NEXT-TO-EIKONAL EXPONENTIATION

Eric Laenen

EL, L. Magnea, G. Stavenga, Phys. Lett.B 669 (2008) 173
EL, G. Stavenga, C. White, JHEP 0903: 054 (2009)
EL, L. Magnea, G. Stavenga, C. White, to appear
E. Gardi, EL, G. Stavenga, C. White, to appear

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OUTLINE

- Introduction
- Extended threshold resummation
- Next-to-eikonal exponentation for matrix elements
  - Path-integral methods
  - Diagrams and induction
- Multiple colored lines
- Conclusions

LARGE X BEHAVIOR

- For DY, DIS, Higgs, singular behavior when x→ 1

$$\delta(1-x) \qquad \left[\frac{\ln^i(1-x)}{1-x}\right] \qquad \ln^k(1-x)$$

  - singularity structure for plus distributions is organizable to all orders, perhaps also for divergent logarithms?

- After Mellin transform

$$\text{Constants} \qquad \ln^i(N) \qquad \frac{\ln^k(N)}{N}$$

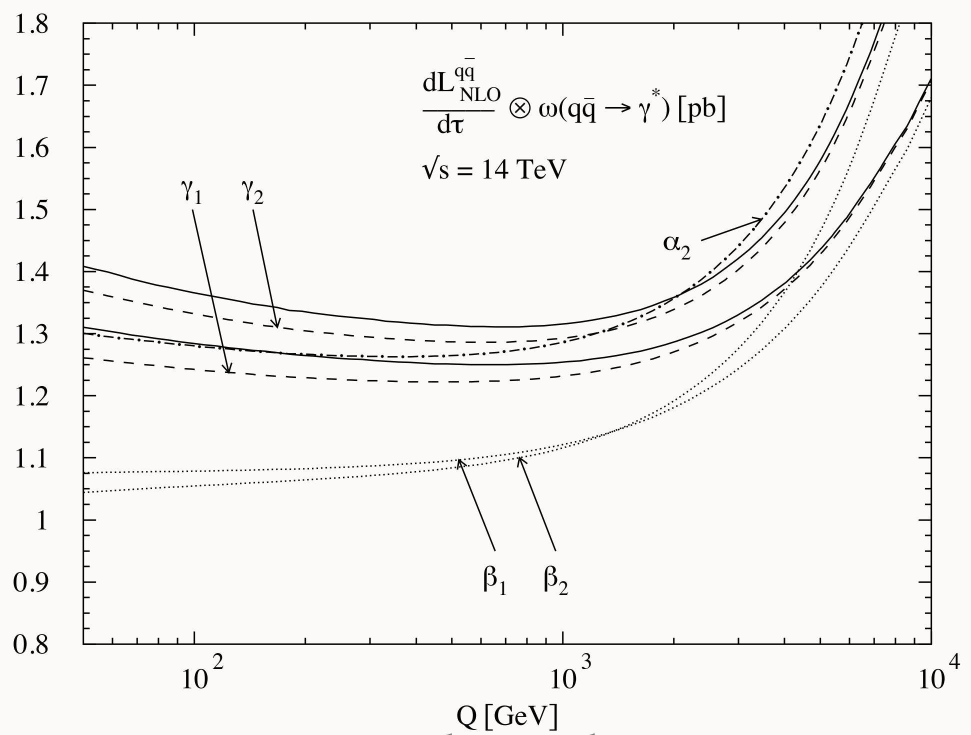
- We know a lot about logs and constants, very little about 1/N

LN(N)/N TERMS

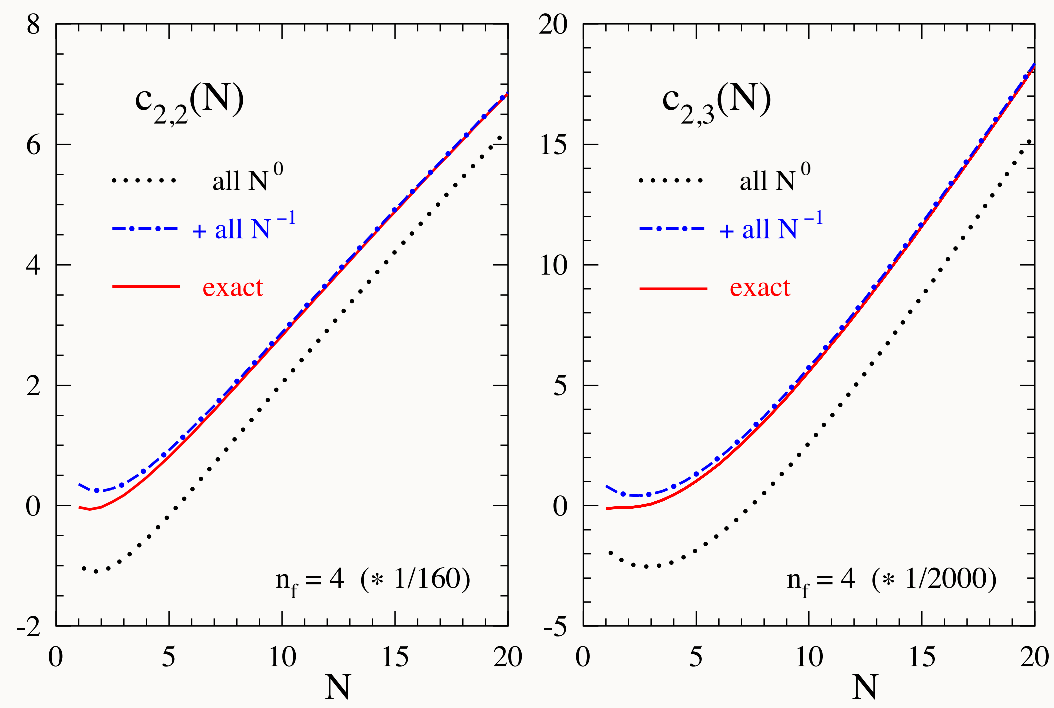
Kraemer, EL, Spira; Catani, De Florian, Grazzini; Kilgore, Harlander

- Can be numerically important

Kraemer, EL, Spira



Moch,Vogt



- We know that the leading series $\ln^i(N)/N$ exponentiates
  - by replacing in resummation formula

$$\frac{1+z^2}{1-z} \longrightarrow \frac{2}{1-z} - 2$$

SUCCESFUL LN(N)/N ORGANIZATION

Dokshitzer, Marchesini, Salam
Basso, Korchemsky

$$\gamma_{qq}(N) = A(\alpha_s)\ln N + B(\alpha_s) + C(\alpha_s)\frac{\ln N}{N} + \ldots$$

- Moch, Vermaseren, Vogt noted an remarkable relation

$$C_2 = A_1^2 \qquad C_3 = 2A_2A_1$$

- DMS reproduced this by changing DGLAP equation

$$\mu^2 \frac{\partial}{\partial \mu^2}\psi(x,\mu^2) = \int_x^1 \frac{dz}{z}\,\psi\left(\frac{x}{z}, z\mu^2\right)\mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right)$$

$$\mathcal{P}\left(z,\alpha_s\right) = \frac{A\left(\alpha_s\right)}{(1-z)_+} + B_\delta\left(\alpha_s\right)\delta(1-z) + \mathcal{O}\left((1-z)\right)$$

- Can this be reproduced in threshold resummation?

