \quad (45)$$

$$= \sigma_{noise} \frac{0.4\lambda}{d} \frac{1}{1+1.155 \log\frac{d}{\lambda}} \; t_p \times (0.59, 0.57, 0.54, 0.51) \quad \text{for} \quad n = 2,3,4,5 \quad (46)$$

$$= \sigma_{noise} \frac{0.4\lambda}{d} \frac{1}{1+1.155 \log\frac{d}{\lambda}} \frac{1}{f_{bw}} \times (0.10, 0.12, 0.13, 0.14) \quad \text{for} \quad n = 2,3,4,5 \quad (47)$$

For an average cluster distance of $\lambda = 0.25\,\mu$m an amplifier with $n = 2$, this expression becomes a sensor thickness of $d = 50\,\mu$m,

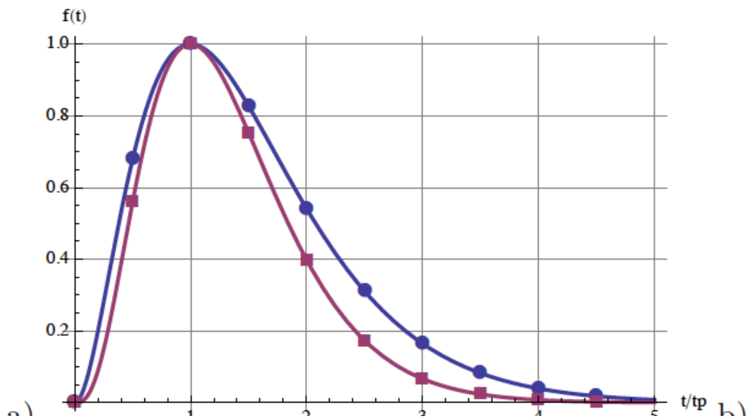
$$\sigma_t = \sigma_{noise}[\text{electrons}] \times 1.6 \times 10^{-4}\, t_p \qquad d = 50\mu m \quad (48)$$

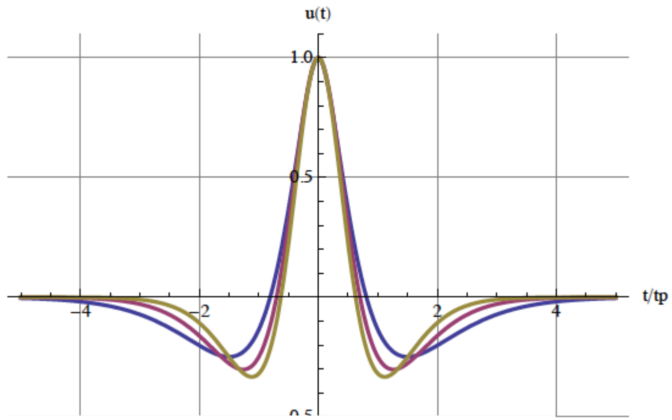
$$= \sigma_{noise}[\text{electrons}] \times 3.3 \times 10^{-5}\, t_p \qquad d = 200\mu m \quad (49)$$

$$(50)$$

so for a peaking time of 2 ns and an equivalent noise charge of 50 electrons the noise contribution to the time resolution is 16.6 ps. There are many methods for realizing a discrimination at constant fraction of the signal, in the following we illustrate the method of continuos sampling.

Noise





N measurement points at tracker layers at position $x_1, x_2, ..., x_N$. We proceed as outlined in [11] where the problem is stated as a χ^2 minimization according to

4. For times

$$\chi^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}[S_i - \alpha_1 f(t_i) + \alpha_2 f'(t_i)]\, V_{ij}\, [S_j - \alpha_1 f(t_j) + \alpha_2 f'(t_j)] \tag{52}$$

The matrix V_{ij} is the inverse of the autocorrelation matrix $R_{ij} = R(t_i - t_j)$ with $R(t)$ being the autocorrelation function of the noise. The series noise of an amplifier for a given white series noise spectral density e_n^2 and detector capacitance C is given by

$$\sigma_{noise}^2 = \frac{1}{2} e_n^2 C^2 \int_{-\infty}^{\infty} f'(t)^2 dt = \frac{1}{2} e_n^2 C^2 \frac{n^2\,(2n-2)!}{t_p} \left(\frac{e}{2n}\right)^{2n} \tag{53}$$

The autocorrelation function of this series noise is

$$R(t) = \sigma_{noise}^2 \int_{-\infty}^{\infty} f'(t+u)\, f'(u) du = \sigma_{noise}^2\, n! \left(\frac{2n|t|}{t_p}\right)^n \frac{2t_p K_{n-1/2}(n|t|/t_p) - tK_{n+1/2}(n|t|/t_p)}{(2n-2)!\,\sqrt{2n|t|\,t_p\pi}} \tag{54}$$

with $K_\nu(x)$ being the modified Bessel function of the second kind. For $n = 2, 3$ evaluates to

$$R(t) = \sigma_{noise}^2 U(t) = \sigma_{noise}^2\, e^{-2|t|/t_p} \left[1 + 2\frac{|t|}{t_p} - 4\left(\frac{|t|}{t_p}\right)^2\right] \qquad n = 2 \tag{55}$$

$$= \sigma_{noise}^2\, e^{-3|t|/t_p} \left[1 + 3\frac{|t|}{t_p} - 9\left(\frac{|t|}{t_p}\right)^3\right] \qquad n = 3 \tag{56}$$

Noise

The autocorrelation function is shown in Fig. 5b), and we see that for time intervals smaller than $t_p/2$ the samples become highly correlated. In the following we us n_s samples within the peaking time t_p, so we have sampling time bins of $\Delta t = t_p/n_s$. We sample the signal in the range of $0 < t < 5\,t_p$, giving $t_i = i\,\Delta t$ with $0 < i < 5\,n_s$. Defining

$$Q_1(n_s) = \sum_{ij} f(t_i) U_{ij}^{-1} f(t_j) \quad Q_2(n_s) = \sum_{ij} f'(t_i) U_{ij}^{-1} f'(t_j) \quad Q_3(n_s) = \sum_{ij} f'(t_i) U_{ij}^{-1} f(t_j) \tag{57}$$

where U_{ij}^{-1} is the inverse of the matrix $U_{ij} = U(t_i - t_j)$, the covariance matrix elements ε_{ij} for α_1, α_2 are then

$$\varepsilon_{11} = \sigma_A^2 = \frac{\sigma_{noise}^2 Q_2}{Q_1 Q_2 - Q_3^2} \qquad \varepsilon_{22} = A^2 \frac{\sigma_\tau^2}{t_p^2} = \frac{\sigma_{noise}^2 Q_1}{Q_1 Q_2 - Q_3^2} \qquad \varepsilon_{12} = \frac{\sigma_{noise}^2 Q_3}{Q_1 Q_2 - Q_3^2} \tag{58}$$

So for the time resolution we finally have

$$\frac{\sigma_\tau}{t_p} = \frac{\sigma_{noise}}{A} \sqrt{\frac{Q_1(n_s)}{Q_1(n_s) Q_2(n_s) - Q_3(n_s)^2}} = \frac{\sigma_{noise}}{A}\, c(n_s) \tag{59}$$

Expressing again the signal amplitude in terms of the number of electrons deposited in a sensor of thickness d we get

$$\overline{\sigma_t} = \sigma_{noise}[\text{electrons}] \frac{0.4\lambda}{d} \frac{1}{1 + 1.155 \log \frac{d}{\lambda}}\, t_p\, c(n_s) \tag{60}$$

This expression represents the optimum time resolution that can be achieved for a given sampling frequency. Fig. 6 shows the function $c(n_s)$ assuming an amplifier with $n = 2, 3$. The horizontal lines correspond to the numbers of 0.59 and 0.57 from Eq. 44 when using constant fraction discrimination.

