

# Recent progress in NNLO calculations

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**Introduction:** High precision for hard processes

I.: Two-loop amplitudes

II.: Infrared structure

III.: Numerical results for  $e^+e^- \rightarrow 3$  jets

## Objectives for the LHC

- Focus on infrared-safe observables: They depend only mildly on non-perturbative physics.
- Extract fundamental quantities like  $\alpha_s$  to high precision ( $e^+e^- \rightarrow 3$  jets at LEP).
- Extract non-perturbative parameters (pdf's) to high precision (HERA).
- Precise predictions for standard hard  $pp$  processes like  $W$ ,  $Z$ , jets, top, Higgs (NNLO).
- Reliable predictions for multi-particle final states for new physics searches (NLO).

## Fully differential NNLO calculations

The best what you can get from theory:  
NNLO calculations with flexible cuts at the parton level.

- Three-loop splitting functions S. Moch, J. Vermaseren and A. Vogt '04;
- $pp \rightarrow W$ , Anastasiou, Dixon, Melnikov, Petriello '03,
- $pp \rightarrow H$ , Anastasiou, Dixon, Melnikov, Petriello '05, Catani, Grazzini '08
- $e^+e^- \rightarrow 2 \text{ jets}$ , Anastasiou, Melnikov, Petriello '04, S.W. '06
- $e^+e^- \rightarrow 3 \text{ jets}$ , Gehrmann-De Ridder, Gehrmann, Glover, Heinrich '07, S.W. '08

# Challenges

What are the bottle-necks ?

- **Amplitudes:** At one-loop and beyond, the occurring integrals cannot be simply looked up in an integral table.
- **Divergences:** At NLO and beyond, infrared divergences occur in intermediate stages, if massless particles are involved.
- **Numerics:** Stable and efficient numerical methods are required for the Monte Carlo integration.

# The complexity of two-loop amplitudes

Rule of thumb: Complexity increases with the number of ratios of invariants.

- 1 ratio:

$$gg \rightarrow gg: \quad x = \frac{s}{t}.$$

- 2 ratios:

$$e^+ e^- \rightarrow qg\bar{q}: \quad x_1 = \frac{s_{12}}{s_{123}}, \quad x_2 = \frac{s_{23}}{s_{123}}.$$

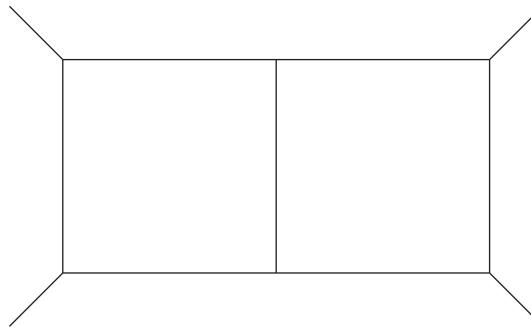
$$gg \rightarrow t\bar{t}: \quad x_1 = \frac{s}{t}, \quad x_2 = \frac{m_t^2}{t}.$$

# The calculation of two-loop integrals

- Techniques to calculate two-loop integrals
  - Mellin-Barnes transformation, Smirnov '99, Tausk '99.
  - Differential equations, Gehrmann, Remiddi '00.
  - Nested sums, Moch, Uwer, S.W. '01.
  - Sector decomposition (numerical), Binoth, Heinrich, '00.
- Methods to reduce the work-load:
  - Integration-by-parts, Chetyrkin, Kataev, Tkachov '81.
  - Reduction algorithms, Tarasov '96, Laporta '01.
  - Cut technique Bern, Dixon, Kosower, '00

## The double-box integral

Two-loop amplitudes for  $2 \rightarrow 2$  processes involve the double-box integral:



- First calculated by Smirnov (planar) and Tausk (non-planar) in 1999.
- Calculation based on [Mellin-Barnes representation](#).
- Result expressed in [harmonic polylogarithms](#).

$$H_{m_1, \dots, m_k}(x) = \sum_{i_1 > i_2 > \dots > i_k > 0} \frac{x^{i_1}}{i_1^{m_1} i_2^{m_2} \dots i_k^{m_k}}, \quad x = \frac{s}{t}.$$

# Multiple polylogarithms

- Definition:

$$\text{Li}_{m_1, \dots, m_k}(x_1, \dots, x_k) = \sum_{i_1 > i_2 > \dots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

(Goncharov; Borwein, Bradley, Broadhurst and Lisonek)

- Special subsets: Harmonic polylogs, Nielsen polylogs, classical polylogs  
(Remiddi and Vermaseren, Gehrmann and Remiddi).
- Have also an integral representation.
- Obey two Hopf algebras (Moch, Uwer, S.W.).
- Can be evaluated numerically for all complex values of the arguments  
(Gehrmann and Remiddi, Vollinga and S.W.).

