NNLO with local subtractions

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in collaboration with
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Outline

- Motivation
- Recipe for a general subtraction scheme at NNLO
- Integrating the counterterms
- Results
- Conclusions

Motivation

Why NNLO?

- Precise predictions for 'standard candles': V (+ jet), top pair
- Missing piece for precise determination of pdf's
- NLO corrections are often large (e.g.>50%): H production
- Main source of uncertainty in experimental results is often due to theory: α_s measurement from shapes, jet rates
- NLO is effectively LO: energy distribution inside jets
- For reliable estimate of theory uncertainty

Why NNLO?

Less sophisticated answer:

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Less sophisticated answer:

Many matrix elements are known,
but yet vaguely used

Status

- NNLO corrections have been known to processes with fully inclusive final states for almost 30 years

  Chetyrkin et al, Van Neerven et al, Harlander-Kilgore

- Dedicated approches for simple final state
  - 2jet electroproduction, H and V hadroproduction with SD

    Anastasiou, Melnikov and Petriello
  - H and V production with NLO + constrained-NNLO subtraction

    Catani and Grazzini

- Antennae subtraction for two- and three-jet production in e^+e^- annihilation

  Gehrmann et al, Weinzierl

  (extension to include coloured initial state is in progress) Daleo et al, Pires and Glover

Problem

$$\begin{aligned}\sigma^{\mathrm{NNLO}} &= \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} \\ &\equiv \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}\end{aligned}$$

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  - in σ^{RR} kinematical singularities as one or two partons become unresolved yielding ϵ-poles at $O(\epsilon^{-4}, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$ after integration over phase space, no explicit ϵ-poles
  - in σ^{RV} kinematical singularities as one parton becomes unresolved yielding ϵ-poles at $O(\epsilon^{-2}, \epsilon^{-1})$ after integration over phase space + explicit ϵ-poles at $O(\epsilon^{-2}, \epsilon^{-1})$
  - in σ^{VV} explicit ϵ-poles at $O\,(\epsilon^{-4}, \epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1})$

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general solution is not yet available

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✓ First method to yield physical cross sections
✓ Calculation is fully numerical
- Cancellation of poles also and depends on the jet function
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CS dipole subtraction

- ✓ Clear concept
- ✓ Explicit documentation for any process
- – Cannot be extended to NNLO for arbitrary processes

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- ✓ fully local counterterms (efficiency and mathematical rigour)
- ✓ explicit expressions including colour (colour space natation is used)
- ✓ completely algorithmic construction (valid in any order of perturbation theory)
- ✓ option to constrain subtraction near singular regions (important check)

Recipe for a general subtraction scheme at NNLO

G. Somogyi, ZT hep-ph/0609041, hep-ph/0609043
G. Somogyi, ZT, V. Del Duca hep-ph/0502226, hep-ph/0609042
Z. Nagy, G. Somogyi, ZT hep-ph/0702273

Structure

of subtractions is governed by jet functions

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

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- Universal IR structure of QCD (squared) matrix elements
  - ϵ-poles of one-loop amplitudes:

$$|\mathcal{M}_m^{(1)}(\{p\})\rangle = -\frac{1}{2}\boldsymbol{I}_1^{(0)}(\epsilon;\{p\})|\mathcal{M}_m^{(0)}(\{p\})\rangle + \mathrm{O}(\varepsilon^0)$$

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  - ϵ-poles of two-loop amplitudes:

$$|\mathcal{M}_m^{(2)}(\{p\})\rangle = -\frac{1}{2}\left(\boldsymbol{I}_1^{(0)}(\epsilon;\{p\})|\mathcal{M}_m^{(1)}(\{p\})\rangle + \boldsymbol{I}_1^{(1)}(\epsilon;\{p\})|\mathcal{M}_m^{(0)}(\{p\})\rangle\right) + \mathrm{O}(\varepsilon^0)$$

