3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering and the VFNS

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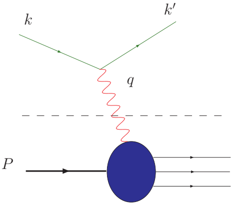
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P, s \rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2) \,.$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.

Introduction

Why are Heavy Flavor Contributions important ?

- They form a significant contribution to F_2 and F_L particularly at small x and high Q^2
- concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching

NNLO: S. Alekhin, J. Blümlein, S. Moch and R. Placakyte, Phys. Rev. D **96** (2017) no.1, 014011 [arXiv:1701.05838 [hep-ph]].

$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$

$m_c(m_c) = 1.252 \pm 0.018(exp)\ {}^{+0.03}_{-0.02}\ (scale)\ {}^{+0.00}_{-0.07}\ (thy)\mathrm{GeV}$,

$m_b(m_b) = 3.84 \pm 0.12\mathrm{GeV}$

$m_t(m_t) = 160.9 \pm 1.1\mathrm{GeV}$ [all in $\overline{\mathrm{MS}}$ scheme.]

Yet approximate NNLO treatment H. Kawamura et al. Nucl. Phys. B **864** (2012) 399 [arXiv:1205.5727].

PS corrections are exact J. Ablinger et al. Nucl. Phys. B **890** (2014) 48 [arXiv:1409.1135 [hep-ph]].

Both the LHeC and the EIC can lead to essential new results here.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j \rangle .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009] mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments N $= 2,4,6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N:

Topic of this talk [single & two mass cases]

The Wilson Coefficients at large Q^2

$$\text{2014} \quad L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\,\delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\,\delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\right]$$

$$\text{2010} \quad L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^3\left[\,A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\,\delta_2+A_{gq,Q}^{(2)}(N_F)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F\right.$$

$$\text{2010} \quad L_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\,A_{qg,Q}^{(3)}(N_F+1)\,\delta_2 +A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+ + A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\,,$$

$$\text{2014} \quad H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^2\left[\,A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\,\delta_2+\,\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\right]+a_s^3\Big[\,A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\,\delta_2 +\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\,\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) +A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\,C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\,,$$

$$H_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s\left[\,A_{Qg}^{(1)}(N_F+1)\,\delta_2+\,\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]+a_s^2\Big[\,A_{Qg}^{(2)}(N_F+1)\,\delta_2 + A_{Qg}^{(1)}(N_F+1)\,C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\,A_{gg,Q}^{(1)}(N_F+1)\,\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\,a_s^3\Big[\,A_{Qg}^{(3)}(N_F+1)\,\delta_2+\,A_{Qg}^{(2)}(N_F+1)\,C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+ + A_{gg,Q}^{(2)}(N_F+1)\,\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\,A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\,A_{gg,Q}^{(1)}(N_F+1)\,\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\,\tilde{C}_{g,(2,L)}^{(3)}(N_F+1$$

All logarithmic corrections are known since 2010.

Variable Flavor Number Scheme

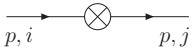
$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\overline{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ . \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ . \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes \Sigma(n_f,\mu^2) \\
&\quad + \left[n_f \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

There are generalizations necessary in the 2-mass case.

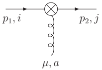
J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]

Calculation of the 3-loop operator matrix elements

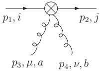
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



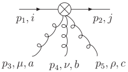
$$\delta^{ij} \not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}, \quad N \ge 1$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \ge 2$$



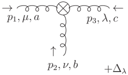
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\Big[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \Big],$$
$$N \ge 3$$



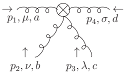
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not\Delta \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\; (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big],$$
$$N \ge 4$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\Big[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \Big], \quad N \ge 2$$



$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \Big] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \Big]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg), \quad N \ge 2$$



$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg),$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \; p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \; \lambda \leftrightarrow \sigma \end{smallmatrix} \right\}, \quad N \ge 2$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1233

A FORM [Vermaseren 2000] program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow 7426 \text{ scalar integrals.} \\ A_{gq,Q}^{(3)} &\rightarrow 12529 \text{ scalar integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\rightarrow 5470 \text{ scalar integrals.} \end{aligned}$$

$\Longrightarrow$ Need to use integration by parts identities.
$\Longrightarrow$ The reduction for all OMEs has been completed.

$\Longrightarrow$ Use special computers: 12 units with overall 5.5 TB RAM, 170 TB fast disc, hundreds of mathematica lic. ; IBP: several TB of final relations.