Noise

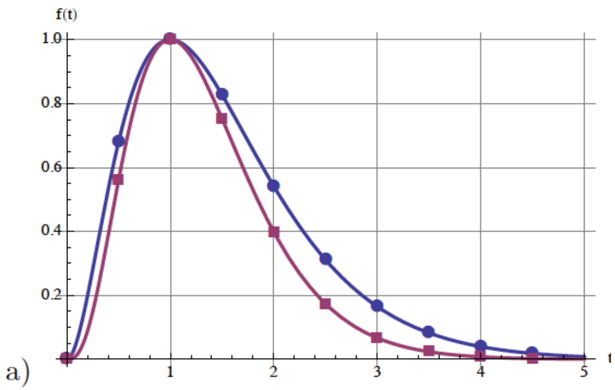
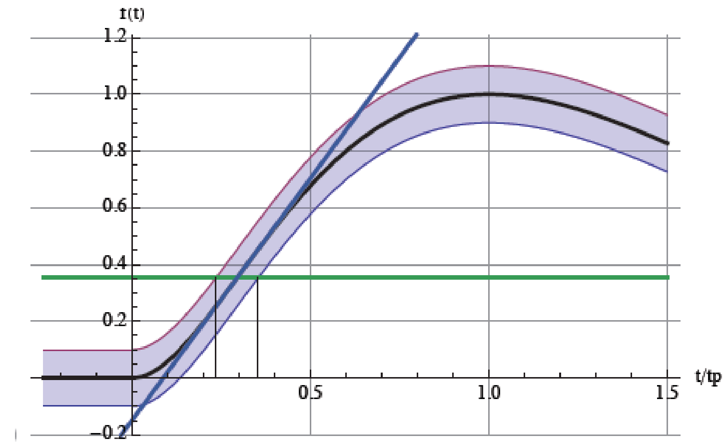


Figure 5: a) Sampling the signal at constant frequency.



$$= \sigma_{noise} \frac{0.4\lambda}{d} \frac{1}{1 + 1.155 \log \frac{d}{\lambda}} \quad t_p \quad \times (0.59, 0.57, 0.54, 0.51)$$

$$n = 2, 3, 4, 5$$

$$\overline{\sigma_t} = \sigma_{noise}[\text{electrons}] \frac{0.4\lambda}{d} \frac{1}{1 + 1.155 \log \frac{d}{\lambda}} t_p \, c(n_s)$$

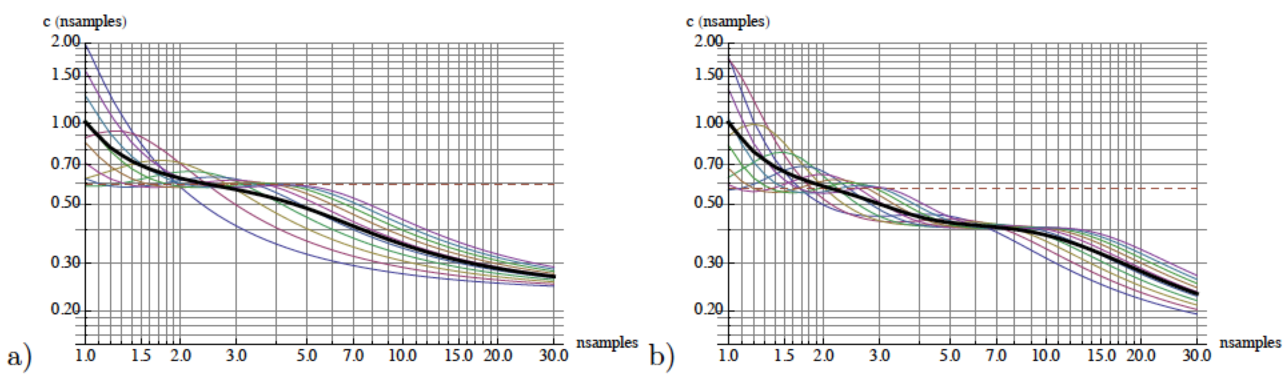


Figure 6: The function $c(n_s)$ for an amplifier with $n = 2$(a) and $n = 3$(b). The horizontal line is the result when discriminating the signal at the maximum slope.

For 2-3 samples within the peaking time, the contribution is similar to the cosntant fraction discrimination.

Beyond that one can improve by about a factor 2-3 for very high frequency smapling.

Weighting field fluctuations

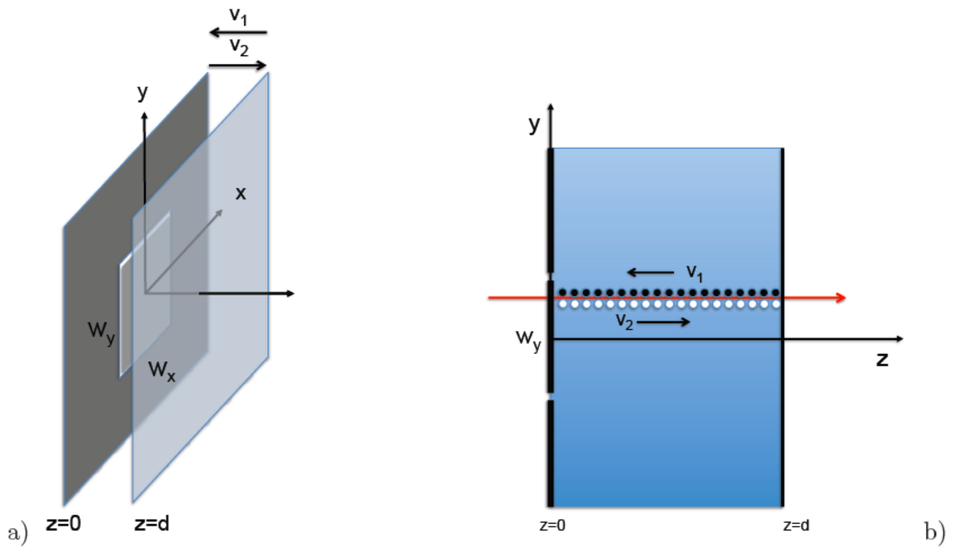


Figure 7: a) A pixel of dimension w_x, w_y centred at $x = y = z = 0$ in a parallel plate geometry of plate distance d. b) Uniform charge deposit of a particle passing the silicon sensor.

$$\Delta_\tau^2 = \overline{\tau^2} - \overline{\tau}^2 = d^2 \left(\frac{c_{11}}{v_1^2} + \frac{c_{12}}{v_1 v_2} + \frac{c_{22}}{v_2^2} \right) = c_{11} T_1^2 + c_{12} T_{12} + c_{22} T_2^2$$

$$\begin{aligned}
c_{11} &= \frac{1}{w_x\, w_y} \iint a_1^2 dx dy - \left(\frac{1}{w_x\, w_y} \iint a_1 dx dy \right)^2 \\
c_{12} &= \frac{2}{w_x\, w_y} \iint a_1 a_2 dx dy - \frac{2}{(w_x\, w_y)^2} \iint a_1 dx dy \iint a_2 dx dy \\
c_{22} &= \frac{1}{w_x\, w_y} \iint a_2^2 dx dy - \left(\frac{1}{w_x\, w_y} \iint a_2 dx dy \right)^2
\end{aligned}$$

$$\begin{aligned}
a_1(x,y) &= \frac{1}{2} - \frac{1}{d^2} \int_0^d (d-z) \phi_w(x,y,z) dz \\
a_2(x,y) &= \frac{1}{d^2} \int_0^d z \phi_w(x,y,z) dz
\end{aligned}$$

Point charge potential and weighting field of a pixel or pad in a plane condenser

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ABSTRACT

We derive expressions for the potential of a point charge as well as the weighting potential and weighting field of a rectangular pad for a plane condenser, which are well suited for numerical evaluation. We relate the expressions to solutions employing the method of image charges, which allows discussion of convergence properties and estimation of errors, providing also an illuminating example of a problem with an infinite number of image charges.

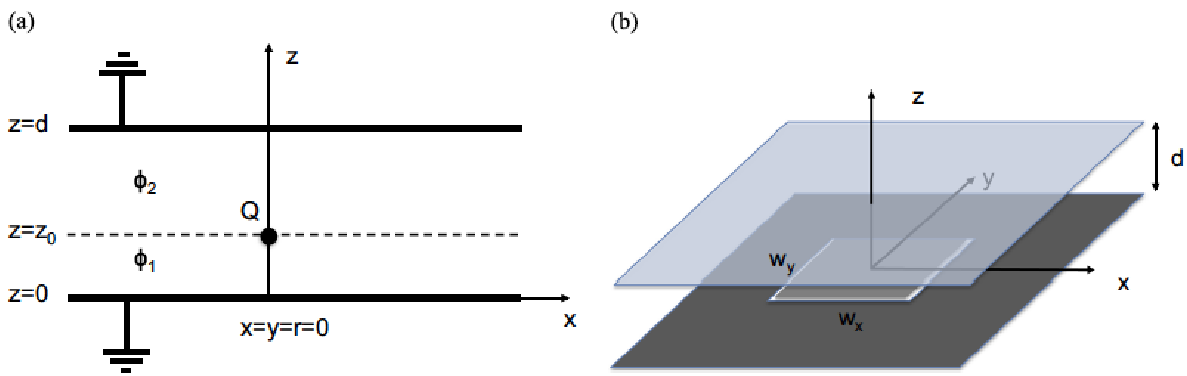


Fig. 1. (a) Point charge Q between two grounded metal planes. (b) Readout pad or pixel of dimension w_x and w_y centred at the origin.

