The all-order infrared structure of massless gauge theories

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Outline

Introduction

Practicalities

- Higher order calculations at colliders cross hinge upon cancellation of divergences between virtual corrections and real emission contributions.
  - Cancellation must be performed analytically before numerical integrations.
  - Need local counterterms for matrix elements in all singular regions.
  - State of the art: NLO multileg. NNLO available only for e^+e^- annihilation.
- Cancellations leave behind large logarithms: they must be resummed.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)\,,$$

  - For inclusive observables: analytic resummation to high logarithmic accuracy.
  - For exclusive final states: parton shower event generators, $(N)LL$ accuracy.
- Resummation probes the all-order structure of perturbation theory.
  - Power-suppressed corrections to QCD cross sections can be studied
  - Power corrections are often essential for phenomenology: event shapes, jets.

Theoretical concerns

- Understanding long-distance singularities to all orders provides a window into non-perturbative effects.
  - IR singularities have a universal structure for all massless gauge theories.
  - Links to the strong coupling regime can be established for SUSY gauge theories.
- A very special theory has emerged as a theoretical laboratory: $\mathcal{N} = 4$ Super Yang-Mills.
  - It is conformal invariant: $\beta_{\mathcal{N}=4}(\alpha_s) = 0$.
  - Exponentiation of IR/C poles in scattering amplitudes simplifies.
  - AdS/CFT suggests a 'simple' description at strong coupling, in the planar limit.
  - Exponentiation has been observed for MHV amplitudes up to five legs.
  - Higher-point amplitudes are strongly constrained by (super)conformal symmetry.
  - A string calculation at strong coupling matches perturbative results.
  - Amplitudes admit a dual description in terms of polygonal Wilson loops.
  - Integrability leads to possibly exact expressions for anomalous dimensions.

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(Anastasiou, Bern, Dixon, Kosower, Smirnov; Alday, Maldacena; Brandhuber, Heslop, Spence, Travaglini; Drummond, Ferro, Henn, Korchemsky, Sokatchev; Beisert, Eden, Staudacher; Del Duca, Duhr, Smirnov; ...)

Tools: dimensional regularization

Nonabelian exponentiation of IR/C poles requires d-dimensional evolution equations. The running coupling in $d = 4 - 2\epsilon$ obeys

$$\mu \frac{\partial \overline{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \overline{\alpha}) = -2\,\epsilon\,\overline{\alpha} + \hat{\beta}(\overline{\alpha}) \ , \quad \hat{\beta}(\overline{\alpha}) = -\frac{\overline{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\overline{\alpha}}{\pi}\right)^n \ .$$

The one-loop solution is

$$\overline{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon}\right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} \ .$$

The β function develops an IR free fixed point, so that $\overline{\alpha}(0, \epsilon) = 0$ for $\epsilon < 0$. The location of the Landau pole acquires an imaginary part for $\epsilon < -\, b_0 \alpha_s / (4\pi)$,

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)}\right)^{-1/\epsilon} \ .$$

- Integrations over the scale of the coupling can be analytically performed.
- All infrared and collinear poles arise by integration of $\alpha_s(\mu^2, \epsilon)$.

Tools: factorization

All factorizations separating dynamics at different energy scales lead to resummations.

- Collinear logarithms: Mellin moments of partonic DIS structure functions factorize

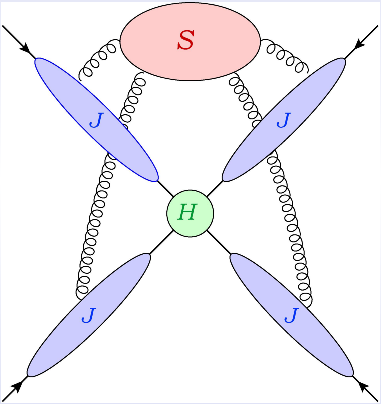
$$\widetilde{F}_2\left(N,\frac{Q^2}{m^2},\alpha_s\right)=\widetilde{C}\left(N,\frac{Q^2}{\mu_F^2},\alpha_s\right)\widetilde{f}\left(N,\frac{\mu_F^2}{m^2},\alpha_s\right)$$

$$\frac{d\widetilde{F}_2}{d\mu_F}=0\quad\rightarrow\quad\frac{d\log\widetilde{f}}{d\log\mu_F}=\gamma_N\left(\alpha_s\right)\ .$$

  Altarelli-Parisi evolution resums collinear logarithms into evolved parton distributions.