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm.

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

$\Longrightarrow$ additional propagator.

Number of master integrals:

$$\begin{aligned}
A_{qq,Q}^{(3),\mathrm{NS}} &\to 35 \ \text{master integrals} \ \surd. \\
A_{gq,Q}^{(3)} &\to 41 \ \text{master integrals} \ \surd. \\
A_{Qq}^{(3),\mathrm{PS}} &\to 66 \ \text{master integrals} \ \surd. \\
A_{gg,Q}^{(3)} &\to 205 \ \text{master integrals} \ \surd. \\
A_{Qg}^{(3)} &\to 340 \ \text{master integrals. (224 done by June 2015.)}
\end{aligned}$$

116 master integrals to be done $\Longrightarrow$ elliptic (and higher) integrals contribute.

24 integral families are required and implemented in Reduze.

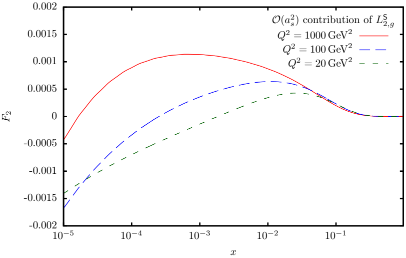
Analytic Computational Methods

- (generalized) hypergeometric functions.
- Summation techniques in difference fields; packages `Sigma`, `EvaluateMultiSums`, `SumProduction` [C. Schneider, 2005–].
- Operation with classes of new special functions; package `HarmonicSums` [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013], Ablinger [2014-].
- Mellin-Barnes representations.
- Hyperlogarithms (finite integrals) [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- Solve large Systems of Differential Equations, [Ablinger et al. 2015]
- Almkvist-Zeilberger Theorem as Integration Method; package `MultiIntegrate` [Ablinger 2012]
- The method of arbitrarily large Mellin moments [JB, Schneider 2017]; package: `SolveCoupledSystem`.

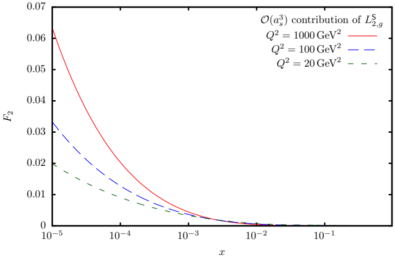
  We calculated 8000 moments for $A_{Qg}^{T_F^2,(3)}$ and obtained all the difference equation [very large in size].
  - Till now five new function spaces for nested sums and iterated integrals introduced.
  - Most recently: Non-iterative Iterative Integrals; iterated elliptic integrals and their generalization [2016].

Predictions for Experiment

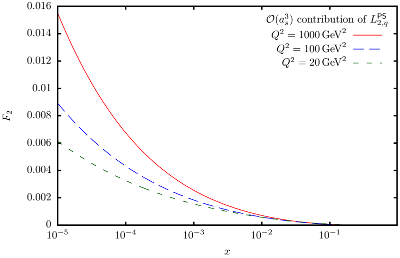
Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$