EXTENDED THRESHOLD RESUMMATION

EL, Magnea, Stavenga

Ansatz: modified resummed expression

$$\ln\left[\sigma(N)\right] = \mathcal{F}_{\rm DY}\left(\alpha_s(Q^2)\right) + \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z}\, D\left[\alpha_s\left(\frac{(1-z)^2Q^2}{z}\right)\right]\right.$$
$$\left. + 2\int_{Q^2}^{(1-z)^2Q^2/z}\frac{dq^2}{q^2}\, P_s\left[z,\alpha_s(q^2)\right]\right\}_+$$

where $$P_s^{(n)}(z) = \frac{z}{1-z}A^{(n)} + C_\gamma^{(n)}\ln(1-z) + \overline{D}_\gamma^{(n)}$$

(We constructed a similar expression for DIS). Structure:

$$\sigma(N) = \sum_{n=0}^{\infty}\left(g^2\right)^n\left[\sum_{m=0}^{2n} a_{nm}\ln^m N + \sum_{m=0}^{2n-1} b_{nm}\frac{\ln^m N}{N}\right] + \mathcal{O}\left(N^{-2}\right)$$

	C_F^2		C_AC_F		n_fC_F	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

EXTENDED THRESHOLD RESUMMATION

DIS

	C_F^2		$C_A C_F$		$n_f C_F$	
d_{23}	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
d_{22}	$\frac{39}{16}$	$\frac{55}{16}$	$\frac{11}{48}$	$\frac{11}{48}$	$-\frac{1}{24}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{1}{4}\zeta_2 - \frac{105}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$	$-\frac{107}{144}$	$-\frac{47}{144}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{3}{4}\zeta_3 + \frac{53}{16}\zeta_2 - \frac{21}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$	$\frac{1}{24}\zeta_2 - \frac{1699}{864}$	$-\frac{1}{8}\zeta_2 + \frac{73}{864}$

Almost works, but not quite. Similar at 3 loop.

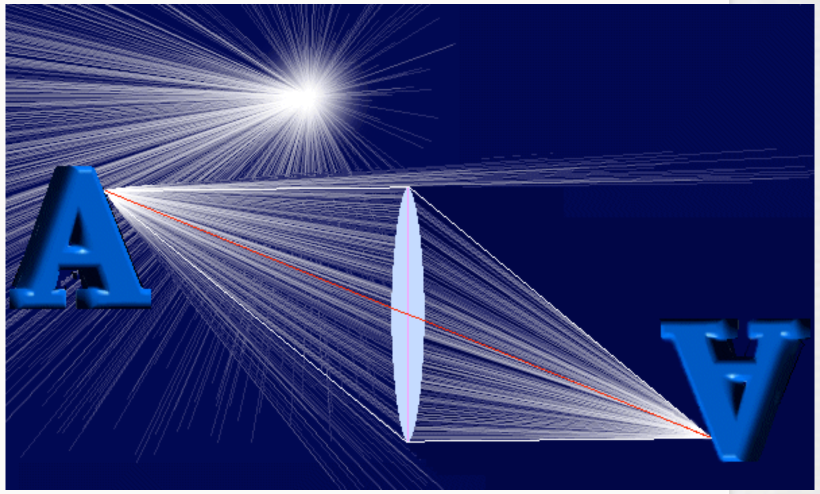
More general approach by Grunberg, Ravindran. Does not work fully either.

Other approach: use physical evolution kernels Moch, Soar, Vermaseren, Vogt

For deeper understanding we must go beyond the eikonal approximation

HISTORY OF EIKONAL APPROXIMATION

- "Eikon" originally from Greek ειχεναι [to resemble]
  - leading to ειχον [icon, image]
- Predates quantum mechanics, and even Maxwell
  - also known in optics as "ray optics"
- Can describe formation of images/eikons
- Cannot describe diffraction, polarization etc
  - these are wave phenomena

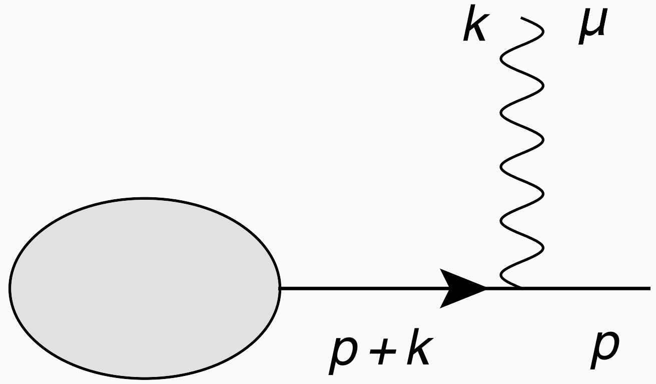


EIKONAL APPROXIMATION IN QFT

- At amplitude level
  - Reveals new symmetries, new structures in gauge theory
  - Intuitive interpretation
  - Practical
    - Coherence, resummation, EFT, ...