- Factorization is the difficult step. It requires a diagrammatic analysis
  - all-order power counting (UV, IR, collinear ...);
  - implementation of gauge invariance via Ward identities.
- Sudakov double logarithms are more difficult.
  - A double factorization is required: hard vs. collinear vs. soft. Gauge invariance plays a key role in the decoupling.
  - After identification of relevant modes, effective field theory can be used (SCET).

Sudakov factorization



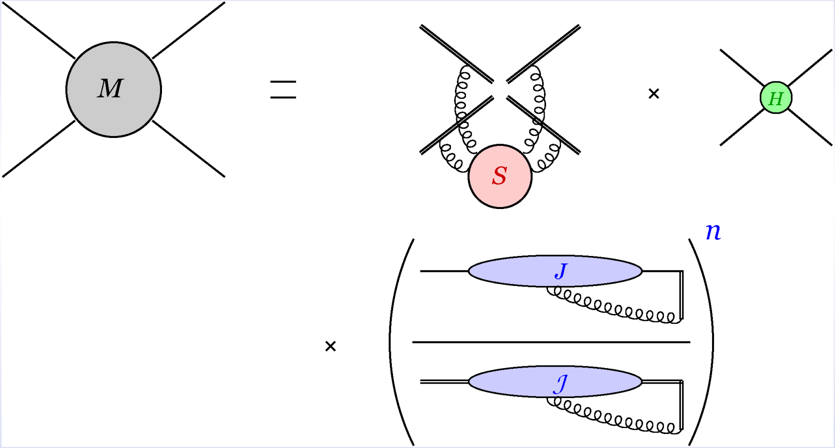
Leading regions for Sudakov factorization.

- Divergences arise in fixed-angle amplitudes from leading regions in loop momentum space.
- Soft gluons factorize both form hard (easy) and from collinear (intricate) virtual exchanges.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: $S \propto \mathbf{I}$.
- In the planar limit S can be reabsorbed defining jets J as square roots of elementary form factors.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- Phenomenological applications to jet and heavy quark production at hadron colliders.

The dipole formula

(with Einan Gardi)

Factorization: pictorial



Operator factorization for fixed-angle scattering amplitudes, with subtractions.

Operator definitions

The functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right] ,$$

We introduced factorization vectors n_i^μ, with $n_i^2 \neq 0$, to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 \,|\Phi_n(\infty, 0)\, \psi(0)\, |p\rangle \, .$$

where Φ_n is the Wilson line operator along the direction n^μ.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A(\lambda n)\right] \, .$$

The jet J has collinear divergences only along p.

Eikonal functions

The soft function $\mathcal{S}$ is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)_{\{\alpha_k\}}\, \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = \sum_{\{\eta_k\}} \langle 0| \prod_{i=1}^{n} \left[\Phi_{\beta_i}(\infty, 0)_{\alpha_k, \eta_k}\right] |0\rangle \; (c_K)_{\{\eta_k\}} \; ,$$

Soft-collinear regions are subtracted dividing by eikonal jets $\mathcal{J}$.

$$\mathcal{J}\left(\frac{(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0| \Phi_n(\infty, 0)\, \Phi_\beta(0, -\infty)\, |0\rangle \; ,$$