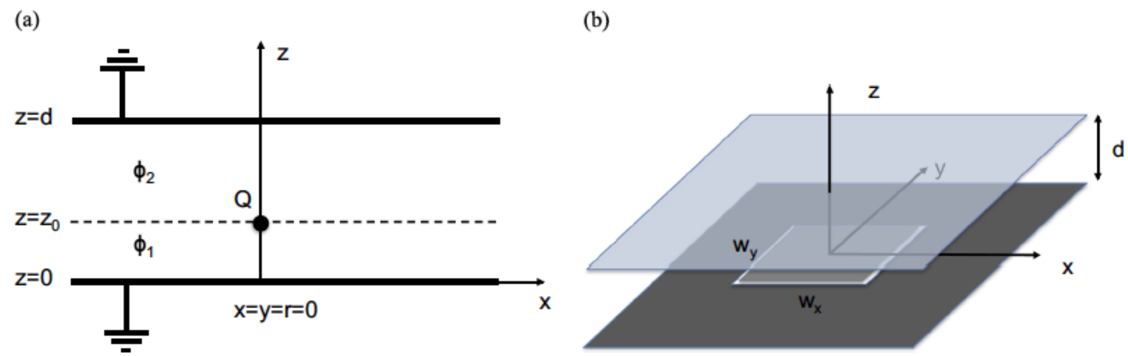


Fig. 1. (a) Point charge Q between two grounded metal planes. (b) Readout pad or pixel of dimension w_x and w_y centred at the origin.

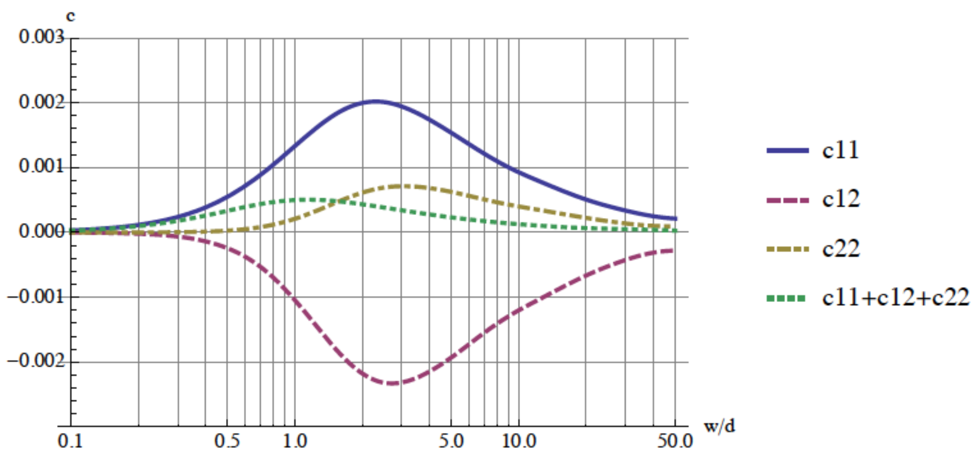
$$\frac{4\pi\varepsilon_0}{Q}\phi(r,z)=\frac{1}{\sqrt{r^2+(z-z_0)^2}}-\frac{1}{\sqrt{r^2+(z+z_0)^2}}$$
$$+\sum_{n=1}^{N}\left[\frac{1}{\sqrt{r^2+(z+2nd-z_0)^2}}+\frac{1}{\sqrt{r^2+(z-2nd-z_0)^2}}-\frac{1}{\sqrt{r^2+(z-2nd+z_0)^2}}-\frac{1}{\sqrt{r^2+(z+2nd+z_0)^2}}\right]$$
$$-\int_0^\infty 2J_0(kr)e^{-k(2N+1)d}\frac{\sinh(kz)\,\sinh(kz_0)}{\sinh(kd)}dk$$

$$\frac{\phi_w(x,y,z)}{V_w}=\frac{1}{2\pi}f(x,y,z)-\frac{1}{2\pi}\sum_{n=1}^{N}[f(x,y,2nd-z)-f(x,y,2nd+z)]$$
$$-\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(k_x x)\,\sin\left(k_x\frac{w_x}{2}\right)\,\cos(k_y y)$$
$$\times\sin\left(k_y\frac{w_y}{2}\right)\frac{e^{-k(2N+1)d}}{k_x k_y}\frac{\sinh(kz)}{\sinh(kd)}dk_x dk_y$$

with

$$f(x,y,u)=\int_{x-w_x/2}^{x+w_x/2}\int_{y-w_y/2}^{y+w_y/2}\frac{u}{(x'^2+y'^2+u^2)^{3/2}}dx'\,dy'$$
$$=\arctan\left(\frac{x_1 y_1}{u\sqrt{x_1^2+y_1^2+u^2}}\right)+\arctan\left(\frac{x_2 y_2}{u\sqrt{x_2^2+y_2^2+u^2}}\right)$$

Weighting field fluctuations for a square pixel



In order to minimize the effect, the 'fast' electrons should move towards the pad while the 'slow' holes should move away from the pad.

Figure 9: The coefficients c_{11}, c_{12}, c_{13} for different values of w/d, where w is the width of the square pad and d is the silicon thickness.

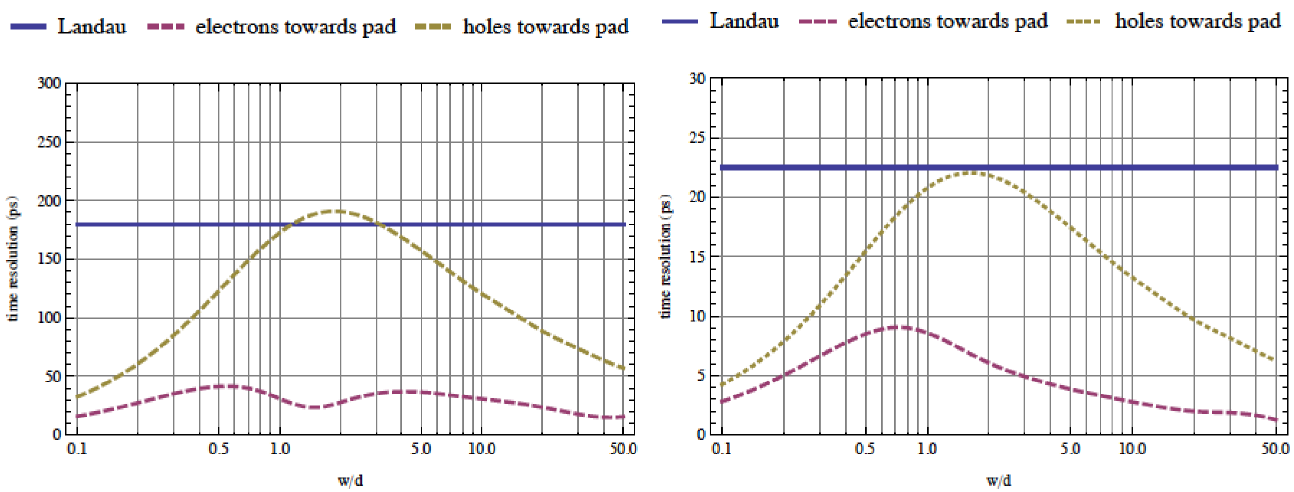


Figure 10: Standard deviation for the c.o.g. time for values of $d = 200\,\mu$m (a) and $d = 50\,\mu$m (b) and $V = 200\,$V, assuming uniform charge deposit and a square readout pad. The horizontal line represents the Landau fluctuations form Eq. 37. The two curves in the plots represent the cases where either the electrons or the holes move towards the readout pad.