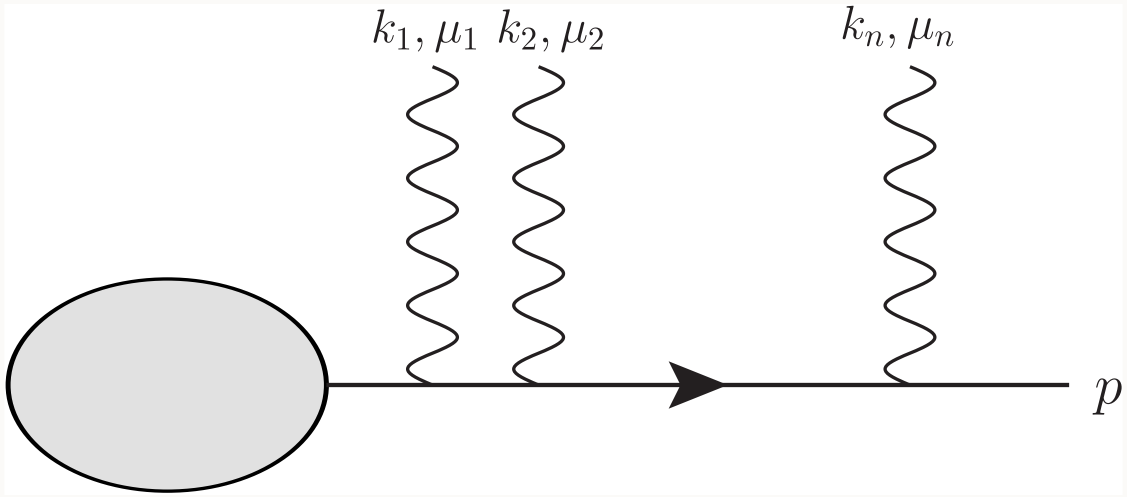
BASICS, QED

- Soft emission by charged particle
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p \cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p \cdot k}$$

BASICS QED, CONT'D



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\ldots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i = \sum_{m=i}^{n} k_m.$$

Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\ldots\frac{1}{2pK_n}2p^{\mu_n}$$

Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

NON-ABELIAN EIKONAL APPROXIMATION

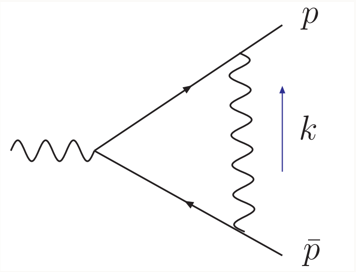
- Same methods as for QED, but organization harder: SU(3) generator at every vertex
  - no obvious decorrelation

Order the T_a according to λ

- Key "object": Wilson line $\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A^a(\lambda n)\, T_a\right]$
  - Order by order in "g", it generates QCD eikonal Feynman rules

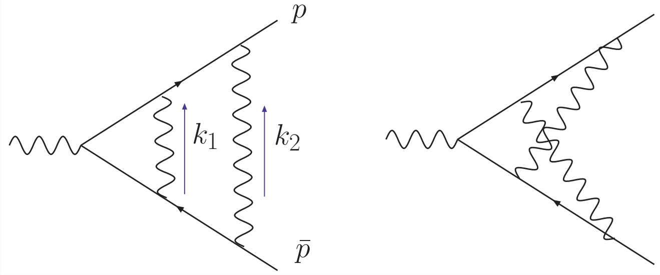
EXPONENTIATON

One loop vertex correction, in eikonal approximation



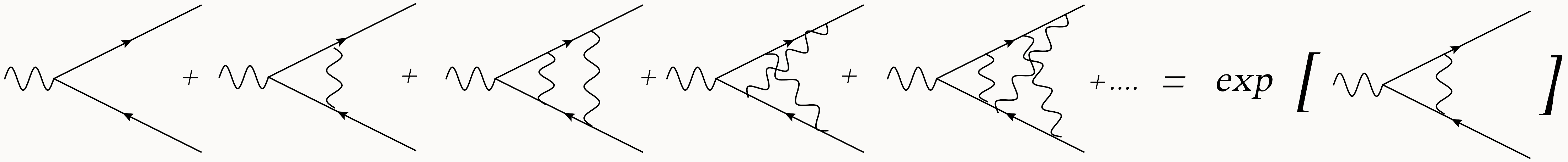
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \, \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

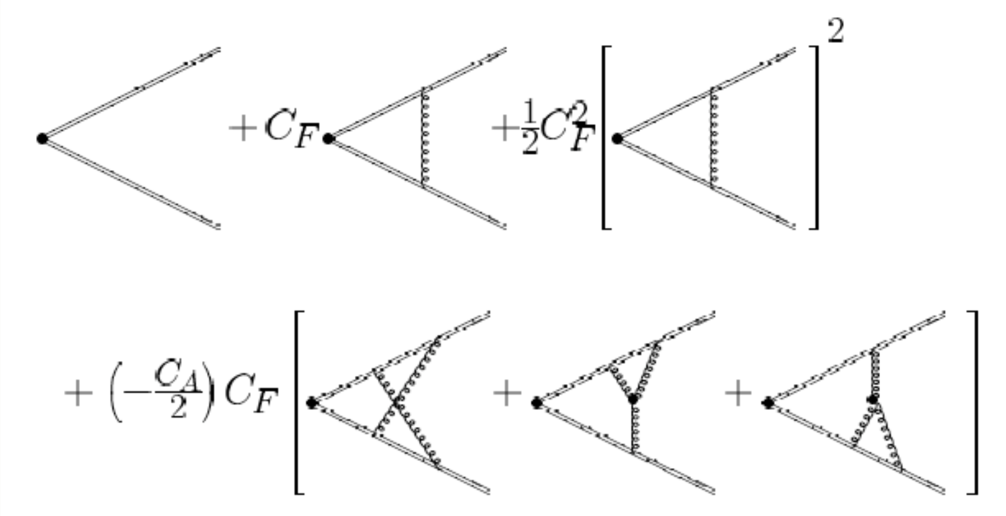
Exponential series



NON-ABELIAN EXPONENTIATION: WEBS

Gatheral; Frenkel, Taylor; Sterman

- Take quark - antiquark line, connect with soft gluons in all possible ways, use eikonal approximation
- Exponentiation still occurs, without path ordering!
  - A selection of diagrams in exponent, but with modified color weights: "webs"



  - Webs are two-eikonal line irreducible
  - Proof by induction; recursive definition of color weights
- How can we extend this to include next-to-eikonal terms?