# Infrared divergences: General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

- Phase space slicing

- $e^+e^-$ : W. Giele and N. Glover, (1992)
- initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
- massive partons, fragmentation: S. Keller and E. Laenen, (1999)

- Subtraction method

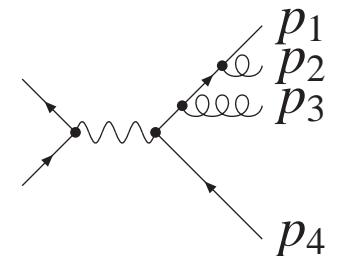
- residue approach: S. Frixione, Z. Kunszt and A. Signer, (1995)
- dipole formalism: S. Catani and M. Seymour, (1996)
- massive partons: L. Phaf and S.W. (2001), S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

# Infrared divergences at NNLO

The leading-colour contributions to  $e^+e^- \rightarrow qgg\bar{q}$ .

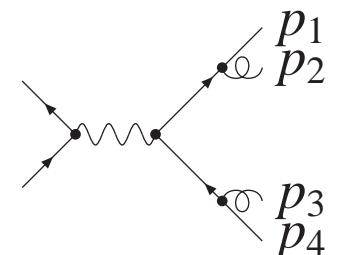
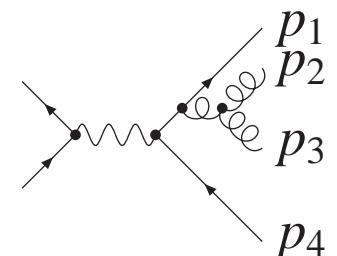
Double unresolved configurations:

- Two pairs of separately collinear particles
- Three particles collinear
- Two particles collinear and a third soft particle
- Two soft particles
- Coplanar degeneracy



Single unresolved configurations:

- Two collinear particles
- One soft particle



## The complexity of the infrared structure at NNLO

Rule of thumb: Complexity increases with the number of hard coloured partons.

- 2 coloured partons:

$$pp \rightarrow W$$

$$pp \rightarrow H$$

$$e^+ e^- \rightarrow 2 \text{ jets}$$

- 3 coloured partons:

$$e^+ e^- \rightarrow 3 \text{ jets}$$

$$pp \rightarrow W + \text{jet}$$

- 4 coloured partons:

$$pp \rightarrow t\bar{t}$$

# The subtraction method at NNLO

- Singular behaviour
  - Factorization of tree amplitudes in double unresolved limits, Berends, Giele, Cambell, Glover, Catani, Grazzini, Del Duca, Frizzo, Maltoni, Kosower '99
  - Factorization of one-loop amplitudes in single unresolved limits, Bern, Del Duca, Kilgore, Schmidt, Kosower, Uwer, Catani, Grazzini, '99
- Extension of the subtraction method to NNLO Kosower; S.W.; Kilgore; Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Frixione, Grazzini; Somogyi, Trócsányi, Del Duca, Aglietti, Duhr, Bolzoni, Moch;
- Cancellation based on sector decomposition Anastasiou, Melnikov, Petriello; Heinrich;

## Spin and colour correlations

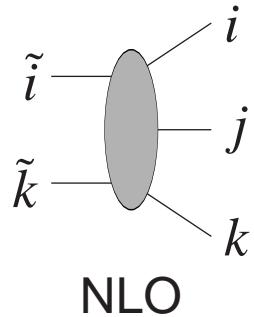
- In the soft limit, amplitudes factorize completely in spin space, but colour correlations remain.
- In the collinear limit, amplitudes factorize completely in colour space, but spin correlations remain.

Spin-correlations occur for the splittings  $g \rightarrow gg$  and  $g \rightarrow q\bar{q}$ , but not for  $q \rightarrow qg$ .

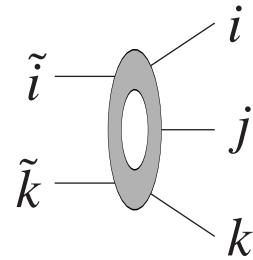
If one uses spin-averaged subtraction terms, one has a local counterterm only after the average over the azimuthal angle.

Alternative: Use combination of subtraction and slicing.