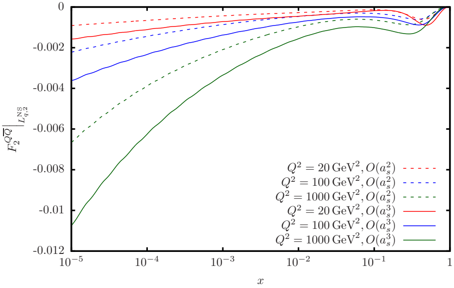


$O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$

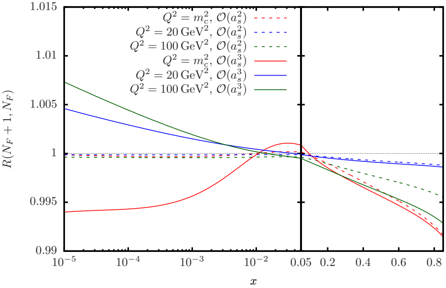


$L_{q,2}^{\mathrm{PS}}$

$L_{q,2}^{\rm NS}$

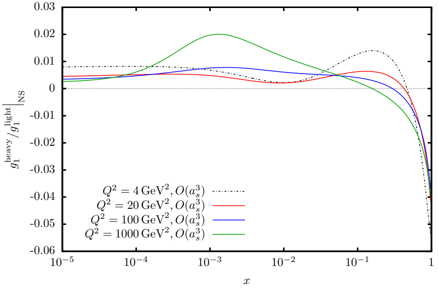


Contribution to $F_2(x, Q^2)$

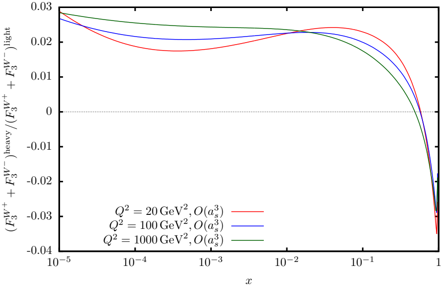


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



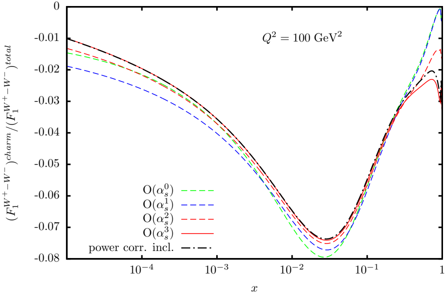
$g_1(x, Q^2)$



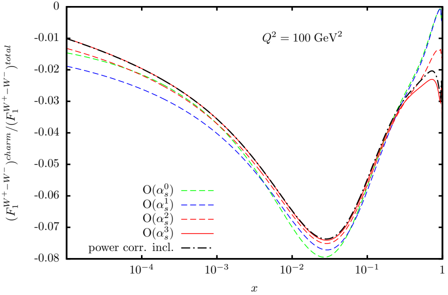
$xF_3^{W^+ + W^-}(x, Q^2)$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

NS corrections to $F_1^{W^+-W^-}$ and $F_2^{W^+-W^-}$



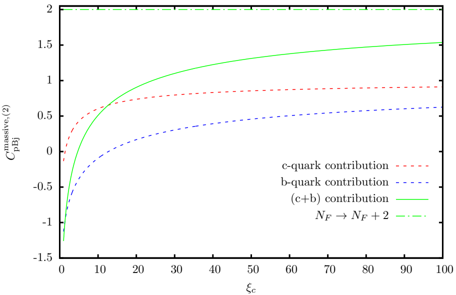
$F_1^{W^+-W^-}(x, Q^2)$



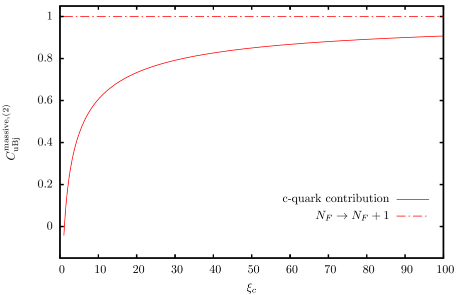
$F_2^{W^+-W^-}(x, Q^2)$

The massless corrections are due to [A. Vogt et al. arXiv:1606.08907 [hep-ph].]
from [A. Behring et al. Phys. Rev. D **94** (2016) no.11, 114006 [arXiv:1609.06255 [hep-ph]]].

$O(\alpha_s^2)$ Complete NS corrections



pol BJ sum rule

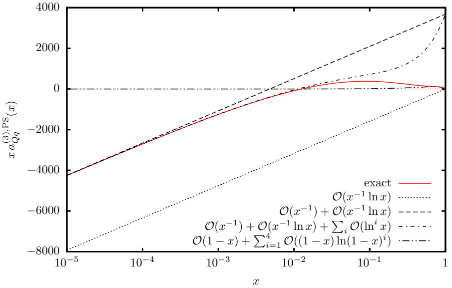


unp. BJ sum rule

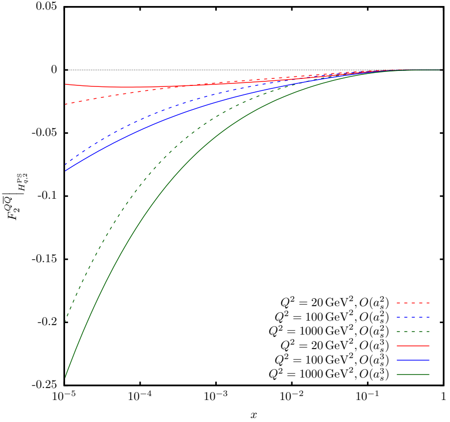
Note the negative corrections at low Q^2!
Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