PATH INTEGRAL METHOD

EL, Stavenga, White

Represent propagator as particle path integral, between coord. and momentum states

$$\tilde{\Delta}_F(p_f^2) = \frac{1}{2}\int_0^\infty dT \frac{\langle p_f|U(T)|x_i\rangle}{\langle p_f|x_i\rangle} = -\frac{i}{p_f^2 + m^2 - i\varepsilon}$$

where

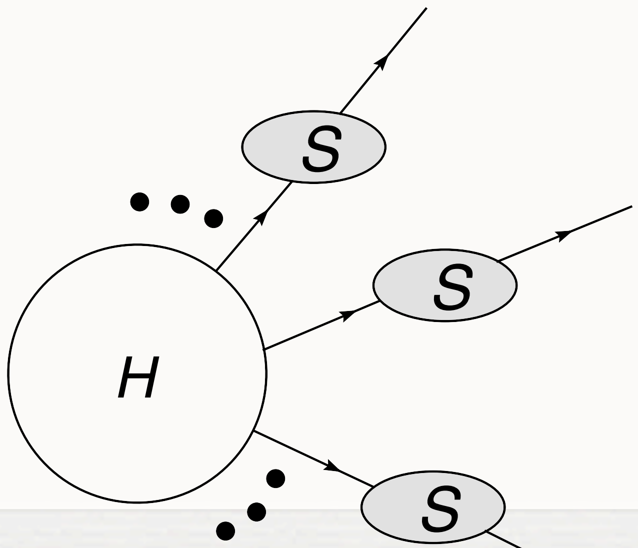
$$\langle p_f|U(T)|x_i\rangle = e^{-ip_f x_i - i\frac{1}{2}(p_f^2+m^2)T}\int_{x(0)=0}^{p(T)=0} \mathcal{D}p\mathcal{D}x\, e^{i\int_0^T dt(p\dot{x}-\frac{1}{2}p^2)}$$

Add an (abelian) gauge field

$$\langle p_f|U(T)|x_i\rangle = \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}p\mathcal{D}x \exp\Big[-ip(T)x(T) + i\int_0^T dt(p\dot{x} - \frac{1}{2}(p^2 + m^2) + p\cdot A + \frac{i}{2}\partial\cdot A - \frac{1}{2}A^2)\Big]$$

n-point Green's function

$$G(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu\, H(x_1,\ldots,x_n)$$
$$\times \langle p_1|((p-A_s)^2 - i\varepsilon)^{-1}|x_1\rangle \ldots \langle p_n|(p-A_s)^2 - i\varepsilon)^{-1}|x_n\rangle$$



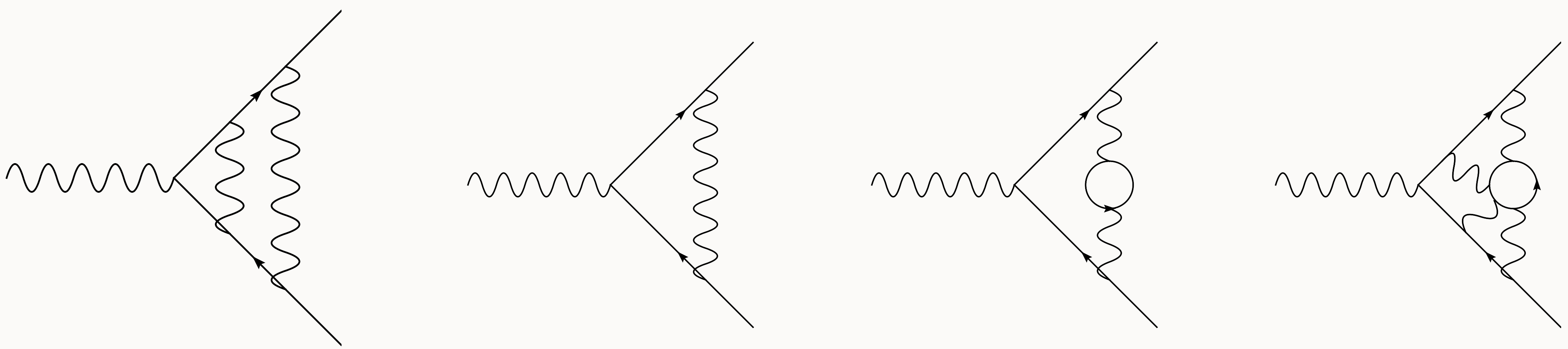
PATH INTEGRAL METHOD

Truncate external lines for S-matrix element $i(p_f^2+m^2)\langle p_f| - i((p-A)^2 - i\varepsilon)^{-1}|x_i\rangle = e^{-ip_f x_i} f(\infty)$

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu H(x_1,\ldots,x_n)\, e^{-ip_1x_1} f_1(\infty)\ldots e^{-ip_nx_n} f_n(\infty)\, e^{iS[A_s]}$$

$$f(\infty) = \int_{x(0)=0} \mathcal{D}x\, e^{i\int_0^\infty dt\left(\frac{1}{2}\dot{x}^2 + (p_f+\dot{x})\cdot A(x_i+p_f t+x(t)) + \frac{i}{2}\partial\cdot A(x_i+p_f t+x)\right)}$$

Eikonal vertices act as sources for gauge bosons along path



Disconnected

Connected

QED: exponentiation now textbook result:
all diagrams = exp (connected diagrams)

REPLICA TRICK

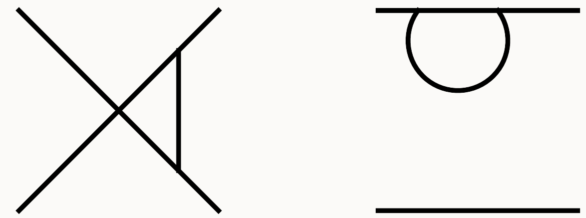
- Can relate exponentiation of soft gauge fields to that of connected diagrams in QFT. Proof: replica trick (from stat. mech.)
- Consider a N copies of a scalar theory

$$Z[J]^N = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N e^{iS[\phi_1]+..+iS[\phi_N]+J\phi_1+..J\phi_N}$$

  - If Z is exponential, find out what contributes to log Z

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$

  - Amounts to diagrams that allow only one replica → connected!



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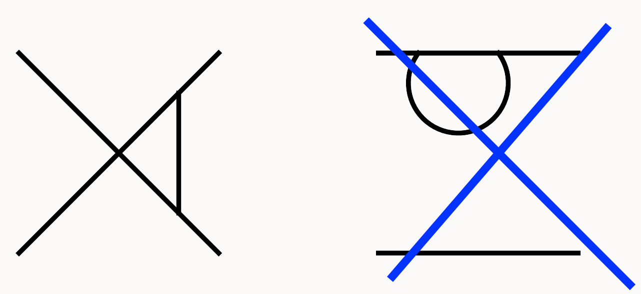
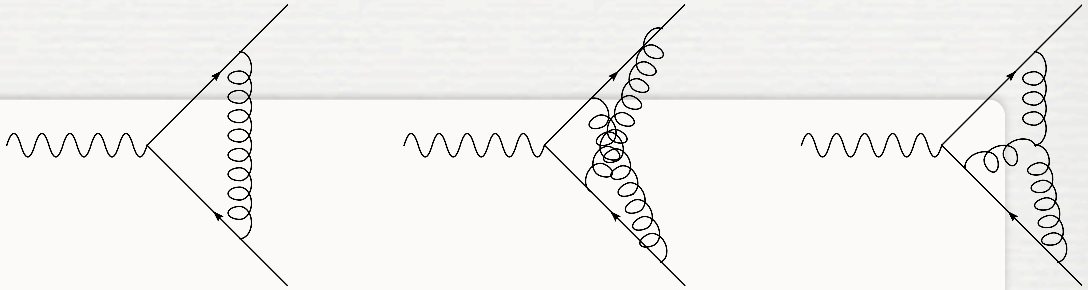