Combined weighting field and Landau fluctuations

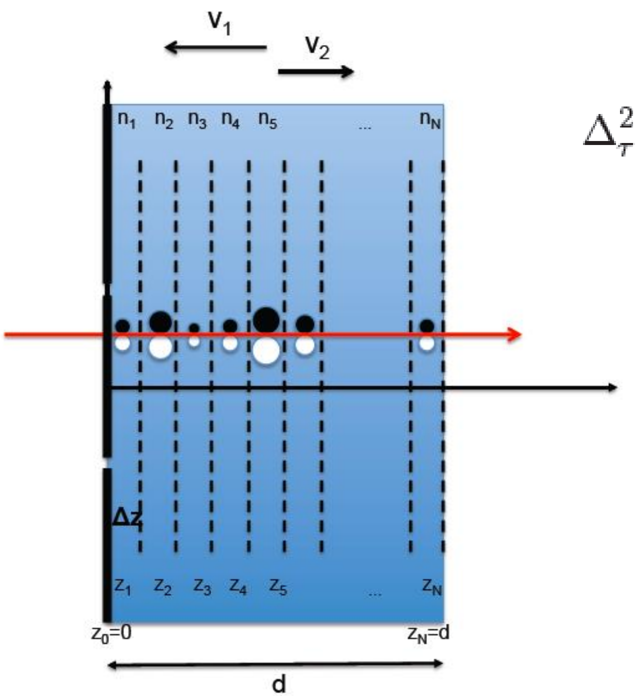


Figure 11: Silicon sensor with a readout pad centered at $x = y$

$$\Delta_\tau^2 = \frac{b_N\, d}{\Delta z} d^2 \left(\frac{k_{11}}{v_1^2} + \frac{k_{12}}{v_1 v_2} + \frac{k_{22}}{v_2^2} \right) + d^2 \left(\frac{c_{11}}{v_1^2} + \frac{c_{12}}{v_1 v_2} + \frac{c_{22}}{v_2^2} \right)$$

$$k_{11} = \frac{1}{w_x w_y} \iint (b_{11} - a_1^2)\, dx dy$$

$$k_{12} = \frac{2}{w_x w_y} \iint (b_{12} - a_1 a_2)\, dx dy$$

$$k_{22} = \frac{1}{w_x w_y} \iint (b_{22} - a_2^2)\, dx dy$$

$$a_1(x,y) = \frac{1}{2} - \frac{1}{d^2} \int_0^d (d-z)\phi_w(x,y,z)\,dz$$

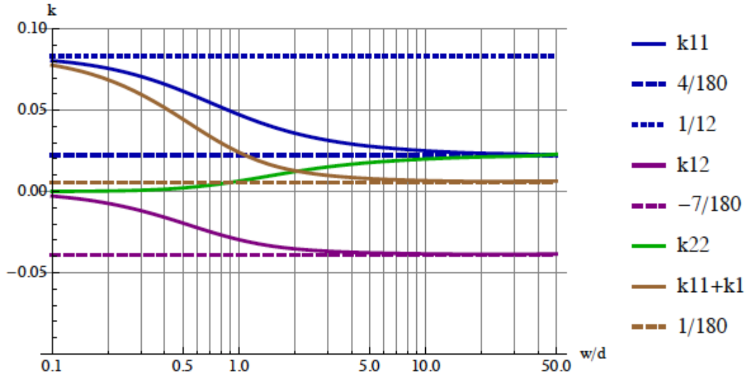
$$a_2(x,y) = \frac{1}{d^2} \int_0^d z\phi_w(x,y,z)\,dz$$

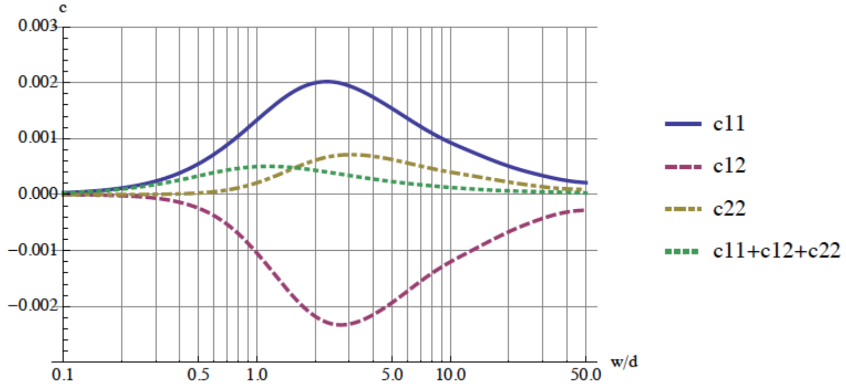
$$b_{11}(x,y) = \frac{1}{d} \int_0^d \left[\frac{z}{d} - \frac{1}{d} \int_0^z \phi_w(x,y,z')\,dz' \right]^2 dz \qquad (77)$$

$$b_{12}(x,y) = \frac{1}{d} \int_0^d \left[\frac{z}{d} - \frac{1}{d} \int_0^z \phi_w(x,y,z')\,dz' \right] \left[\frac{1}{d} \int_z^d \phi_w(x,y,z')\,dz' \right] dz$$

$$b_{22}(x,y) = \frac{1}{d} \int_0^d \left[\frac{1}{d} \int_z^d \phi_w(x,y,z')\,dz' \right]^2 dz$$

Combined weighting field and Landau fluctuations





$$\Delta_\tau^2 = \frac{b_N\, d}{\Delta z}\, d^2 \left(\frac{k_{11}}{v_1^2} + \frac{k_{12}}{v_1 v_2} + \frac{k_{22}}{v_2^2} \right) + d^2 \left(\frac{c_{11}}{v_1^2} + \frac{c_{12}}{v_1 v_2} + \frac{c_{22}}{v_2^2} \right)$$

$$\Delta_\tau^2 = \overline{\tau^2} - \overline{\tau}^2 = \frac{b_N}{\Delta z} \frac{d^3(4v_1^2 - 7v_1 v_2 + 4v_2^2)}{180 v_1^2 v_2^2}$$

Combined weighting field and Landau fluctuations

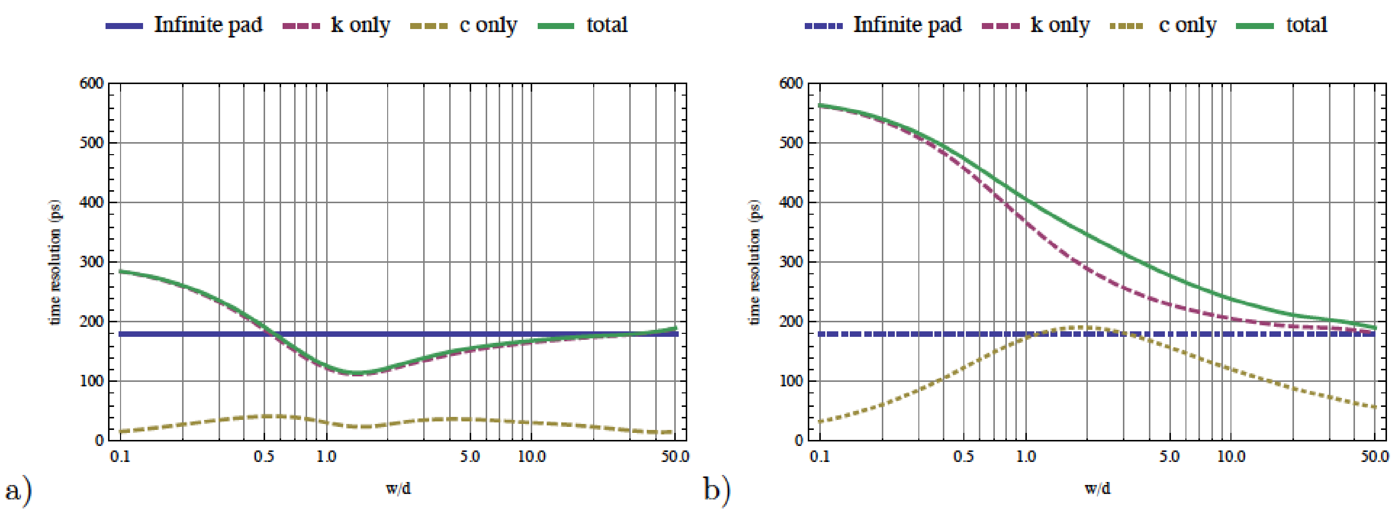


Figure 13: C.o.g. time resolution for values of $d = 200\,\mu\text{m}$ and $V = 200\,\text{V}$. The 'c only' curve refers to the effect from a uniform line charge. In a) the electrons move towards the pixel while in b) the holes move towards the pixel.