- $\mathcal{S}$ and $\mathcal{J}$ are pure counterterms in dimensional regularization.
  $\Rightarrow$ Infrared poles are mapped to ultraviolet singularities.
- Functional dependence of jet and soft factors on the vectors n_i^μ is restricted by the classical invariance of Wilson lines under velocity rescalings, $n_i^\mu \to \kappa_i n_i^\mu$.
- Rescaling invariance for light-like velocities, $\beta_i^2 = 0$ is broken by quantum corrections.
  $\Rightarrow$ UV counterterms contain collinear poles, corresponding to soft-collinear singularities.
- Double poles are determined by the cusp anomalous dimension $\gamma_K(\alpha_s)$.
  $\Rightarrow$ $\gamma_K(\alpha_s)$ governs the renormalization of Wilson lines with light-like cusps.

Soft matrices

The soft function $\mathcal{S}$ obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = -\,\Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \mathcal{S}_{JK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \,,$$

- $\Gamma^{\mathcal{S}}$ is singular due to overlapping UV and collinear poles.

$\mathcal{S}$ is a pure counterterm. In dimensional regularization, using $\alpha_s(\mu^2 = 0, \epsilon) = 0$, one finds

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left[-\frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right)\right] .$$

Double poles cancel in the reduced soft function

$$\overline{\mathcal{S}}_{LK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

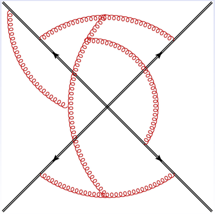
- $\overline{\mathcal{S}}$ must depend on rescaling invariant variables,

$$\rho_{ij} \equiv \frac{n_i^2\, n_j^2\, (\beta_i \cdot \beta_j)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2} \,.$$

- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

Surprising simplicity

- $\Gamma^{\mathcal{S}}$ can be computed from UV poles of $\mathcal{S}$
- Non-abelian eikonal exponentiation selects the relevant diagrams: webs
- $\Gamma^{\mathcal{S}}$ appears highly complex at high orders.
- g-loop webs directly correlate color and kinematics of up to $g+1$ Wilson lines.



A web contributing to $\Gamma^{\mathcal{S}}$.

The two-loop calculation (M. Aybat, L. Dixon, G. Sterman) leads to a surprising result: for any number of light-like eikonal lines

$$\Gamma^{(2)}_{\mathcal{S}} = \frac{\kappa}{2}\,\Gamma^{(1)}_{\mathcal{S}} \qquad \kappa = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F C_F\,.$$

- No new kinematic dependence; no new matrix structure.
- κ is the two-loop coefficient of $\gamma_K(\alpha_s)$, rescaled by the appropriate quadratic Casimir,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)}\left[2\,\frac{\alpha_s}{\pi} + \kappa\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right)\right]\,.$$

Factorization constraints

Recall the origin of kinematic dependence for eikonal functions

- The classical rescaling symmetry of Wilson line correlators under $\beta_i \to \kappa_i \beta_i$ is violated only through the cusp anomaly.

  $\Rightarrow$ For eikonal jets, no β_i dependence is possible at all except through the cusp.

- In the reduced soft function $\overline{\mathcal{S}}$ the cusp anomaly cancels.

  $\Rightarrow$ $\overline{\mathcal{S}}$ must depend on β_i only through rescaling-invarant combinations such as ρ_{ij}, or, for $n \geq 4$ legs, the cross ratios $\rho_{ijkl} \equiv (\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)/(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)$.

Consider then the anomalous dimension for the reduced soft function

$$\Gamma_{IJ}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s(\mu^2)\right) = \Gamma_{IJ}^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}\left(\frac{(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s(\mu^2), \epsilon\right) .$$

This poses strong constraints on the soft matrix. Indeed

- Singular terms in $\Gamma^{\mathcal{S}}$ must be diagonal and proportional to γ_K.
- Finite diagonal terms must conspire to construct ρ_{ij}'s combining $\beta_i \cdot \beta_j$ with x_i.
- Off-diagonal terms in $\Gamma^{\mathcal{S}}$ must be finite, and must depend only on the cross-ratios ρ_{ijkl}.