APPLICATION TO QCD



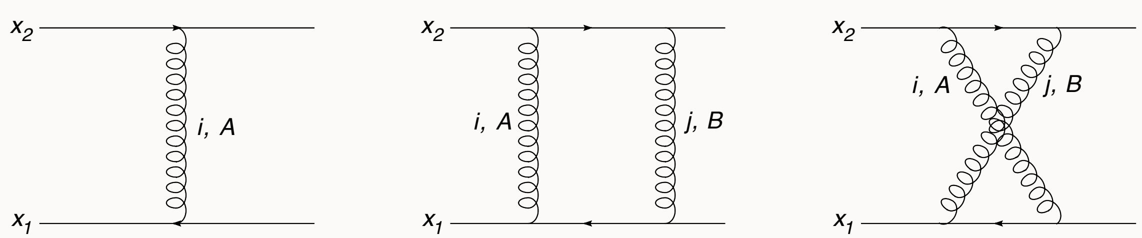
Amplitude for two colored lines

$$S(p_1, p_2) = H(p_1, p_2) \int \mathcal{D}A_s f(\infty) e^{iS[A_s]}$$

Replicate, and introduce ordering operator

$$f(\infty) = \mathcal{P} \exp\left[\int dx \cdot A(x)\right] \qquad \prod_{i=1}^{N} \mathcal{P} \exp\left[\int dx \cdot A_i(x)\right] = \mathcal{R}\mathcal{P} \exp\left[\sum_{i=1}^{N} \int dx \cdot A_i(x)\right]$$

Look for diagrams of replica order N. These will go into exponent



(a) (b) (c)

(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2 C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N: $N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$

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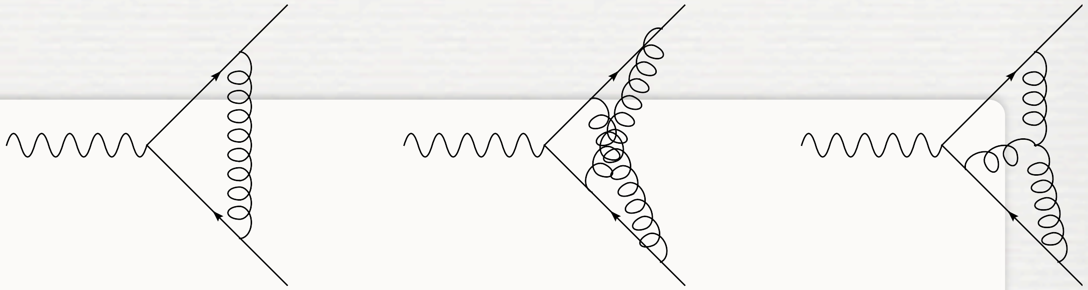

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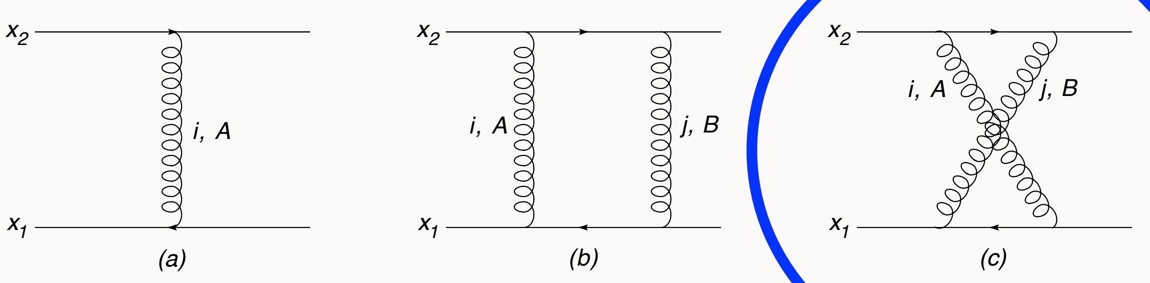


Web
Modified color factor

(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2 C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N:

$$N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$$

WEBS TO ALL ORDERS

- Can even give an all-order, *non-recursive* formula for modified color factors
  - Consider general diagram G, with a number of connected pieces
  - Distribute replica numbers in all possible ways, and count for each such partition P the multiplicity to linear order in N

$$\overline{C}(G) = \sum_{P} (-)^{n(P)-1} (n(P)-1)! \prod_{g} C(g)$$

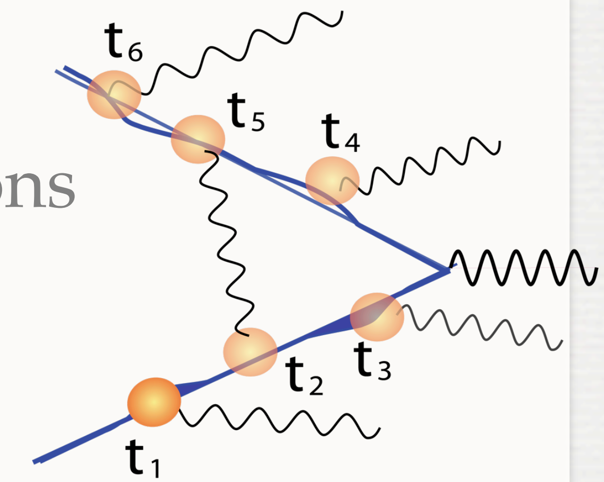
  - Examples

$$\begin{aligned}\overline{C}(X) &= C(X) - C(I)C(I) \\ &= (C_F^2 - \frac{1}{2}C_A C_F) - C_F^2\end{aligned}$$

$$\overline{C}(IX) = C(IX) - C(I)C(X) = 0$$

NEXT-TO-EIKONAL EXPONENTIATION

- Wilson lines are classical solutions of path integral
- Fluctuations around classical path lead to NE corrections
  - This class of NE corrections exponentiates
  - Keep track via scaling variable λ $p^\mu = \lambda n^\mu$



$$f(\infty) = \int_{x(0)=0} \mathcal{D}x \, \exp\left[i \int_0^\infty dt \left(\frac{\lambda}{2}\dot{x}^2 + (n+\dot{x})\cdot A(x_i + nt + x) + \frac{i}{2\lambda}\partial\cdot A(x_i + p_f t + x) \right) \right]$$