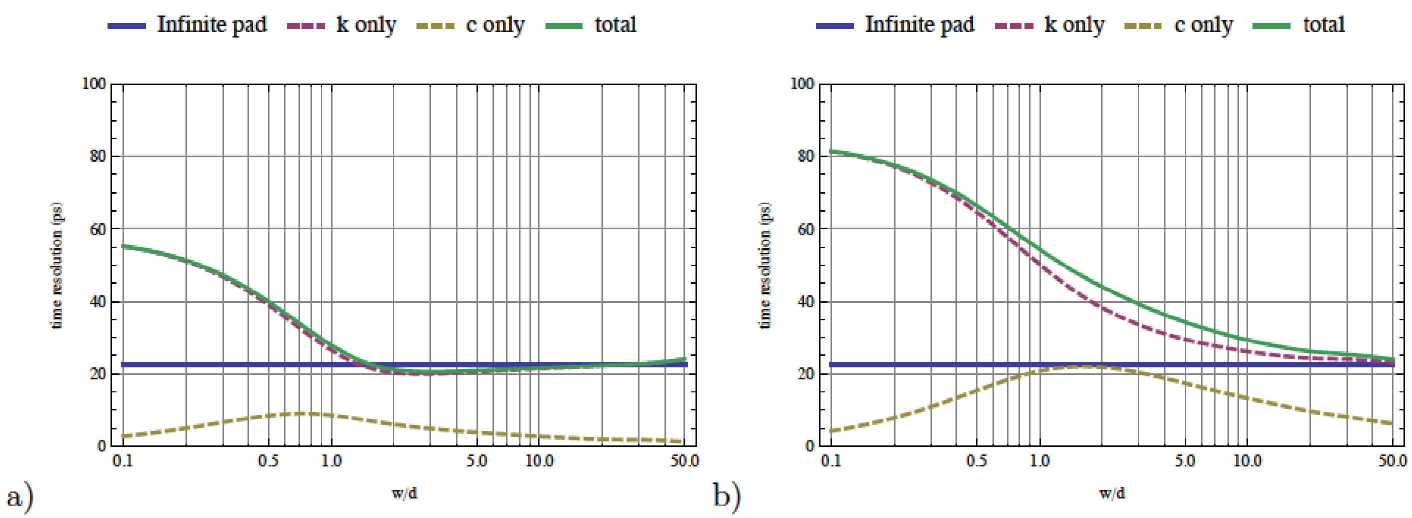


Figure 14: Time resolution for values of $d = 50\,\mu\text{m}$ and $V = 200\,\text{V}$. The 'c only' curve refers to the effect from a uniform line charge. In a) the electrons move towards the pixel while in b) the holes move towards the pixel.

Leading edge discrimination

The current signal for a single electron hole pair at position x, y, z is

$$i_0(x,y,z,t) = e_0\left[v_1\, E_w(x,y,z - v_1 t)\Theta(z/v_1 - t) + v_2\, E_w(x,y,z+v_2 t)\Theta((d-z)/v_2 - t)\right] \qquad (80)$$

The current signal for having n_1 e/h pairs at $z = \Delta z$, n_2 e/h pairs at $z = 2\Delta z$ etc. is given by

$$i(n_1, n_2, ..., n_N, x, y, t) = \sum_{k=1}^{N} n_k i_0(x, y, k\Delta z, t) \qquad (81)$$

We now send this signal through an amplifier with delta response $cf(t/t_p)$ where t_p is the peaking time, $f(1) = 1$ and c has units of V/C

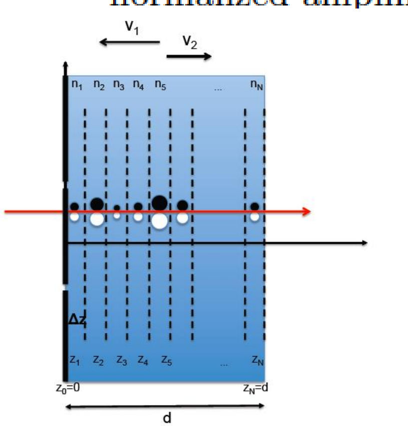
$$f(x) = x^n\, e^{n(1-x)} \qquad (82)$$

so the output signal becomes

$$s(n_1, n_2, ..., n_N, x, y, t) = c\int_0^t f\left(\frac{t-t'}{t_p}\right) i(n_1, n_2, ..., n_N, x, y, t')dt' \qquad (83)$$

To perform slewing corrections we divide the signal by the total charge $e_0 \sum n_k$ and we finally have the normalized amplifier output signal

$$h(n_1, n_2, ..., n_N, x, y, t) = \frac{c}{\sum_{k=1}^{N} n_k} \sum_{k=1}^{N} n_k\, g(x, y, k\Delta z, t) \qquad (84)$$



gure 11: Silicon sensor with a readout pad centered at $x = y = z = ($

Leading edge discrimination

$$h(n_1, n_2, ..., n_N, x, y, t) = \frac{c}{\sum_{k=1}^{N} n_k} \sum_{k=1}^{N} n_k \, g(x, y, k\Delta z, t) \tag{84}$$

where $g(x, y, z, t)$ is given by

$$\begin{aligned} g(x,y,z,t) &= \Theta(z - v_1 t) \int_{\frac{z-v_1 t}{d}}^{\frac{z}{d}} f\left(\frac{v_1 t - z + ud}{v_1 t_p}\right) E_w(x/d, y/d, u, w_x/d, w_y/d, 1) du \\ &+ \Theta(v_1 t - z) \int_0^{\frac{z}{d}} f\left(\frac{v_1 t - z + ud}{v_1 t_p}\right) E_w(x/d, y/d, u, w_x/d, w_y/d, 1) du \\ &+ \Theta[(d - z) - v_2 t] \int_{\frac{z}{d}}^{\frac{z+v_2 t}{d}} f\left(\frac{v_2 t + z - ud}{v_2 t_p}\right) E_w(x/d, y/d, u, w_x/d, w_y/d, 1) du \\ &+ \Theta[v_2 t - (d - z)] \int_{\frac{z}{d}}^{1} f\left(\frac{v_2 t + z - ud}{v_2 t_p}\right) E_w(x/d, y/d, u, w_x/d, w_y/d, 1) du \end{aligned}$$

The average normalized signal is then given by

$$\overline{h}(t) = \frac{c}{w_x \, w_y} \iint \left[\int_0^1 g(x, y, sd, t) ds\right] dx dy \tag{85}$$

The variance of the signal at time t is given by

$$\begin{aligned} \Delta_h^2(t) &= \frac{b_N \, d}{\Delta z} \frac{c^2}{w_x w_y} \iint \left[\int_0^1 g(x, y, sd, t)^2 ds - \left(\int_0^1 g(x, y, sd, t) ds\right)^2\right] dx dy \\ &+ \frac{c^2}{w_x w_y} \iint \left(\int_0^1 g(x, y, sd, t) ds\right)^2 dx dy - \left[\frac{c}{w_x w_y} \iint \left(\int_0^1 g(x, y, sd, t) ds\right) dx dy\right]^2 \end{aligned}$$

Leading edge discrimination

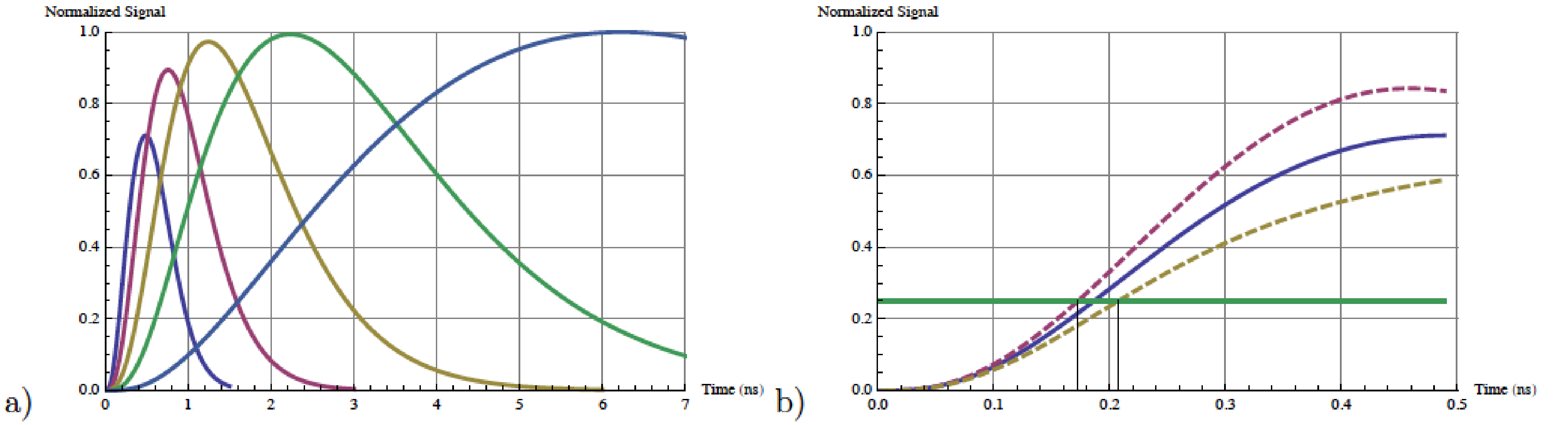


Figure 15: a) Average normalized signal $\overline{h}(t)$ for amplifier peaking times $t_p = 0.25, 0.5, 1, 2, 6$ ns for a 50μm sensor and V=200 V. b) The normalized average signal $\overline{h}(t)$ for $t_p = 0.25$ ns together with the curves $\overline{h}(t) + \Delta_h(t)$ and $\overline{h}(t) - \Delta_h(t)$. (86)