Factorization constraints

The constraints can be formalized simply by using the chain rule: $\Gamma^{\overline{\mathcal{S}}}$ can depend on the factorization vectors n_i only through eikonal jets, which are color diagonal.

Defining $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$ one finds

$$x_i \frac{\partial}{\partial x_i} \Gamma^{\overline{\mathcal{S}}}_{IJ} \left(\rho_{ij}, \alpha_s \right) = - \, \delta_{IJ} \; x_i \frac{\partial}{\partial x_i} \gamma_{\mathcal{J}} \left(x_i, \alpha_s, \epsilon \right) \; = \; - \, \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s \right) \delta_{IJ} \, .$$

This leads to a linear equation for the dependence of $\Gamma^{\overline{\mathcal{S}}}$ on its proper arguments, ρ_{ij}

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{MN} \left(\rho_{ij}, \alpha_s \right) = \frac{1}{4} \, \gamma_K^{(i)} \left(\alpha_s \right) \, \delta_{MN} \qquad \forall i \, ,$$

- The equation relates $\Gamma^{\overline{\mathcal{S}}}$ to γ_K to all orders in perturbation theory
  $\Rightarrow$ and should remain true at strong coupling as well.
- It correlates color and kinematics for any number of hard partons.
- It admits a unique solution for amplitudes with up to three hard partons.
  $\Rightarrow$ For $n \geq 4$ hard partons, functions of ρ_{ijkl} solve the homogeneous equation.

The dipole formula

The cusp anomalous dimension exhibits Casimir scaling up to three loops.

- $\gamma_K^{(i)}(\alpha_s) = C^{(i)}\,\widehat{\gamma}_K(\alpha_s)$, with $C^{(i)}$ the quadratic Casimir and $\widehat{\gamma}_K(\alpha_s)$ universal.

Denoting with $\widetilde{\gamma}_K^{(i)}$ possible terms violating Casimir scaling, we write

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\left[C^{(i)}\,\widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)\right] \qquad \forall i\,,$$

By linearity, using the color generator notation, the scaling term yields

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\mathrm{Q.C.}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\, \mathrm{T}_i \cdot \mathrm{T}_i\, \widehat{\gamma}_K(\alpha_s)\,, \qquad \forall i$$

An all-order solution is the dipole formula (E. Gardi, LM; T. Becher, M. Neubert)

$$\Gamma^{\overline{\mathcal{S}}}_{\mathrm{dip}}(\rho_{ij}, \alpha_s) = -\frac{1}{8}\,\widehat{\gamma}_K(\alpha_s) \sum_{j \neq i} \ln(\rho_{ij})\, \mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_i \mathbf{T}_i \cdot \mathbf{T}_i\,,$$

as easily checked using color conservation, $\sum_i T_i = 0$.

Note: all known results for massless gauge theories are of this form.

The full amplitude

It is possible to construct a dipole formula for the full amplitude enforcing the cancellation of the dependence on the factorization vectors n_i through

$$\ln\left(\frac{(2p_i\cdot n_i)^2}{n_i^2}\right) + \ln\left(\frac{(2p_j\cdot n_j)^2}{n_j^2}\right) + \ln\left(\frac{(-\beta_i\cdot\beta_j)^2\, n_i^2 n_j^2}{2(\beta_i\cdot n_i)^2\, 2(\beta_j\cdot n_j)^2}\right) = 2\ln\left(-2p_i\cdot p_j\right) .$$

Soft and collinear singularities can then be collected in a matrix Z

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right) = Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right)\,\mathcal{H}\left(\frac{p_i}{\mu},\frac{\mu_f}{\mu},\alpha_s(\mu^2),\epsilon\right) ,$$

satisfying a matrix evolution equation

$$\frac{d}{d\ln\mu_f} Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right) = -\,\Gamma\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2)\right) Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right) .$$