Use 1-D field theory propagators

$$\langle x(t) x(t') \rangle = G(t,t') = \frac{i}{\lambda}\min(t,t')$$

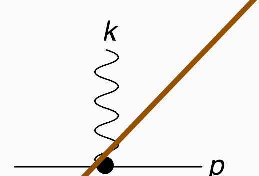
$$\langle \dot{x}(t)\dot{x}(t') \rangle = \frac{i}{\lambda}\delta(t-t')$$

FEYNMAN RULES

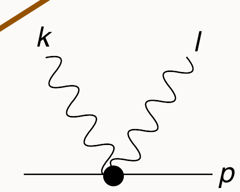
$$f(\infty) = \exp\left[-\int \frac{d^dk}{(2\pi)^d}\frac{n^\mu}{n\cdot k}\tilde{A}_\mu(k) + \frac{1}{2\lambda}\int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{n\cdot k}\tilde{A}_\mu(k) + \sum \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet + \sum \bigcirc \right]$$

Both 2-point correlators and tadpoles contribute

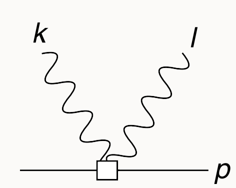
NE Feynman rules



$$\frac{k^\mu}{2p\cdot k} - k^2\frac{p^\mu}{2(p\cdot k)^2}$$



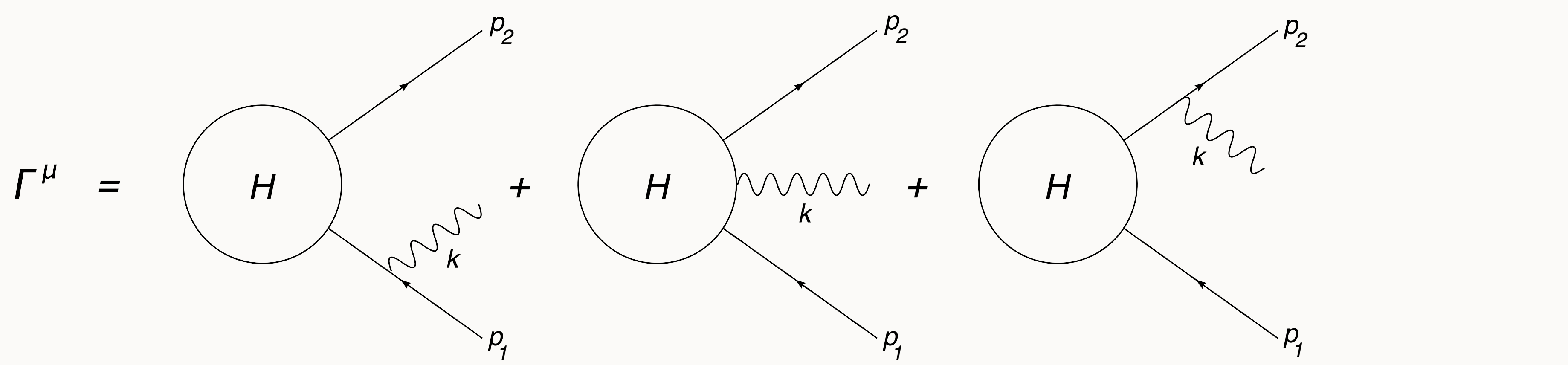
$$+\frac{\eta^{\mu\nu}}{p\cdot(k+l)}$$



$$-\frac{l^\mu p^\nu p\cdot k + k^\nu p^\mu p\cdot l}{p\cdot(k+l)p\cdot kp\cdot l}$$

LOW-BURNETT-KROLL

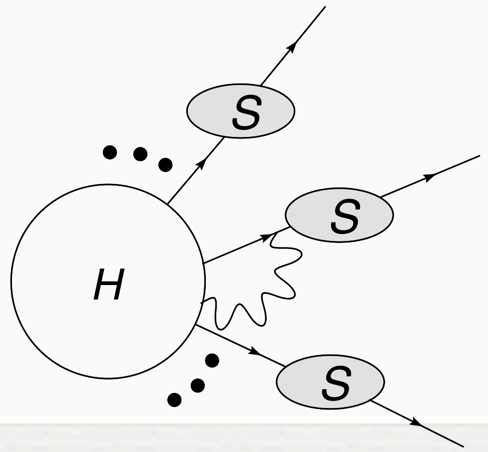
One soft emission determined by elastic amplitude to eikonal and next-to-eikonal order



$$\Gamma^\mu = \left[\frac{(2p_1-k)^\mu}{-2p_1\cdot k} + \frac{(2p_2+k)^\mu}{2p_2\cdot k}\right]\Gamma + \left[\frac{p_1^\mu(k\cdot p_2 - k\cdot p_1)}{p_1\cdot k} + \frac{p_2^\mu(k\cdot p_1 - k\cdot p_2)}{p_2\cdot k}\right]\frac{\partial\Gamma}{\partial p_1\cdot p_2}$$

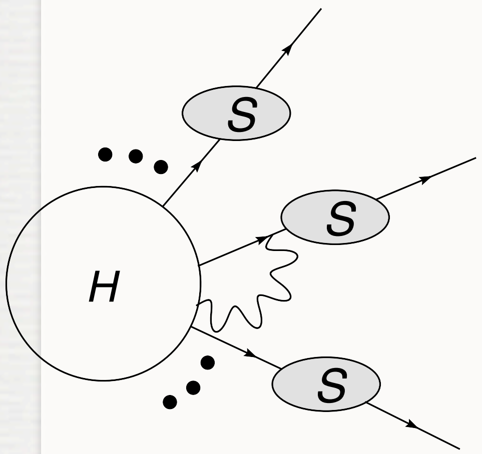
Analyzed in context of jet-soft factorization by Del Duca

One emission from H still missing in our approach



LOW-BURNETT-KROLL

Path integral method provides elegant way to derive Low's theorem



$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s H(x_1,\ldots,x_n;A_s)e^{-ip_1x_1}f(x_1,p_1;A_s)\ldots e^{-ip_nx_n}f(x_1,p_1;A_s)e^{iS[A_s]}$$

Gauge transformation must cancel between f's and H

$$f(x_i,p_f;A) \to f(x_i,p_f;A+\partial\Lambda) = e^{-iq\Lambda(x_i)}f(x_i,p_f;A)$$