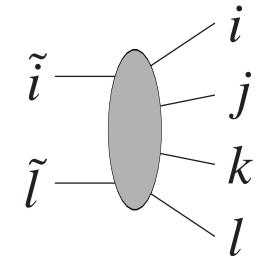
## Antenna subtraction terms at NNLO



NLO



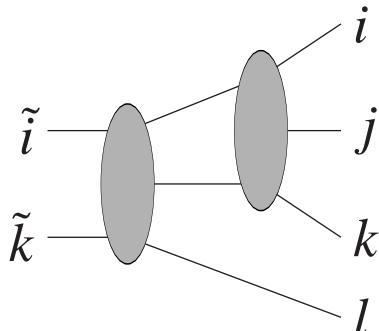
one-loop unresolved



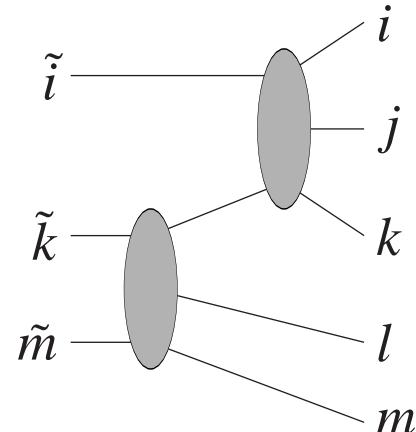
double unresolved

Gehrman-De Ridder, Gehrman, Glover, '05

At NNLO also **iterated structures**:



colour connected



almost colour connected

# The hybrid method

Strict subtraction:

3 parton:

- explicit poles cancel point-by-point

4 parton:

- explicit poles cancel point-by-point
- phase space singularities cancel point-by-point

5 parton:

- phase space singularities cancel point-by-point

Hybrid subtraction:

3 parton:

- explicit poles cancel point-by-point

4 parton:

- explicit poles **cancel after integration over unresolved phase space**
- phase space singularities **cancel after azimuthal average**

5 parton:

- phase space singularities **cancel after azimuthal average**

## Cancellation of explicit poles after integration

$$\int d\phi_3 \left| \mathcal{A}_3^{(0)} \right|^2 \left[ \int d\phi_{unres}^D(i, j, k) X_3^0(i, j, k) \underbrace{\sum \mathcal{Y}_3^0(i, j, k, \dots)}_{1/\varepsilon} \right]$$

Two cases:

1. Expression in the square bracket vanishes after integration to all orders in  $\varepsilon$ .  
That's nice :-)
2. Only the pole terms vanish after integration.  
Warning:  $d\phi_{unres}^D = d\phi_{unres}^4 |\sin \phi|^{-2\varepsilon}$ .

Can arrange subtraction terms such that only the first case occurs !

## The subtraction method at NNLO

5 partons :  $d\sigma_5^{(0)} - d\alpha^{NLO} - d\alpha^{double} - d\alpha^{almost} + d\alpha^{iterated} - d\alpha^{soft}$ ,

4 partons :  $d\sigma_4^{(1)} + d\alpha^{NLO} - d\alpha^{loop} + d\alpha^{almost} - d\alpha^{iterated} - d\alpha^{product} + d\alpha^{soft}$ ,

3 partons :  $d\sigma_3^{(2)} + d\alpha^{double} + d\alpha^{loop} + d\alpha^{product}$ .

$d\alpha^{double}$  contains the four-parton antenna functions,

$d\alpha^{almost}$  contains a product of two three-parton antenna functions,

$d\alpha^{iterated}$  is the approximation of  $d\alpha^{NLO}$ ,

$d\alpha^{loop}$  is the approximation of the one-loop matrix elements,

$d\alpha^{product}$  contains a product of two three-parton antenna functions, both with  $4 \rightarrow 3$  parton kinematics

$d\alpha^{soft}$  is an additional subtraction term due to soft gluons, occurring in processes with three or more hard partons.

## The soft subtraction term: 4 partons

If one ignores the soft subtraction term:

$$\int d\phi_3 \left| \mathcal{A}_3^{(0)} \right|^2 \int d\phi_{unres}^D X_3^0(a, i, b) \frac{1}{\varepsilon} \left[ \ln \frac{s_{\tilde{a}j}s_{j\tilde{b}}}{s_{\tilde{a}\tilde{b}}} - \ln \frac{s_{aj}s_{jb}}{s_{ab}} \right]$$

Integration over the unresolved phase space leads to

$$\frac{1}{\varepsilon} \int_0^{2\pi} d\phi \ln \left( \frac{(1+c_j)(1-c_b)}{2(1-c_b c_j - s_b s_j \cos \phi)} \right) = \frac{2\pi}{\varepsilon} \ln \left( \frac{1-c_b c_j + (c_j - c_b)}{1-c_b c_j + |c_j - c_b|} \right).$$