The dipole structure of $\Gamma^{\overline{S}}$ is inherited by Γ, which reads (T. Becher, M. Neubert)

$$\Gamma_{\rm dip}\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right) = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{j\neq i}\ln\left(\frac{-2\,p_i\cdot p_j}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\mu^2)\right) .$$

Beyond the dipole formula

(with Lance Dixon and Einan Gardi)

Beyond the minimal solution

- The cusp anomalous dimension may violate Casimir scaling starting at four loops. This would add to $\Gamma_{\rm dip}^{\overline{\mathcal{S}}}$ a contribution $\Gamma_{\rm H.C.}^{\overline{\mathcal{S}}}$ satisfying

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{\rm H.C.}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \, \widetilde{\gamma}_K^{(i)} \left(\alpha_s\right) \,, \qquad \forall i \,.$$

- For $n \geq 4$ the constraints do not uniquely determine $\Gamma^{\overline{\mathcal{S}}}$: one may write

$$\Gamma^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) = \Gamma_{\rm dip}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) + \Delta^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) \,,$$

where $\Delta^{\overline{\mathcal{S}}}$ solves the homogeneous equation

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) = 0 \qquad \Leftrightarrow \qquad \Delta^{\overline{\mathcal{S}}} = \Delta^{\overline{\mathcal{S}}} \left(\rho_{ijkl}, \alpha_s\right) \,.$$

- By eikonal exponentiation $\Delta^{\overline{\mathcal{S}}}$ must directly correlate four partons.
  - A nontrivial function of ρ_{ijkl} cannot appear in $\Gamma^{\overline{\mathcal{S}}}$ at two loops.

$$\widetilde{\mathbf{H}}_{[\rm f]} = \sum_{j,k,l} \sum_{a,b,c} {\rm i} f_{abc} \, {\rm T}_j^a {\rm T}_k^b {\rm T}_l^c \, \ln\left(\rho_{ijkl}\right) \, \ln\left(\rho_{iklj}\right) \, \ln\left(\rho_{iljk}\right) \,.$$

  - The minimal solution holds for ‘matter loop’ diagrams at three loops (L. Dixon).

Collinear constraints

Factorization of fixed-angle amplitudes breaks down in collinear limits, as $p_i \cdot p_j \to 0$. New singularities are expected to be captured by a universal splitting function

$$\mathcal{M}_n\left(p_1, p_2, p_j; \mu, \epsilon\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu, \epsilon\right) \mathcal{M}_{n-1}\left(P, p_j; \mu, \epsilon\right) .$$

Infrared poles of the splitting function are generated by a splitting anomalous dimension

$$\mathbf{Sp}(p_1, p_2; \mu, \epsilon) = \mathbf{Sp}^{(0)}_{\mathcal{H}}(p_1, p_2; \mu, \epsilon) \; \exp\left[-\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda)\right] ,$$

related to the soft anomalous dimension matrices of the two amplitudes,

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu_f) \equiv \Gamma_n\left(p_1, p_2, p_j; \mu_f\right) - \Gamma_{n-1}\left(P, p_j; \mu_f\right) .$$

If the dipole formula receives corrections, so does the splitting amplitude

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) = \Gamma_{\mathbf{Sp},\, \mathrm{dip}}(p_1, p_2; \lambda) + \Delta_n\left(\rho_{ijkl}; \lambda\right) - \Delta_{n-1}\left(\rho_{ijkl}; \lambda\right) .$$

Universality of $\Gamma_{\mathbf{Sp}}$ constrains the combination $\Delta_n - \Delta_{n-1}$: it must depend only on the kinematics and color of the collinear parton pair (T. Becher, M. Neubert).