$H_{q,2}^{\mathrm{PS}}$



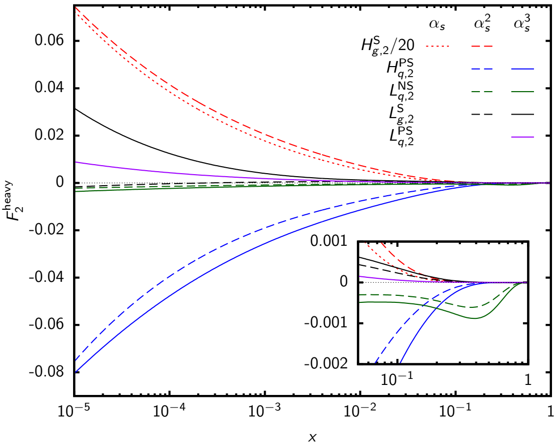
$a_{Qq}^{(3),\mathrm{PS}}$



Contribution to $F_2(x, Q^2)$

The leading small x approximation corresponding to High-energy factorization and small x heavy flavor production S. Catani, M. Ciafaloni, F. Hautmann, Nucl.Phys. B366 (1991) 135 departs from the physical result everywhere except for $x = 1$ (dotted line).

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\mathrm{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

We have calculated 18 of 28 color and ζ-factors of $A_{Qg}^{(3)}$, as well as 2000 moments analytically. (MATAD, 2009: $N \leq 10$).

Here the method of arbitrary high moments proved to be crucial.

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \; & \frac{1+(-1)^N}{2}\Bigg\{ C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} - \frac{4P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128\left(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2}\right)}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22) S_1^2 S_2}{3N(N+1)(N+2)} + \cdots \right) + \cdots \Bigg] \\
& + C_A C_F T_F \Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{32P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14} S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23} S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63} S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg] \\
& + C_A^2 T_F \Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{256P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52} S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36} S_2^2}{9(N-1)N^2(N+1)^2} + \cdots \Bigg] \\
& + C_F T_F^2 \Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N} \frac{4^i S_1(i-1)}{\binom{2i}{i} i^2} - 7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45} S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45} S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots \Bigg] + \cdots \Bigg\}
\end{aligned}
\qquad (1)
$$

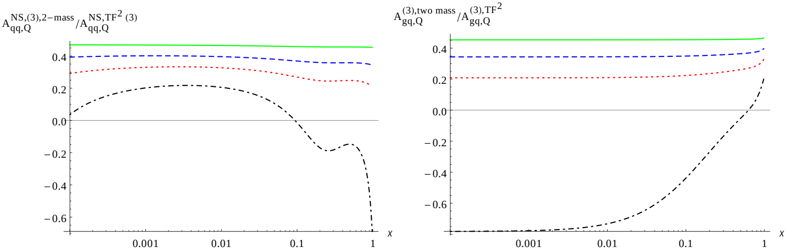
Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature.

Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \;=\; & \frac{1}{2}\Bigg\{T_F^2 C_A\Bigg\{\frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{156992}{6615} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{68332}{6615} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{412808}{19845}\Bigg\} \\
& + T_F^2 C_F\Bigg\{-\frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1230328}{138915}\Bigg\}\Bigg\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

$\rightarrow$ `q2e/exp` [Harlander, Seidensticker, Steinhauser 1999] $x = m_1^2/m_2^2$

Moments for graphs with two massive lines ($m_1 \neq m_2$)



dash-dotted: $Q^2 = 30\ \mathrm{GeV}^2$, dotted: $Q^2 = 50\mathrm{GeV}^2$, dashed: $Q^2 = 100\mathrm{GeV}^2$, full: $Q^2 = 1000\mathrm{GeV}^2$

- Analytic general N and x results are available for $A_{qq,Q}^{\rm NS}$, $A_{qq,Q}^{\rm NS,TR}$, $A_{Qq}^{\rm PS}$ and the scalar integrals of $A_{gg,Q}$. J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]
- Most recently: $A_{Qq}^{\rm PS}$ (2 mass) has been completed; $A_{gg,Q}^{\rm S}$ (2 mass) is nearly finished.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, coefficient functions.
  2010: Coefficient functions $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals).
- 2014 $L_q^{\mathrm{NS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, $A_{qq,Q}^{\mathrm{NS,TR}(3)}$, $H_{2,q}^{\mathrm{PS}(3)}$ and $A_{Qq}^{\mathrm{PS}(3)}$ were completed.
- A method for the calculation of graphs with two massive lines of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.
- The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [$\longrightarrow$ extended renormalization]; for the OMEs $A_{qq,Q}^{NS,(TR)(3)}, A_{gq,Q}^{(3)}, A_{Qq}^{PS,(3)}$ the complete 2-mass structure has been computed; ($A_{gg,Q}^{(3)}$)
  New VFNS relations !
- The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.