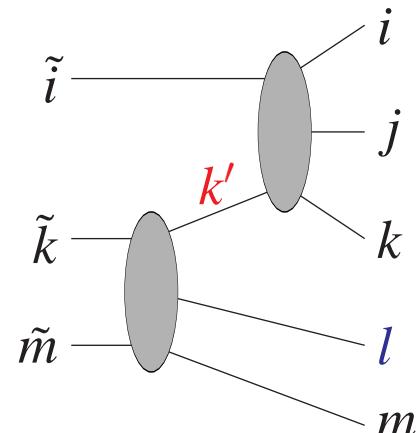
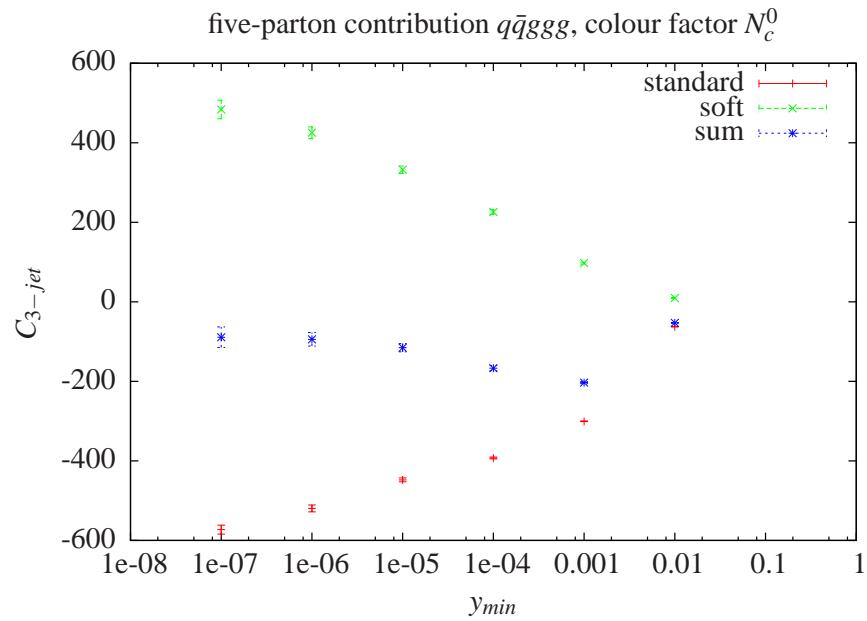
Non-zero for  $c_j < c_b$  !

The explicit poles in the four-parton configuration have to cancel:  $d\alpha^{soft}$  is needed.

# The soft subtraction term: 5 partons

If one ignores the soft subtraction term:

Dependence on the slicing parameter:



Gluon  $l$  soft:  
Eikonal factor  
 $Eik(k', l, m)$

The five-parton contribution has to be independent of the slicing parameter:  
 $-d\alpha^{soft}$  is needed.

$$e^+ e^- \rightarrow 3 \text{ jets at NNLO}$$

Fully differential Monte-Carlo programs for 3-jet observables at NNLO:

- **EERAD3**

Gehrmann-De Ridder, Gehrmann, Glover, Heinrich,

Phys.Rev.Lett.99:132002,2007,

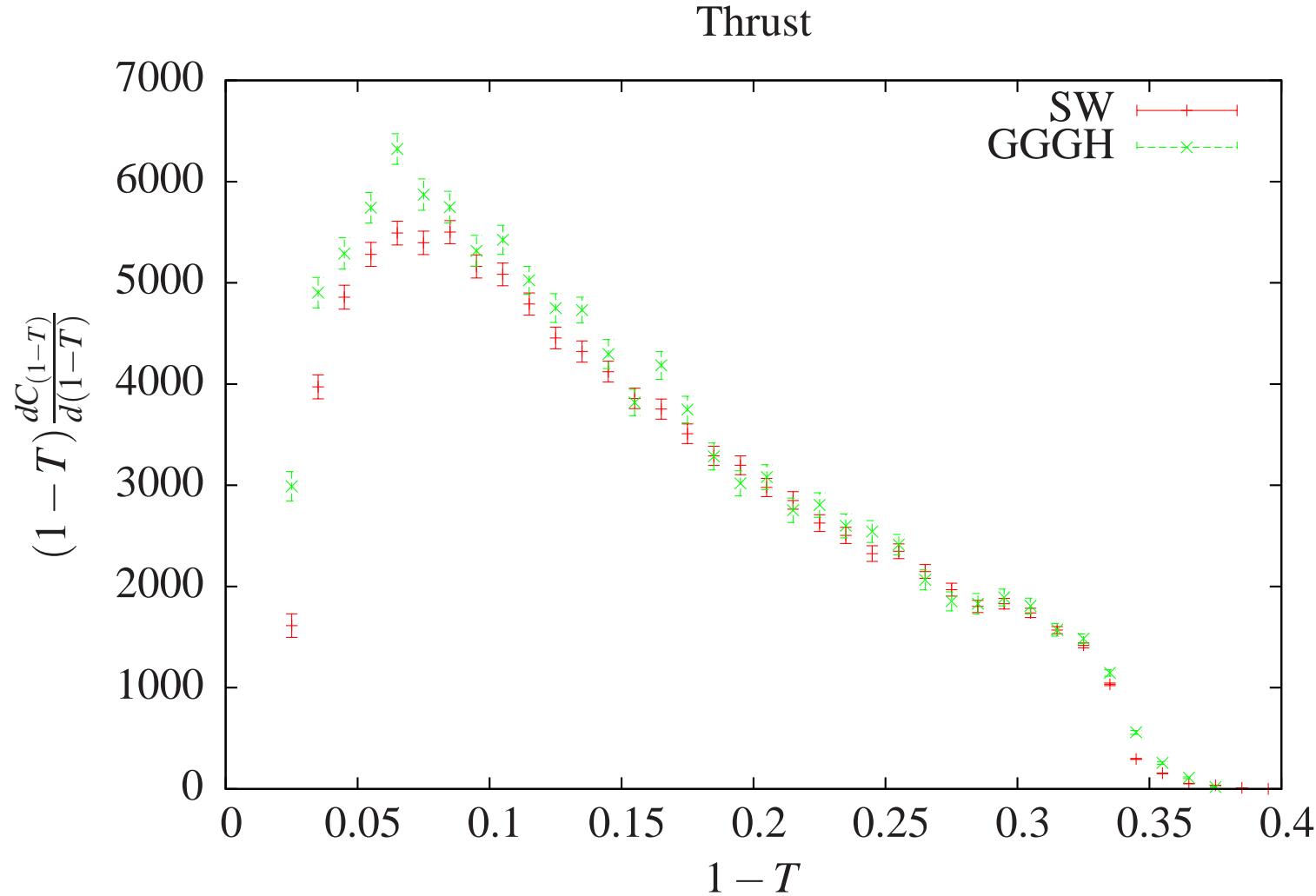
Phys.Rev.Lett.100:172001,2008

- **MERCUTIO2**

S.W.,

Phys.Rev.Lett.101:162001,2008

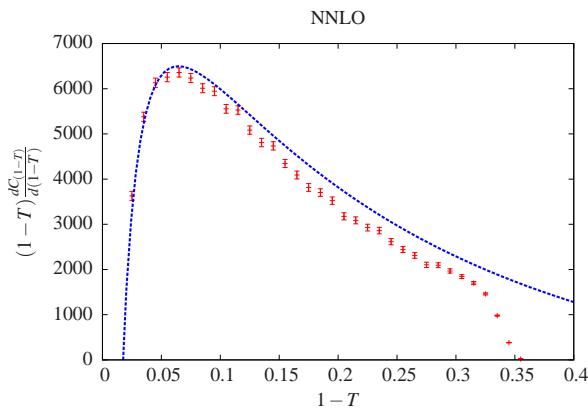
## Comparison with EERAD3



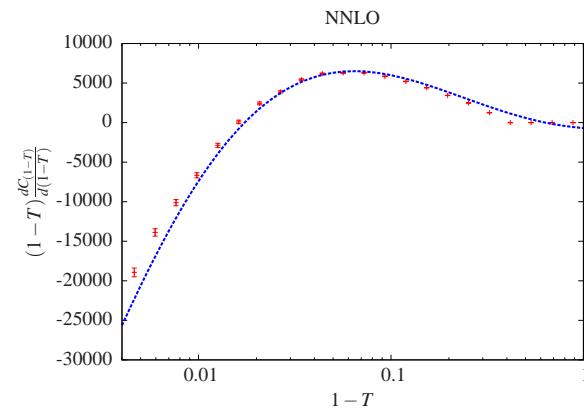
# Comparison of the thrust distribution with Becher/Schwartz

Logarithmic terms:

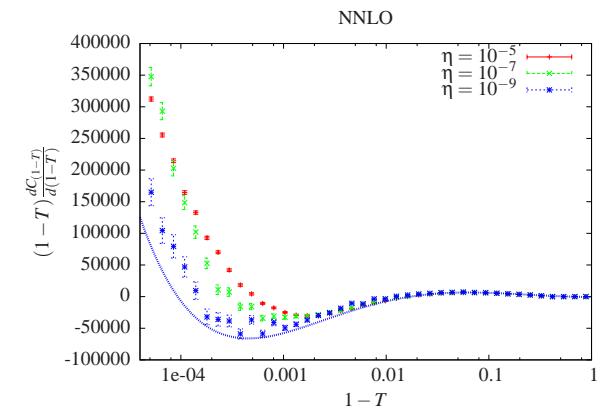
$$\frac{dC_\tau}{d\tau} = \frac{1}{\tau} [a_5 \ln^5 \tau + a_4 \ln^4 \tau + a_3 \ln^3 \tau + a_2 \ln^2 \tau + a_1 \ln \tau + a_0 + O(\tau)], \quad \tau = 1 - T.$$