Bose symmetry, transcendentality

Contributions to $\Delta_n(\rho_{ijkl})$ arise from gluon subdiagrams of eikonal correlators. They must be Bose symmetric. With four hard partons,

$$\Delta_4(\rho_{ijkl}) = \sum_i h^{(i)}_{abcd} \, \mathbf{T}_i^a \, \mathbf{T}_j^b \, \mathbf{T}_k^c \, \mathbf{T}_l^d \; \Delta^{(i)}_{4,\,\mathrm{kin}}(\rho_{ijkl}) \, ,$$

the symmetries of $\Delta^{(i)}_{4,\,\mathrm{kin}}$ must match those of $h^{(i)}_{abcd}$. For polynomials in $L_{ijkl} \equiv \log \rho_{ijkl}$ one easily matches symmetries of available color tensors

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f_{ade} \, f_{cb}{}^{e} \, L_{1234}^{h_1} \left(L_{1423}^{h_2} \, L_{1342}^{h_3} - (-1)^{h_1+h_2+h_3} \, L_{1342}^{h_2} \, L_{1423}^{h_3} \right) + \text{cycl.} \right] ,$$

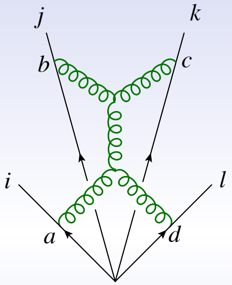
- Transcendentality constrains the powers of the logarithms. At L loops

$$h_{\mathrm{tot}} \equiv h_1 + h_2 + h_3 \leq \tau \leq 2L - 1$$

- For $\mathcal{N} = 4$ SYM, and for any massless gauge theory at three loops, the bound is expected to be saturated.
- Collinear consistency requires $h_i \geq 1$ in any monomial.

Three loops

- Δ_n can first appear at three loops.
- A general Δ_n is a 'sum over quadrupoles'.
- Relevant webs are the same in $\mathcal{N} = 4$ SYM.
- The only available color tensors are $f_{ade}\, f_{cb}{}^{e}$
- Polynomials in L_{ijkl} are severely constrained.
- Using Jacobi identities for color and $L_{1234} + L_{1423} + L_{1342} = 0$ for kinematics, only one structure polynomial in L_{ijkl} survives.



Three-loop web contributing to $\Gamma^{\mathcal{S}}$.

h_1	h_2	h_3	h_{tot}	comment	
1	1	1	3	vanishes identically by Jacobi identity	
2	1	1	4	kinematic factor vanishes identically	
1	1	2	4	allowed by symmetry, excluded by transcendentality	
1	2	2	5	viable possibility	all coincide
3	1	1	5	viable possibility	
2	1	2	5	viable possibility	
1	1	3	5	viable possibility	

Survivors

Just one maximal transcendentality, Bose symmetric, collinear safe polynomial in the logarithms survives all available constraints.

$$\Delta_4^{(122)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; L_{1234}\, \left(L_{1423}\, L_{1342}\right)^2 + f_{cae}\, f_{db}{}^{e}\; L_{1423}\, \left(L_{1234}\, L_{1342}\right)^2 + f_{bae}\, f_{cd}{}^{e}\; L_{1342}\, \left(L_{1423}\, L_{1234}\right)^2 \Big] .$$

Allowing for polylogarithms, structures mimicking the simple symmetries of L_{ijkl} must be constructed. Two examples are

$$\Delta_4^{(122,\,\mathrm{Li}_2)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; L_{1234}\, \Big(\mathrm{Li}_2(1-\rho_{1342}) - \mathrm{Li}_2(1-1/\rho_{1342})\Big) \times \Big(\mathrm{Li}_2(1-\rho_{1423}) - \mathrm{Li}_2(1-1/\rho_{1423})\Big) + \mathrm{cycl.}\Big] .$$

$$\Delta_4^{(311,\,\mathrm{Li}_3)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; \Big(\mathrm{Li}_3(1-\rho_{1342}) - \mathrm{Li}_3(1-1/\rho_{1342})\Big)\; L_{1423}\, L_{1342} + \mathrm{cycl.}\Big] .$$

Higher-order polylogarithms are ruled out by their trancendentality combined with collinear constraints (recall one must have $h_i \geq 1,\ \forall i$).