Since we 'normalized' the signal by the total amount of charge in Eq. 84, before applying threshold discrimination, we also have to normalize the noise σ_{noise} by the total amount of charge to calculate it's impact. The average normalized noise is then

$$\overline{\sigma}_{noise} = \int_0^\infty \frac{\sigma_{noise}}{n} p(n,d)\, dn = \sigma_{noise} \frac{\lambda}{2.50\, d} \frac{1}{1 + 1.155 \log \frac{d}{\lambda}} \qquad (87)$$

The variance if the signal due to Landau fluctuations from Eq. 20 can therefore be expressed as equivalent noise level expressed in electrons according to

$$\sigma_{landau}(t) = \Delta_h(t) \frac{2.50\, d}{\lambda} (1 + 1.155 \log \frac{d}{\lambda}) \qquad (88)$$

Leading edge discrimination 200um sensor

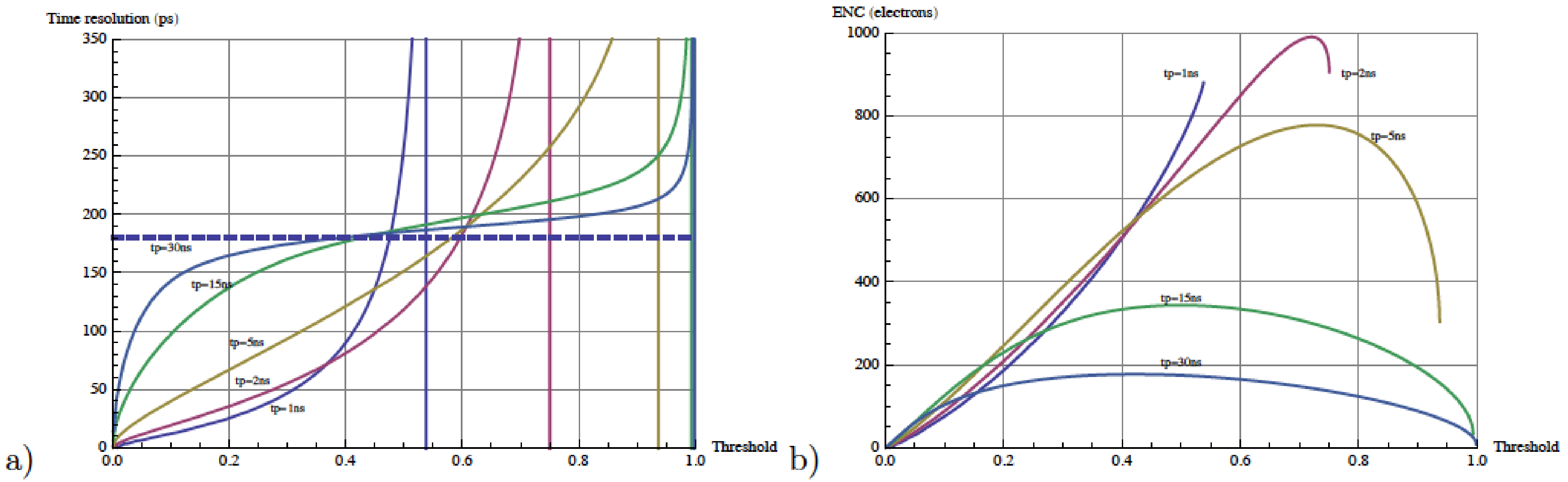


Figure 17: Time resolution for a sensor of 200μm thickness at 200V bias voltage. The slewing correction is performed by dividing the signal by the total charge and applying the threshold as a fraction of this charge. The plot on the right shows the ENC needed to match the noise effect of the time resolution to the effect from the Landau fluctuations.

Leading edge discrimination 50um sensor

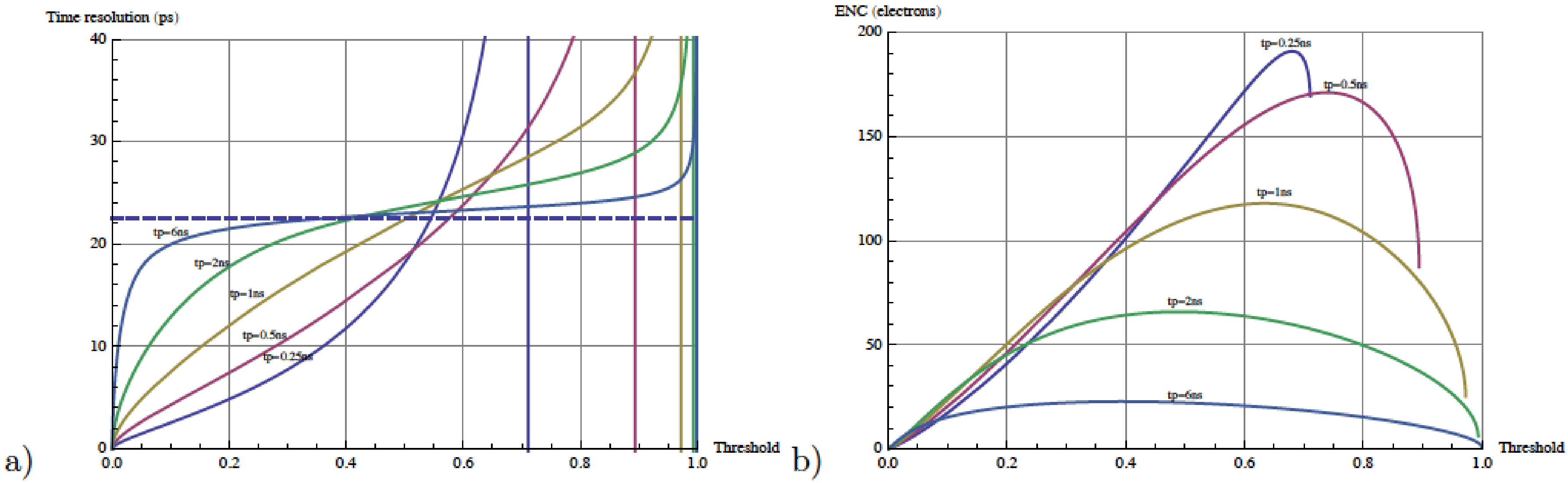


Figure 16: Time resolution for a sensor of 50μm thickness at 200V bias voltage. The slewing correction is performed by dividing the signal by the total charge and applying the threshold as a fraction of this charge. The plot on the right shows the ENC needed to match the noise effect of the time resolution to the effect from the Landau fluctuations.

E.g. for a 50um sensor with tp=1ns, a threshold at 20% of the most probable value and a noise value of 50 electrons one should manage to stay <20ps time resolution.

Sensors with internal gain, c.o.g. time resolution

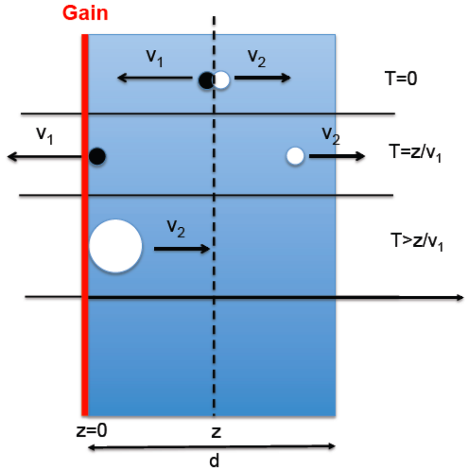


Figure 18: Silicon sensor with internal gain. An e-h par is produced at position z, the electron arrives at $z = 0$ at time $T = z/v_1$, the electron multiplies is a high field layer at $z = 0$ and the holes move back to $z = d$, inducing the dominant part of the current signal.

process are moving back from $z = 0$ to $z = d$ through the entire sensor thickness d. If we assume the gain G to be sufficiently large such that the signal from the primary electron and hole movement is negligible, if we assume the amplification structure to be infinitely thin and if in addition we assume a sensor with depletion voltage $V_{dep} = 0$, the signal from a single e-h pair created at position z is simply a 'box' of duration $T = d/v_2$, shifted by the time $T_e = z/v_1$

$$i(t) = -G\,\frac{q\,v_2}{d}\left[\Theta(t - z/v_1) - \Theta(t - z/v_1 - d/v_2)\right] \tag{90}$$

The c.o.g. time of this signal is

$$\tau = \frac{d}{2v_2} + \frac{z}{v_1} \tag{91}$$

The signal for the case of $n_1, n_2, ..., n_N$ clusters at positions $z_1, z_2, ..., z_N$ is then

$$\tau(n_1, n_2, ..., n_N) = \frac{1}{\sum_{k=1}^{N} n_k}\sum_{k=1}^{N} n_k\left(\frac{d}{2v_2} + \frac{z_k}{v_1}\right) = \frac{d}{2v_2} + \frac{1}{\sum_{k=1}^{N} n_k}\sum_{k=1}^{N} n_k\frac{z_k}{v_1} \tag{92}$$