Hard region

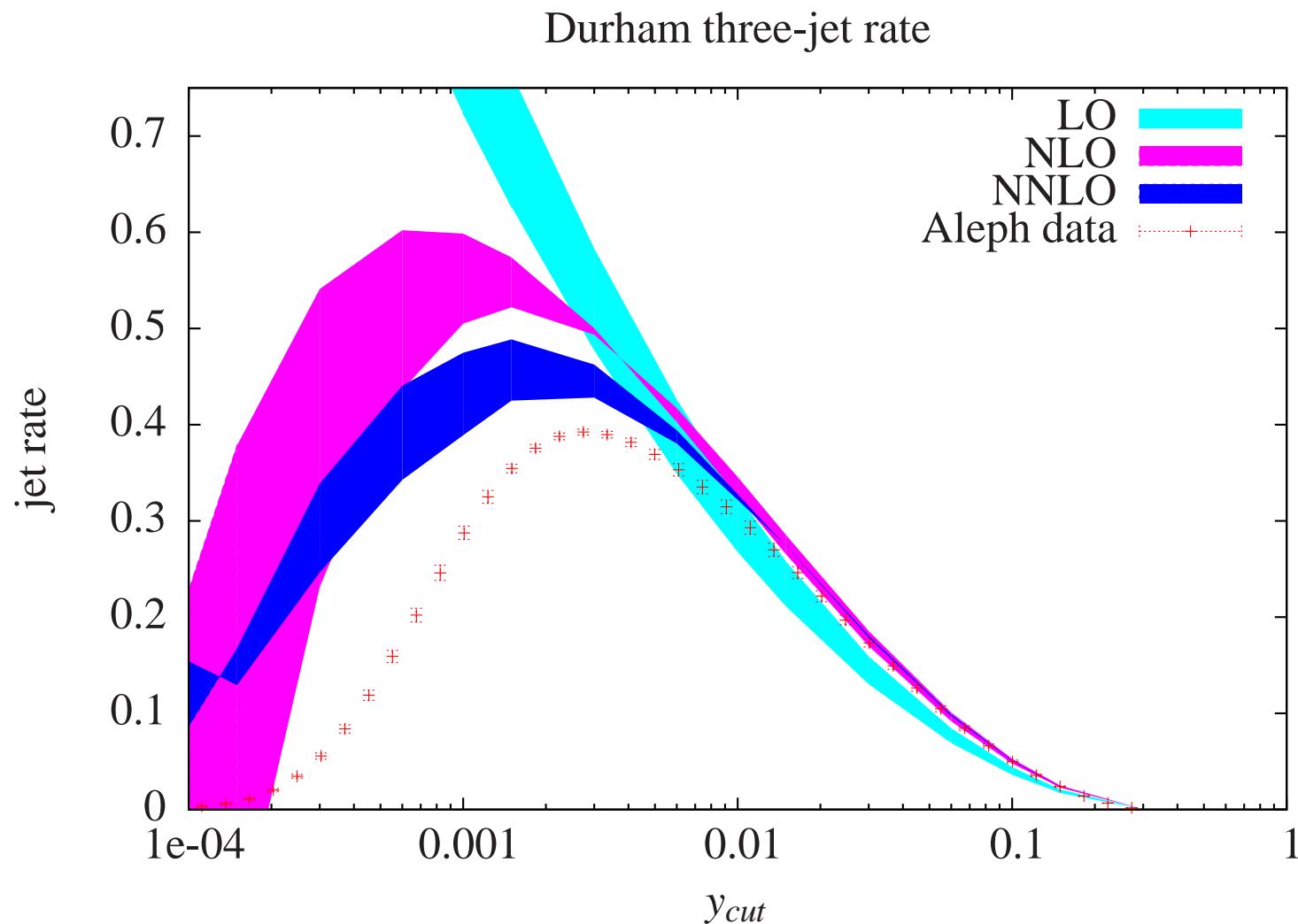


Peak region

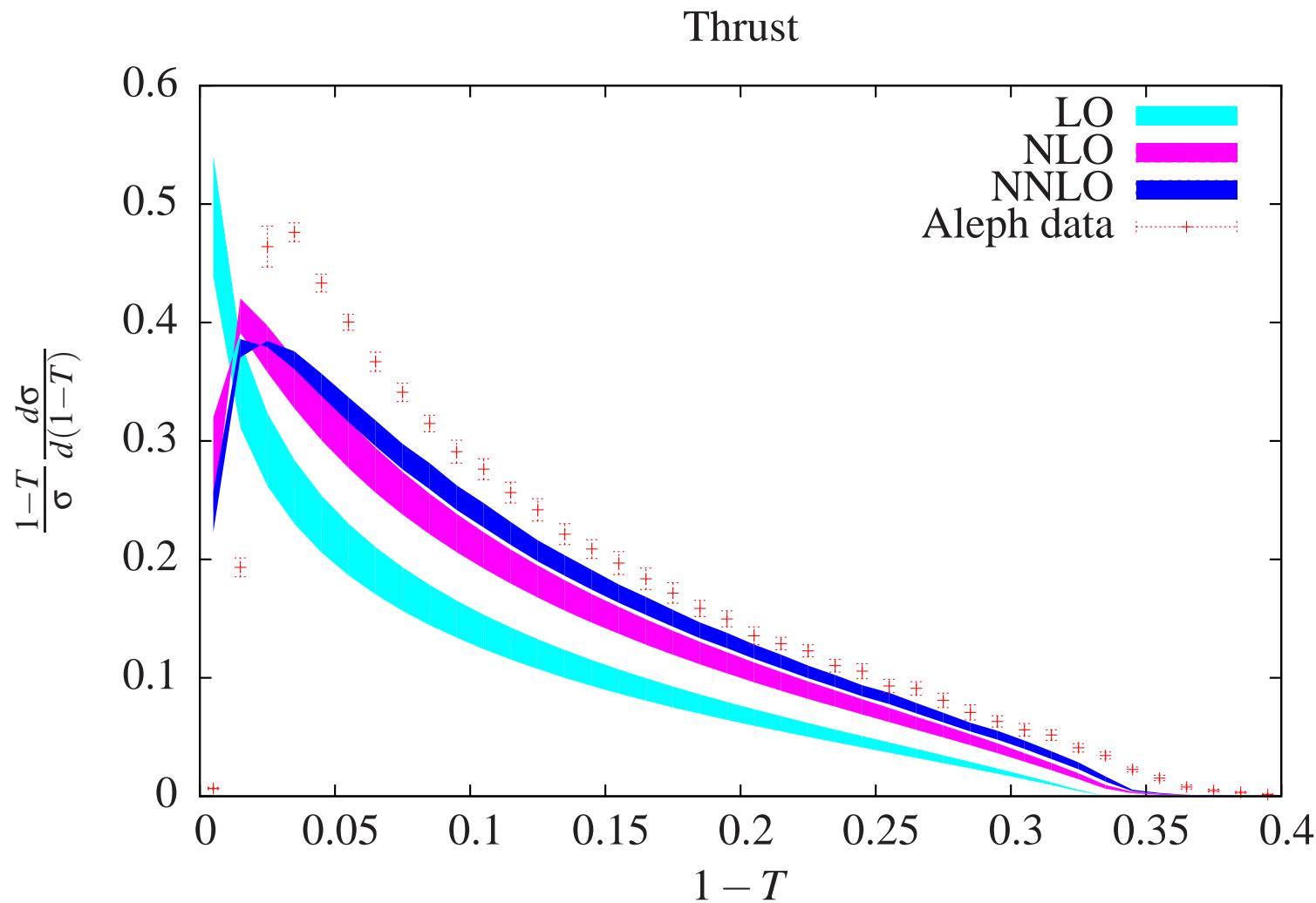


Extreme two-jet region

# Results for the three-jet rate in electron-positron annihilation



## Results for the thrust distribution



## Further refinements

**Soft-gluon resummation:** Perturbative expansion is of the form

$$1 + c_0 \alpha_s + c_1 \alpha_s \ln y_{cut} + c_2 \alpha_s \ln^2 y_{cut} + O(\alpha^2)$$

In the region where  $\alpha_s \ln^2 y_{cut} \approx 1$  resum the large logarithms.