Conclusions

- All corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS- and the qg-case independently for the first time.
- In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.
- All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC, and future projects as the LHeC and EIC.

Publications: Physics

JB, A. De Freitas, S. Klein, W.L. van Neerven, Nucl. Phys. B755 (2006) 272
I. Bierenbaum, JB, S. Klein, Nucl. Phys. B780 (2007) 40; Nucl.Phys. B820 (2009) 417; Phys.Lett. B672 (2009) 401
JB, S. Klein, B. Tödtli, Phys. Rev. D80 (2009) 094010
I. Bierenbaum, JB, S. Klein, C. Schneider, Nucl. Phys. B803 (2008) 1
J. Ablinger, JB, S. Klein, C. Schneider, F. Wißbrock, Nucl. Phys. B844 (2011) 26
JB, A. Hasselhuhn, S. Klein, C. Schneider, Nucl. Phys. B866 (2013) 196
J. Ablinger et al., Nucl. Phys. B864 (2012) 52; Nucl. Phys. B882 (2014) 263; Nucl. Phys. B885 (2014) 409; Nucl. Phys. B885 (2014) 280; Nucl. Phys. B886 (2014) 733; Nucl.Phys. B890 (2014) 48
A. Behring et al., Eur.Phys.J. C74 (2014) 9, 3033; Nucl. Phys. B897 (2015) 612; Phys. Rev. D92 (2015) 11405
JB, G. Falcioni, A. De Freitas Nucl. Phys. B910 (2016) 568.
J. Ablinger et al., Nucl.Phys. B922 (2017) 1
J. Ablinger et al., Nucl.Phys. B921 (2017) 585

Publications: Mathematics

JB, S. Kurth, Phys. Rev. D **60** (1999) 014018
JB, Comput. Comput.Phys.Commun. 133 (2000) 76
JB, Comput. Phys. Commun. 159 (2004) 19
JB, Comput. Phys. Commun. 180 (2009) 2143; 0901.0837
JB, D. Broadhurst, J. Vermaseren, Comput. Phys. Commun. 181 (2010) 582
JB, M. Kauers, C. Schneider, Comput. Phys. Commun. 180 (2009) 2143
JB, S.Klein, C. Schneider, F. Stan. J. Symbolic Comput. 47 (2012) 1267
J. Ablinger, JB, C. Schneider, J. Math. Phys. 52 (2011) 102301, J. Math. Phys. 54 (2013) 082301
J. Ablinger, JB, 1304.7071 [Contr. to a Book: Springer, Wien]
J. Ablinger, JB, C. Raab, C. Schneider, J. Math. Phys. 55 (2014) 112301
J. Ablinger, A. Behring, JB, A. De Freitas, A. von Manteuffel, C. Schneider, Comp. Phys. Commun. 202 (2016) 33
JB, C. Schneider, Phys. Lett. B 771 (2017) 31.
A. Ablinger et al., DESY 16-147, arXiv:1706.01299.
JB, M. Round, C. Schneider arXiv:1706.03677 [cs.SC].

Backup Slides

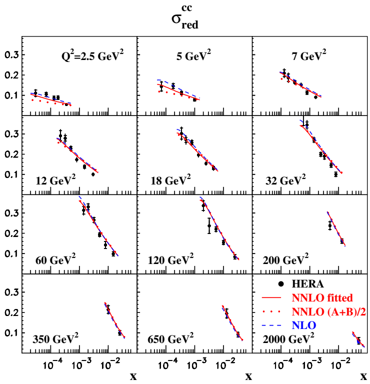
$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses

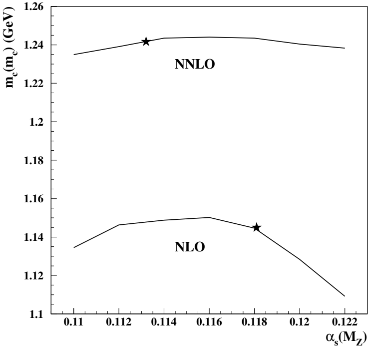
	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1162 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
Thorne	0.1136	[DIS+DY+HT*] (2014)
ABM11_J	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
ABM16	0.1147 ± 0.0008	+ new HERA, + $t\bar{t}$
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\ 0.0028}_{-\ 0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, N^3LO

$$\Delta_{\rm TH}\alpha_s = \alpha_s(\text{N}^3\text{LO}) - \alpha_s(\text{NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Deep–Inelastic Scattering (DIS):