Perspective

- After $\mathcal{O}\left(10^2\right)$ years, soft and collinear singularities in massless gauge theories are still a fertile field of study. A definitive solution may be at hand.

  $\Rightarrow$ We are probing the all-order structure of the nonabelian exponent.

  $\Rightarrow$ All-order results constrain, test and help fixed order calculations.

  $\Rightarrow$ Understanding singularities has phenomenological applications through resummation.

- Factorization theorems $\Rightarrow$ Evolution equations $\Rightarrow$ Exponentiation.

- Dimensional continuation is the simplest and most elegant regulator.

  $\Rightarrow$ Transparent mapping UV $\leftrightarrow$ IR for 'pure counterterm' functions.

- Remarkable simplifications in $\mathcal{N} = 4$ SYM point to exact results.

- Factorization and velocity rescaling invariance severely constrain soft anomalous dimensions to all orders and for any number of legs.

- A simple sum-over-dipole formula may encode all infrared singularites for any massless gauge theory, a natural generalization of the planar limit.

- The study of possible corrections to the dipole formula is under way.

- Applications to resummations, subtraction methods and parton showers are possible.

Backup Slides

Jet evolution

The full form factor does not depend on the factorization vectors n_i^μ.
Defining $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$,

$$x_i \frac{\partial}{\partial x_i} \log \Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = 0 \, .$$

This dictates the evolution of the jet J, through a '$K + G$' equation

$$\begin{aligned} x_i \frac{\partial}{\partial x_i} \log J_i \quad &= \quad - \; x_i \frac{\partial}{\partial x_i} \log H \quad + \quad x_i \frac{\partial}{\partial x_i} \log \mathcal{J}_i \\ &\equiv \quad \frac{1}{2} \left[\mathcal{G}_i \Big(x_i, \alpha_s(\mu^2), \epsilon \Big) + \mathcal{K} \Big(\alpha_s(\mu^2), \epsilon \Big) \right] \, , \end{aligned}$$

Imposing RG invariance of the form factor

$$\gamma_{\overline{\mathcal{S}}} \left(\rho_{12}, \alpha_s \right) + \gamma_H \left(\rho_{12}, \alpha_s \right) + 2 \gamma_J \left(\alpha_s \right) = 0 \, .$$

leads to the final evolution equation

$$Q \frac{\partial}{\partial Q} \log \Gamma = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\bar{\mathcal{S}}} - 2 \, \gamma_J + \sum_{i=1}^{2} \left(\mathcal{G}_i + \mathcal{K} \right) \, .$$

Form factor evolution

We can now resum IR poles for form factors, such as the quark form factor

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv \langle 0|J_\mu(0)|p_1,p_2\rangle = \bar{v}(p_2)\gamma_\mu u(p_1)\,\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) .$$

- Form factors obey evolution equations of the form

$$Q^2\frac{\partial}{\partial Q^2}\log\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = \frac{1}{2}\left[K\left(\epsilon,\alpha_s(\mu^2)\right) + G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] ,$$

- Renormalization group invariance requires

$$\mu\frac{dG}{d\mu} = -\mu\frac{dK}{d\mu} = \gamma_K\left(\alpha_s(\mu^2)\right) .$$

$\gamma_K(\alpha_s)$ is the cusp anomalous dimension.

- Dimensional regularization provides a trivial initial condition for evolution if $\epsilon < 0$ (for IR regularization).

$$\overline{\alpha}(\mu^2=0,\epsilon<0)=0 \;\;\rightarrow \Gamma\left(0,\alpha_s(\mu^2),\epsilon\right) = \Gamma\left(1,\overline{\alpha}\left(0,\epsilon\right),\epsilon\right) = 1 .$$

Results for form factors

- The counterterm function K is determined by γ_K.

$$\mu \frac{d}{d\mu} K(\epsilon, \alpha_s) = -\gamma_K(\alpha_s) \quad \Longrightarrow \quad K\left(\epsilon, \alpha_s(\mu^2)\right) = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K\left(\bar{\alpha}(\lambda^2, \epsilon)\right) \, .$$