Proceeding as before we get for the average and variance of the c.o.g. time the expression

$$\overline{\tau} = \frac{d}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right) \qquad \Delta_\tau = \sqrt{\frac{b_N\,d}{\Delta z}\,\frac{d^2}{12v_1^2}} = \frac{1}{\sqrt{1 + 1.155\log d/\lambda}}\,\frac{T_1}{\sqrt{12}} \tag{93}$$

Sensors with internal gain, c.o.g. time resolution

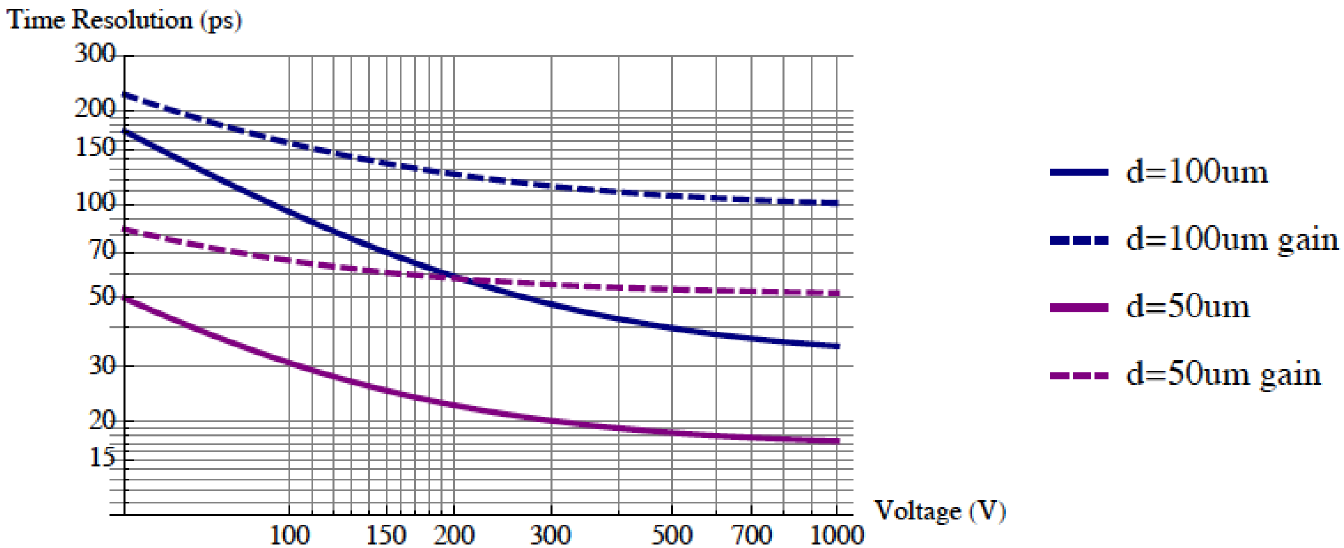


Figure 19: Standard deviation of the c.o.g. time from Eq. 37 for 50μm and 100 μm thickness for standard sensors (solid) and from Eq. 93 for an internal gain of electrons with a signal only from gain holes (dashed).

$$\overline{\tau} = \frac{d}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right) \qquad \Delta_\tau = \sqrt{\frac{b_N\, d}{\Delta z}\,\frac{d^2}{12 v_1^2}} = \frac{1}{\sqrt{1 + 1.155 \log d/\lambda}}\,\frac{T_1}{\sqrt{12}} \qquad (93)$$

The c.o.g. time resolution of sensors with internal gain is significantly worse than the time resolution without gain. The 'time distribution' essentially becomes and arrival time distribution of electrons.

→ When using internal gain one is tied to fast electronics in order to catch the first arriving electrons.

Sensors with internal gain, weighting field effect on the c.o.g. time

10. Weighting field effect on the c.o.g. time for silicon sensors with gain

A single electron arriving at time $t = 0$ at the pixel at $z = 0$ will be create a number of holes with a gain of G and the signal due to the holes moving back to $z = d$ with velocity v_2 is

$$i(t) = -G\, q\, v_2\, E_w[x, y, v_2(t - z/v_1)]\,[\Theta(t - z/v_1) - \Theta(t - z/v_1 - d/v_2)] \tag{94}$$

The c.o.g. time for the signal of a cluster of holes moving from the readout pixel at $z = 0$ to $z = d$ with velocity v_2 is given by

$$\tau(x, y, z) = \frac{z}{v_1} + \frac{d}{v_2}\int_0^1 \phi_w(z, y, s\,d)ds \tag{95}$$

Assuming a uniform charge deposit along the track, the c.o.g. time becomes

$$\tau(x, y) = \frac{1}{d}\int_0^d \tau(x, y, z)dz = \frac{d}{2v_1} + \frac{d}{v_2}\int_0^1 \phi_w(x, y, s\,d)ds \tag{96}$$

The variance for uniform irradiation of the pad is the

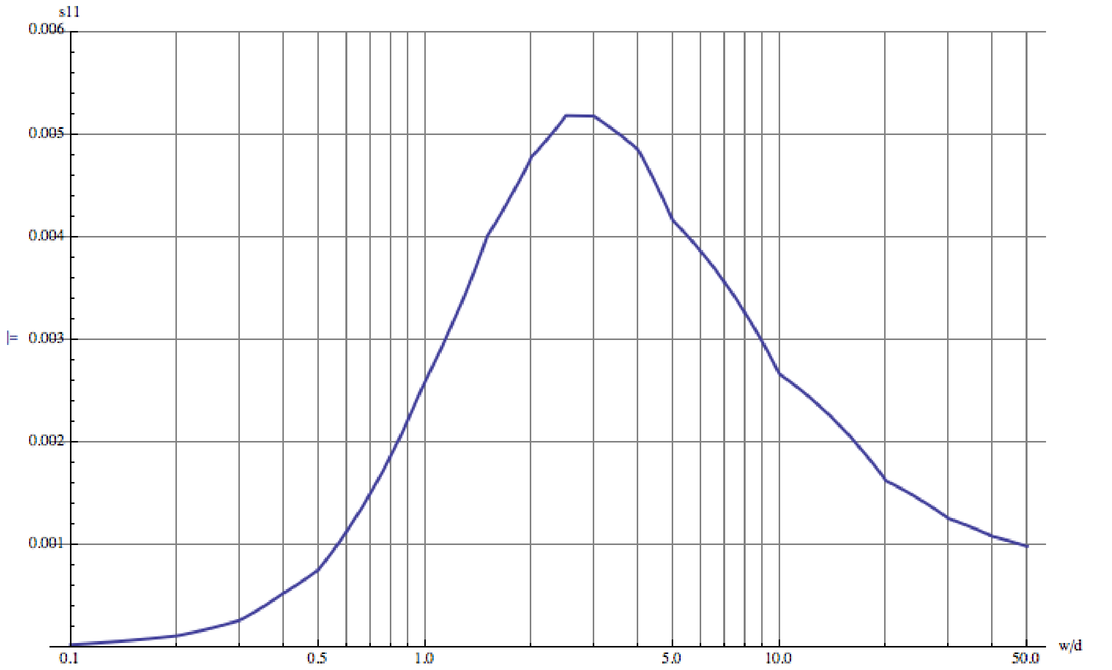
$$\begin{aligned} \Delta_t^2 &= \overline{\tau^2} - \overline{\tau}^2 \qquad (97) \\ &= \frac{d^2}{v_2^2}\left[\frac{1}{w_x w_y}\iint \left(\int_0^1 \phi_w(x, y, s\,d)ds\right)^2 dxdy - \left(\frac{1}{w_x w_y}\iint \left(\int_0^1 \phi_w(x, y, s\,d)ds\right) dxdx\right)^2\right] \\ &= \frac{d^2}{v_2^2}\, s_{22} = T_2^2\, s_{22} \end{aligned}$$