Opposite transformation in H, expand to first order in A and Λ

Low's contribution is then:

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A\Big[\int \frac{d^dk}{(2\pi)^d}\sum_j^n q_j\Big(\frac{n_j^\mu}{n_j\cdot k}k_\nu\frac{\partial}{\partial p_{j_\nu}} - \frac{\partial}{\partial p_{j_\mu}}\Big)H(p_1,\ldots,p_n)A_\mu(k)\Big]$$
$$\times f(0,p_1;A)\ldots f(0,p_n;A)$$

First term is due to displacement of f(x,p,A)

Analagous result in non-abelian case, for n=2

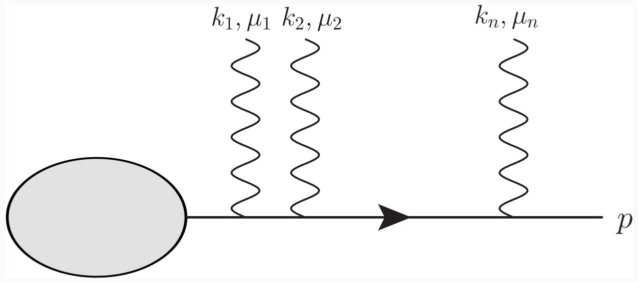
UPSHOT

- Exponentiation of soft emissions for matrix elements as "connectedness"
    - For both eikonal and next-to-eikonal contributions from external lines
    - Replica trick both for exponentiation, and for explicit expression for webs. New NE Webs.
    - 1 emission from hard part also included
    - QCD: 2 lines ok. 3 lines also easy. 4+ (later)
- Can we arrive at the same results using diagrams, and inductive reasoning?
    - Combinatorics challenging

DIAGRAMMATIC APPROACH

EL, Magnea, Stavenga, White

Recall: Abelian case, multiple emission, and sum over permutations



Eikonal identity:

$$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

For many emissions

$$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Non-abelian case requires

- web: two-eikonal irreducible graph
- "group" : projection of web on external line
- analogue of eikonal identity for permutations that leaves ordering in group invariant

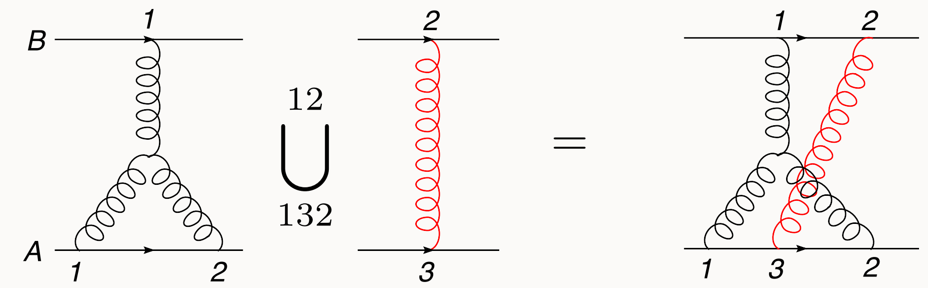
(shuffle product)

$$\sum_{\tilde{\pi}} \frac{1}{2p\cdot k_{\tilde{\pi}_1}}\frac{1}{2p\cdot(k_{\tilde{\pi}_1}+k_{\tilde{\pi}_2})}\cdots\frac{1}{p\cdot(k_{\tilde{\pi}_1}+\ldots+k_{\tilde{\pi}_n})}=\prod_g \frac{1}{2p\cdot k_{g_1}}\frac{1}{2p\cdot(k_{g_1}+k_{g_2})}$$
$$\cdots\frac{1}{2p\cdot(k_{g_1}+\ldots+k_{g_m})}$$

EXPONENTIATION BY INDUCTION

Can also use in reverse, as "merging"

$$E(H_1)\,E(H_2) = \sum_{\pi_A}\sum_{\pi_B} E(H_1 \cup^{\pi_B}_{\pi_A} H_2)$$



Collect identical diagrams $\quad E(H_1)\,E(H_2) = \sum_G E(G)\,N_{G|H_1 H_2}$

$$\exp\left\{\sum_i \bar{c}_H E(H)\right\} = \prod_H \left(\sum_n \frac{1}{n!}[\bar{c}_H E(H)]^n\right) = \sum_G c_G E(G)$$

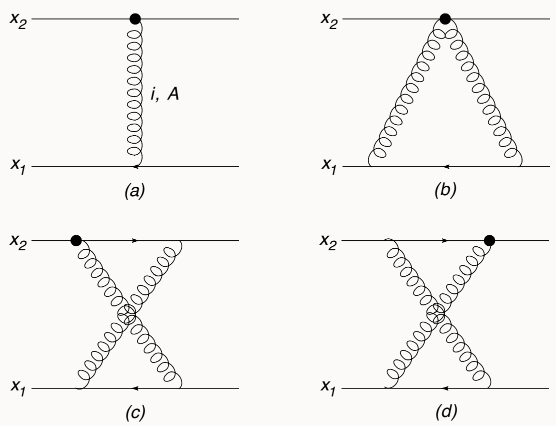
Prove that, for normal color factors on rhs, those on left side are those of webs

Proof uses

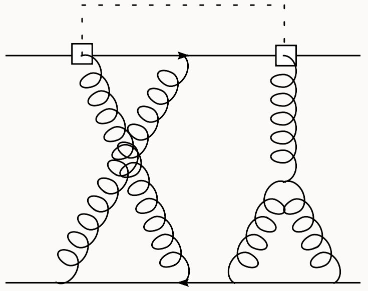
- induction
- combinatorics
- simplicity of color structure

NEXT-TO-EIKONAL CASE

- Identify next-to-eikonal vertices
  - show that they "decorrelate", once summed over all perm's. Use induction again
    - as eikonal webs, but now with a special vertex
      - for fermions: they become spin-sensitive
    - new correlations between eikonal webs → NE webs



- checked precise correspondence with path integral method
- still two-eikonal line irreducible
- Proof of exponentiation as for eikonal case



DRELL-YAN CHECK

- Check use of NE Feynman rules for Drell-Yan double real emission
  - for amplitude, expand