Catani, Trentadue, Turnock, Webber, '93; Becher, Schwartz, '08

**Power corrections:** From the operator product expansion we expect power corrections of the form

$$\frac{\lambda}{Q} + O\left(\frac{1}{Q^2}\right)$$

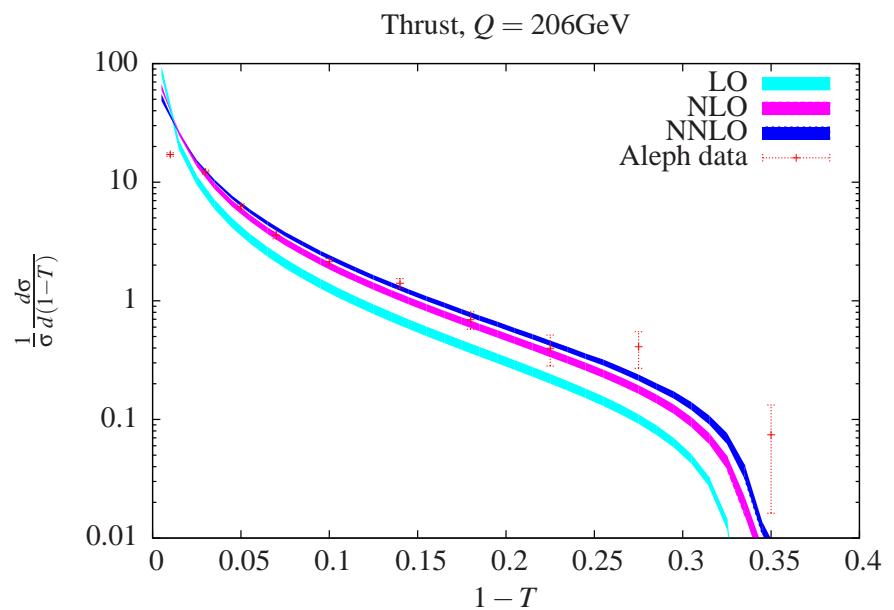
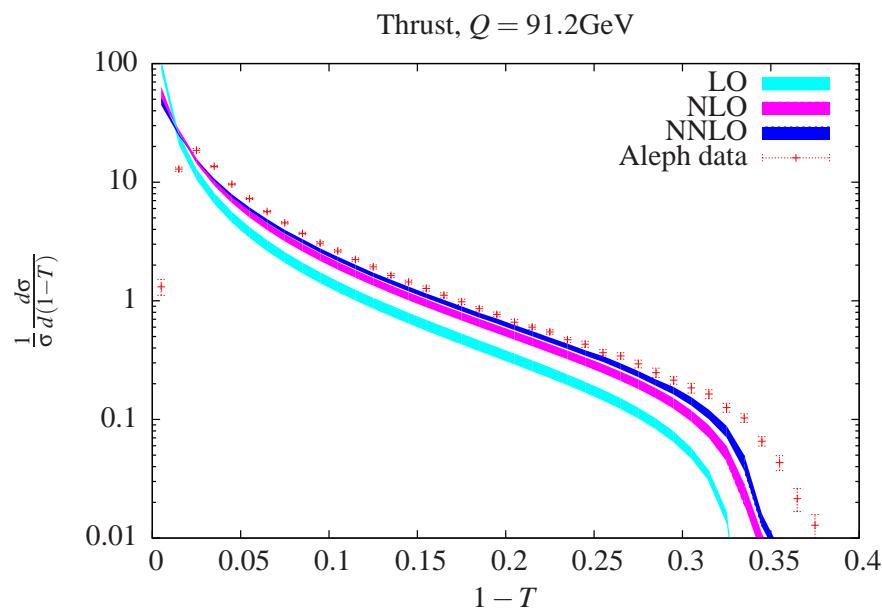
Dokshitzer, Webber, '97; Davison, Webber, '08

**Electroweak corrections:** At the per-cent level the  $O(\alpha^3 \alpha_s)$  electroweak corrections are equally important.

Carloni-Calame, Moretti, Piccinin, Ross, '09; Denner, Dittmaier, Gehrmann, Kurz, '09

# Results for the thrust distribution

Changing the centre-of-mass energy:



## Moments

$$\langle O^n \rangle = \frac{1}{\sigma_{tot}} \int_0^1 O^n \frac{d\sigma}{dO} dO.$$

Moments for  $n \geq 2$  unproblematic.

The first moment can receive sizeable corrections from the two-jet region.

First moment for thrust:

$$\langle 1 - T \rangle_{NNLO} = \underbrace{(1.076 \pm 0.001) \cdot 10^3}_{1-T > 10^{-3}} + \underbrace{(0.029 \pm 0.025) \cdot 10^3}_{1-T < 10^{-3}} = (1.10 \pm 0.03) \cdot 10^3$$

Worst case: First moment of the wide jet broadening

$$\langle B_W \rangle_{NNLO} = \underbrace{(0.356 \pm 0.004) \cdot 10^3}_{B_W > 10^{-3}} + \underbrace{(1.7 \pm 1.2) \cdot 10^3}_{B_W < 10^{-3}} = (2.1 \pm 1.2) \cdot 10^3$$

## Summary

- $\alpha_s$  is one of the fundamental parameters of nature
- Error on  $\alpha_s$  dominated by theory
- NNLO calculations reduce the theoretical uncertainty  
Re-analysis of JADE data, ...
- Calculational techniques developed for  $e^+e^- \rightarrow 3 \text{ jets}$  can be applied to other processes