Sensors with internal gain, weighting field effect on the c.o.g. time

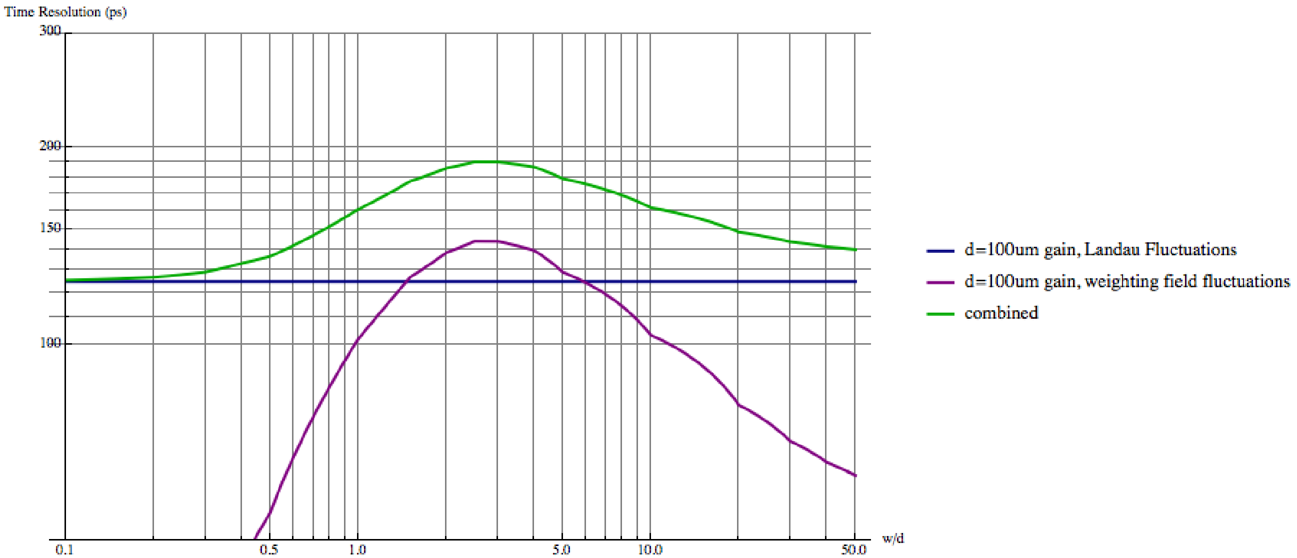
In case we also take into account the Landau fluctuations we have to use Eq. 95 in Eq. 74 and find

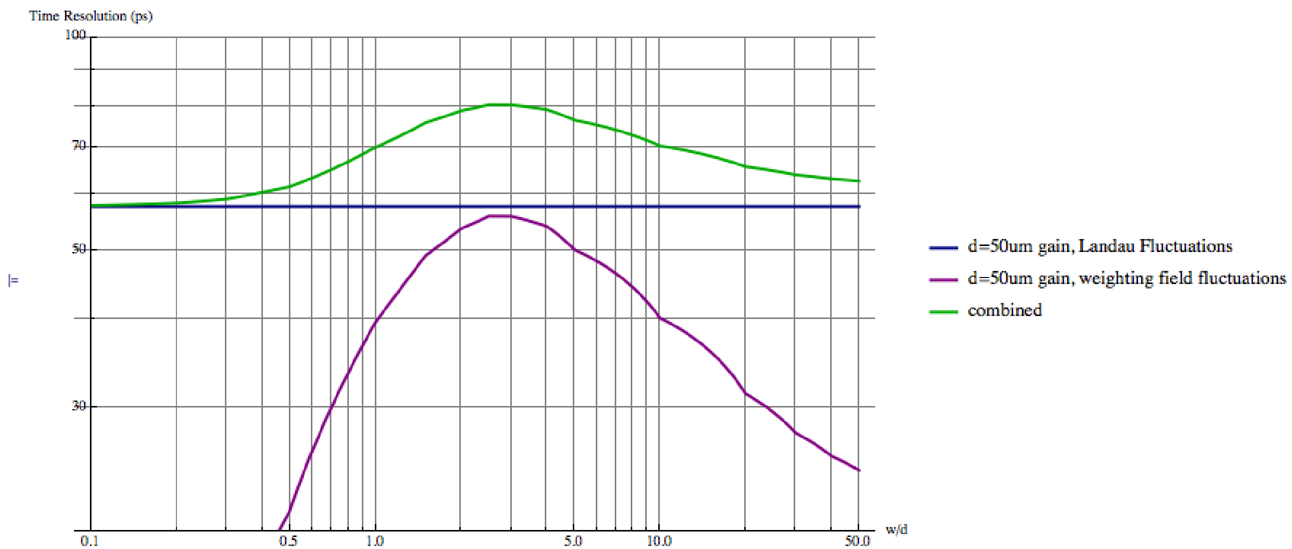
$$\Delta_t^2 = \overline{\tau^2} - \overline{\tau}^2 = \frac{b_N d}{\Delta z}\frac{d^2}{12\, v_1^2} + \frac{d^2}{v_2^2} s_{11} = \frac{b_N d}{\Delta z}\frac{T_1^2}{12} + T_2^2 s_{22} \qquad (98)$$

so we find the interesting result that in this case there is no correlation between the Landau fluctuations and the weighting field fluctuations and Landau fluctuations do just add in squares.



Sensors with internal gain, weighting field effect on the c.o.g. time





Sensors with internal gain, leading edge discrimination of a signal normalized to the total charge.

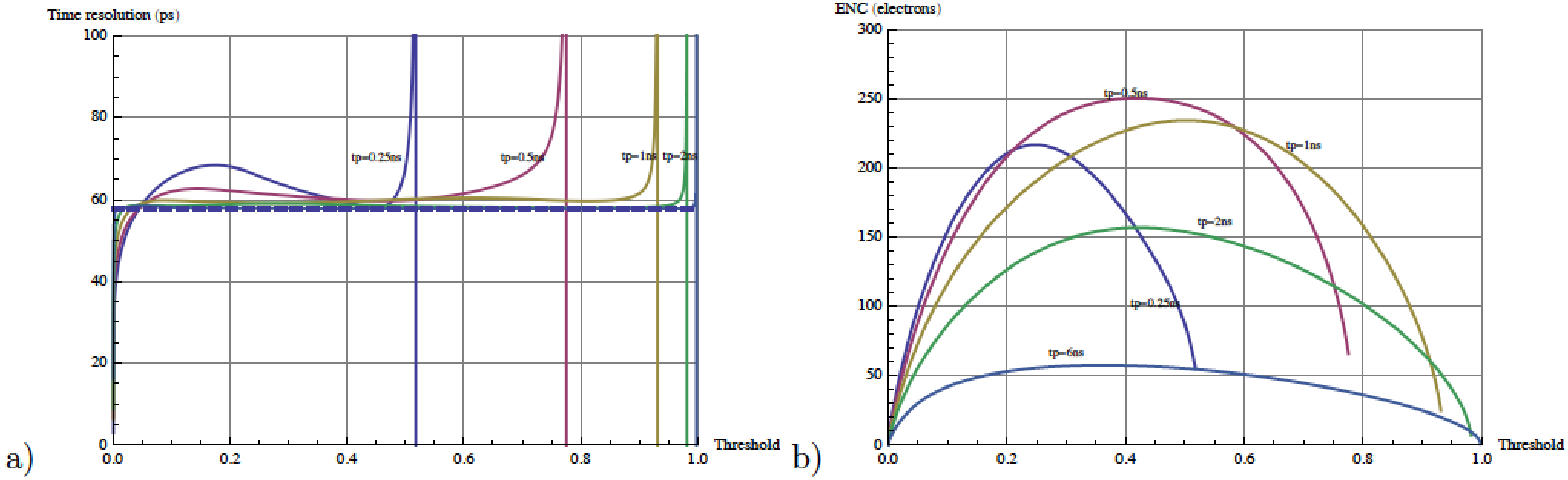


Figure 20: Time resolution for a sensor of $50\mu m$ thickness at 200V bias voltage. The slewing correction is performed by dividing the signal by the total charge and applying the threshold as a fraction of this charge. The plot on the right shows the ENC needed to match the noise effect of the time resolution to the effect from the Landau fluctuations.

There is no more correlation of the time to the total charge, only to the leading edge charge …

Summary

The c.o.g. time of a signal is measured by using an amplifier response time (peaking time) that is longer than the signal duration. It might represent an interesting way to measure timing with relatively 'slow' electronics – if noise allows.

The 'standard deviation' of the center of gravity (c.o.g.) time of a silicon sensor signal of 50/200um thickness due to Landau fluctuations is 23/180ps at 200V, assuming large pads and negligible depletion voltage.

To minimize the weighting field effect on the c.o.g. time resolution, the electrons should move towards the pad while the holes move away from the pad.

For a 50um sensor with 50x50um pixels, the weighting field effect should still not destroy the time resolution. N.B. – the formulas hold for perpendicular tracks neglecting diffusion !

Using internal gain, one is tied to fast electronics and leading edge discrimination to catch the first arriving electrons.

The slewing (from pulseheight variations) in sensors with gain is not related to the total charge but to the 'leading edge charge' i.e. constant fraction techniques using the leading edge.