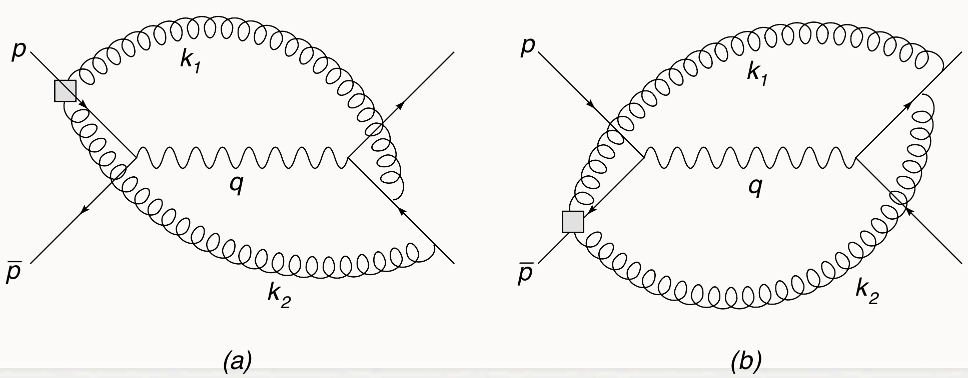
$$\mathcal{A} = \mathcal{A}^{(0)} \exp\left[\bar{\mathcal{A}}^{(1)\mathrm{E}} + \bar{\mathcal{A}}^{(1)\mathrm{NE}} + \bar{\mathcal{A}}^{(2)\mathrm{NE}}\right]$$

  - to 2nd order, and integrate each term with exact 3-particle phase space. Cross term leads to

$$\left(\frac{\alpha_S C_F}{4\pi}\right)^2 \left[\frac{-1024\log^3(1-z)}{3} - \frac{512\log^2(1-z)}{\epsilon} - \frac{512\log(1-z)}{\epsilon^2} - \frac{256}{\epsilon^3}\right]$$

  - Also need special vertex

$$R^{\mu\nu}(p;k_1,k_2) = -\frac{(p\cdot k_2)p^\mu k_1^\nu + (p\cdot k_1)k_2^\mu p^\nu - (p\cdot k_1)(p\cdot k_2)g^{\mu\nu} - (k_1\cdot k_2)p^\mu p^\nu}{p\cdot(k_1+k_2)}$$



DRELL-YAN CHECK

- Combine with exact phase space

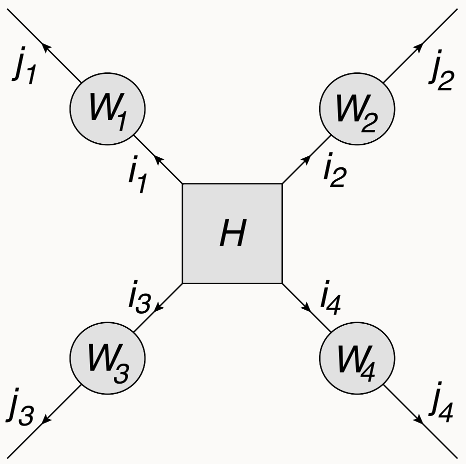
$$K^{(2)NE} = \left(\frac{\alpha_S C_F}{4\pi}\right)^2 \left[\frac{1024\mathcal{D}_3}{3} - \frac{1024\log^3(1-z)}{3} + 640\log^2(1-z) \right.$$
$$+ \frac{512\mathcal{D}_2 - 512\log^2(1-z) + 640\log(1-z)}{\epsilon} + \frac{512\mathcal{D}_1 - 512\log(1-z)}{\epsilon^2}$$
$$\left. + \frac{256\mathcal{D}_0 - 256}{\epsilon^3}\right]$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+$$

  - Agrees with equivalent exact result

MULTIPLE COLORED LINES

Gardi, EL, Stavenga, White

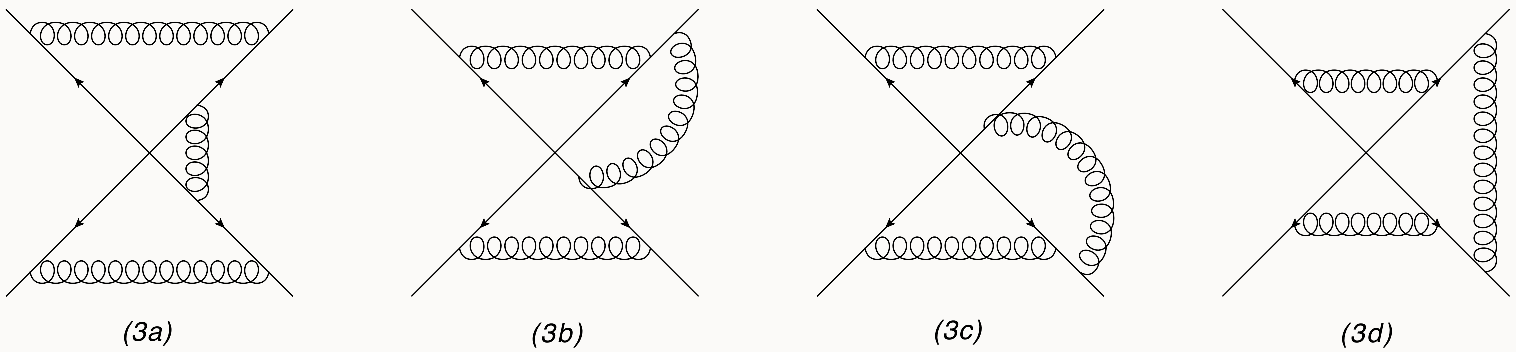


$$Z^N_{IJ} = \int \mathcal{D}\mathbf{A^1}\dots\mathcal{D}\mathbf{A^N} e^{i\sum S[A^i]}$$
$$\times \left[(\mathbf{W}_1^{(1)}\dots\mathbf{W}_N^{(1)})\dots\right]$$

- Replica trick for multiple colored lines; find again order N terms
  - even when "present", these may be kinematically zero   Aybat, Dixon, Sterman

MULTIPLE COLORED LINES

- example: "closed sets" of diagrams



$$\frac{1}{6}\Big[C(3a) - C(3b) - C(3c) + C(3d)\Big] \times \Big[M(3a) - 2M(3b) - 2M(3c) + M(3d)\Big]$$

- Closed form solution for modified color factor

$$\overline{C}(G) = \sum_P \frac{(-)^{n(P)-1}}{n(P)} \sum_\pi C(g_{\pi_1}) \ldots C(g_{\pi_{n(P)}})$$

CONCLUSIONS

- Eikonal approximation important, yields simplification, symmetries, all-order results
- Next-to-eikonal contributions not negligible, but fairly little is known
- Found that certain next-to-eikonal contributions form new webs, and exponentiate
  - using path integrals, or diagrammatics
    - Feynman rules for exponent of scattering amplitude
  - classified “Low’s theorem” contributions
- Outlook:
  - more legs
  - application to cross sections