- The form factor can be written in terms of just G and γ_K,

$$\Gamma\left(Q^2, \epsilon\right) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[G\Big(-1, \overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\Big) - \frac{1}{2} \gamma_K\Big(\overline{\alpha}\left(\xi^2, \epsilon\right)\Big) \log\left(\frac{-Q^2}{\xi^2}\right) \right] \right\} .$$

$\Rightarrow$ In general, poles up to $\alpha_s^n / \epsilon^{n+1}$ appear in the exponent.

- The ratio of the timelike to the spacelike form factor is

$$\log\left[\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right] = \mathrm{i}\frac{\pi}{2} K(\epsilon) + \frac{\mathrm{i}}{2} \int_0^{\pi} \left[G\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\theta} Q^2\right), \epsilon\right) - \frac{\mathrm{i}}{2} \int_0^{\theta} d\phi \, \gamma_K\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\phi} Q^2\right)\right) \right]$$

$\Rightarrow$ Infinities are confined to a phase given by γ_K.

$\Rightarrow$ The modulus of the ratio is finite, and physically relevant.

Form factors in $\mathcal{N} = 4$ SYM

- In $d = 4 - 2\epsilon$ conformal invariance is broken and $\beta(\alpha_s) = -2\,\epsilon\,\alpha_s$.
- All integrations are trivial. The exponent has only double and single poles to all orders (Z. Bern, L. Dixon, A. Smirnov).

$$\log\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n\left(\frac{\mu^2}{-Q^2}\right)^{n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2}+\frac{G^{(n)}(\epsilon)}{n\epsilon}\right]$$
$$= -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \mathrm{e}^{-\mathrm{i}\pi n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2}+\frac{G^{(n)}(\epsilon)}{n\epsilon}\right],$$

- In the planar limit this captures all singularities of fixed-angle amplitudes in $\mathcal{N} = 4$ SYM. The structure remains valid at strong coupling, in the planar limit (F. Alday, J. Maldacena).
- The analytic continuation yields a finite result in four dimensions, arguably exact.

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2 = \exp\left[\frac{\pi^2}{4}\,\gamma_K\left(\alpha_s(Q^2)\right)\right].$$

Characterizing $G(\alpha_s, \epsilon)$

The single-pole function $G(\alpha_s, \epsilon)$ is a sum of anomalous dimensions

$$G(\alpha_s, \epsilon) = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\bar{\mathcal{S}}} - 2\gamma_J + \sum_{i=1}^{2} \mathcal{G}_i \, ,$$

In $d = 4 - 2\epsilon$ finite remainders can be neatly exponentiated

$$C\left(\alpha_s(Q^2), \epsilon\right) = \exp\left[\int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left\{ \frac{d \, \log C\left(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)}{d \ln \xi^2} \right\}\right] \equiv \exp\left[\frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} G_C\left(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)\right]$$

The soft function exponentiates like the full form factor

$$\mathcal{S}\left(\alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \left[G_{\rm eik}\left(\overline{\alpha}\left(\xi^2, \epsilon\right)\right) - \frac{1}{2} \gamma_K\left(\overline{\alpha}\left(\xi^2, \epsilon\right)\right) \log\left(\frac{\mu^2}{\xi^2}\right)\right]\right\} .$$

$G(\alpha_s, \epsilon)$ is then simply related to collinear splitting functions and to the eikonal approximation

$$G(\alpha_s, \epsilon) = 2 \, B_\delta \left(\alpha_s\right) + G_{\rm eik} \left(\alpha_s\right) + G_{\overline{H}} \left(\alpha_s, \epsilon\right) \ ,$$

$\Rightarrow$ $G_{\overline{H}}$ does not generate poles; it vanishes in $\mathcal{N} = 4$ SYM.

$\Rightarrow$ Checked at strong coupling, in the planar limit (F. Alday).