V-Topology

Emergence of new nested sums :

$$\sum_{i=1}^{N}\binom{2i}{i}(-2)^i\sum_{j=1}^{i}\frac{1}{j\binom{2j}{j}}S_{1,2}\left(\tfrac{1}{2},-1;j\right)$$
$$=\int_0^1 dx\frac{x^N-1}{x-1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{w_{17},-1,0}(x)-2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$
$$+\frac{\zeta_2}{2}\int_0^1 dx\frac{(-x)^N-1}{x+1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{12}(x)-2\mathrm{H}^*_{13}(x)\right]$$
$$+c_3\int_0^1 dx\frac{(-8x)^N-1}{x+\frac{1}{8}}\sqrt{\frac{x}{1-x}}\,,$$

$$w_{12}=\frac{1}{\sqrt{x(8-x)}},\qquad w_{13}=\frac{1}{(2-x)\sqrt{x(8-x)}},$$
$$w_{17}=\frac{1}{\sqrt{x(8+x)}},\qquad w_{18}=\frac{1}{(2+x)\sqrt{x(8+x)}}\,.$$

Non-iterative Iterative Integrals

The live after iterative integrals and/or differential equations factorizing completely to 1st order:

- Iterative integrals/nested sums in QFT have been very well understood during the last 19 years since 1998. [J. Vermaseren, E. Remiddi, JB];
- Now even general alphabets (including up to root valued letters).
- Even single-scale Feynman integrals lead beyond that [Sabry's kite, 1962]
- Currently worked out by the community. [Ablinger, Adams, Ananthanarayan, Behring, Bijnes, JB, Bloch, Bogner, Brown, DeFreitas, Gangl, Ghosh, Hebbar, Hoeij, Imamoglu, Laporta, Levin, Müller-Stach, Remiddi, Schneider, Schweitzer, Tancredi, Vidunnas, Weinzierl, Zagier, Zayadeh, ...]

$$\mathbb{H}_{a_1,\ldots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\ldots,a_q}(x) = \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 \ldots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m) F_m[r(y_m)] \times H_{a_{m+1},\ldots,a_q}(y_{m+1}),$$

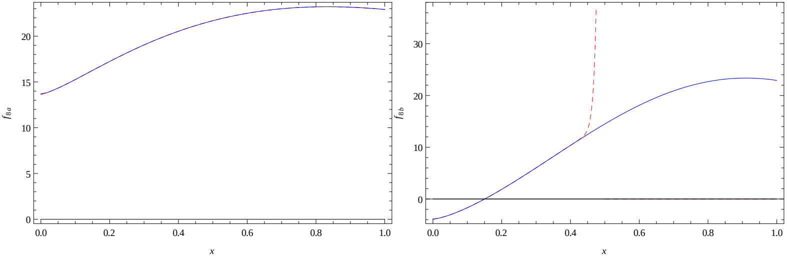
$$F[r(y)] = \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y],$$

$$\psi_{1a}^{(0)}(x) = \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{5}{3}\\2\end{matrix};z\right]$$

$$z = \frac{x^2(x^2-9)^2}{(x^2+3)^3}.$$

CIS-series; In some cases: complete elliptic integrals at very special rational arguments. Highly precise numerical representations already available. On the structural side: Relations to elliptic polylogarithms [in the elliptic case].

Non-iterative Iterative Integrals



- Have to handle branch-points in case.
- Relations due to shuffle algebras.
- Further relations due to triangle group; Important relations between different solutions of the homogeneous equations.

Most of the master integrals infected by the new CIS solutions are iterated integrals over a few of the former ones.

- We have identified the whole respective tree in case of our project.
- It would be interesting to view the corresponding situation in case of $\sigma(pp \to t\bar{t})(\hat{s})$.

Spill-Off:
New Mathematical Function Classes and Algebras

- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Radu, Schneider]
- 2016: Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [since April 2016.] [Ablinger, Behring, JB, De Freitas, van Hoeij, Raab, Schneider, DESY16-147].

Particle Physics Generates NEW Mathematics.