S. Catani 1998, G. Sterman, M.E.Tejeda-Yeomans 2003, S. Moch, M. Mitov 2007

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- Universal IR structure of QCD (squared) matrix elements
  - ϵ-poles of one- and two-loop amplitudes
  - soft and collinear factorization of QCD matrix elements

tree-level 3-parton splitting, double soft current:

J.M. Campbell, E.W.N. Glover 1997, S. Catani, M. Grazzini 1998

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- Extension over whole phase space using momentum mappings

$$\{p\}_{n+s} \to \{\tilde{p}\}_n$$

Momentum mappings

$$\{p\}_{n+s} \to \{\tilde{p}\}_n$$

- implement exact momentum conservation
- recoil distributed democratically

  $\Rightarrow$ can be generalized to any number of s unresolved partons
- different mappings for
  - collinear limit $p_i || p_r$: $\{p\}_{n+1} \xrightarrow{C_{ir}} \{\tilde{p}\}_n^{(ir)}$
  - soft limit $p_s \to 0$: $\{p\}_{n+1} \xrightarrow{S_s} \{\tilde{p}\}_n^{(s)}$

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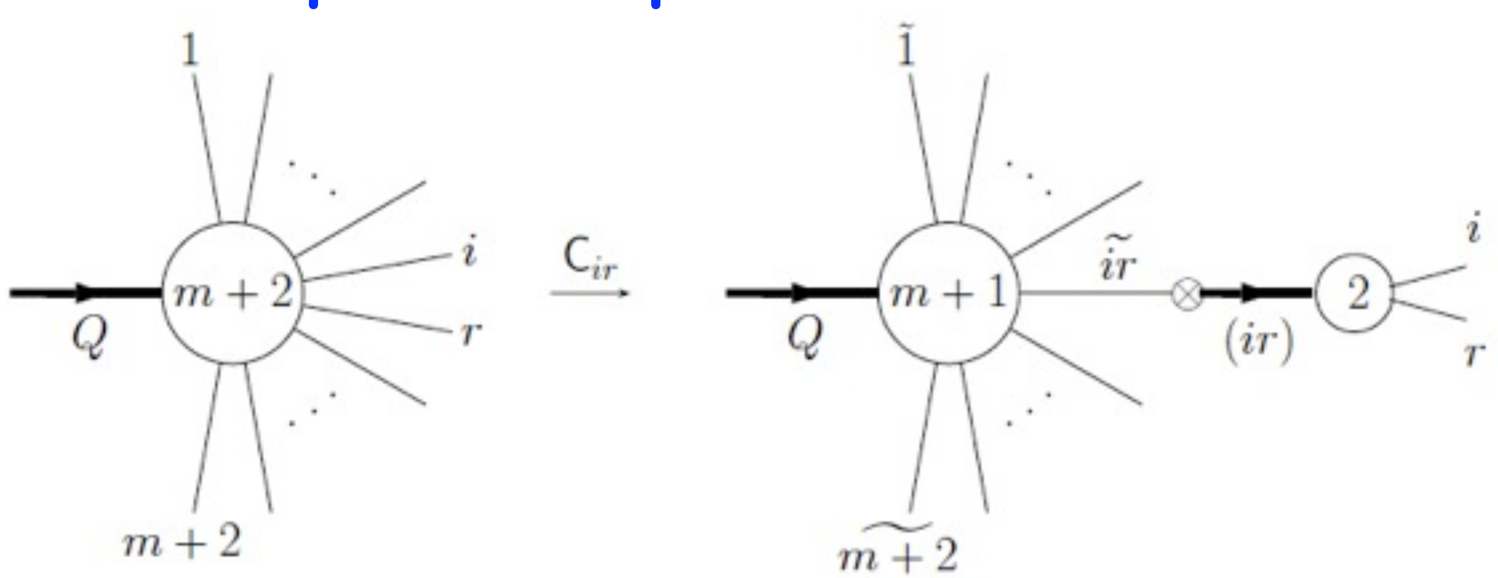

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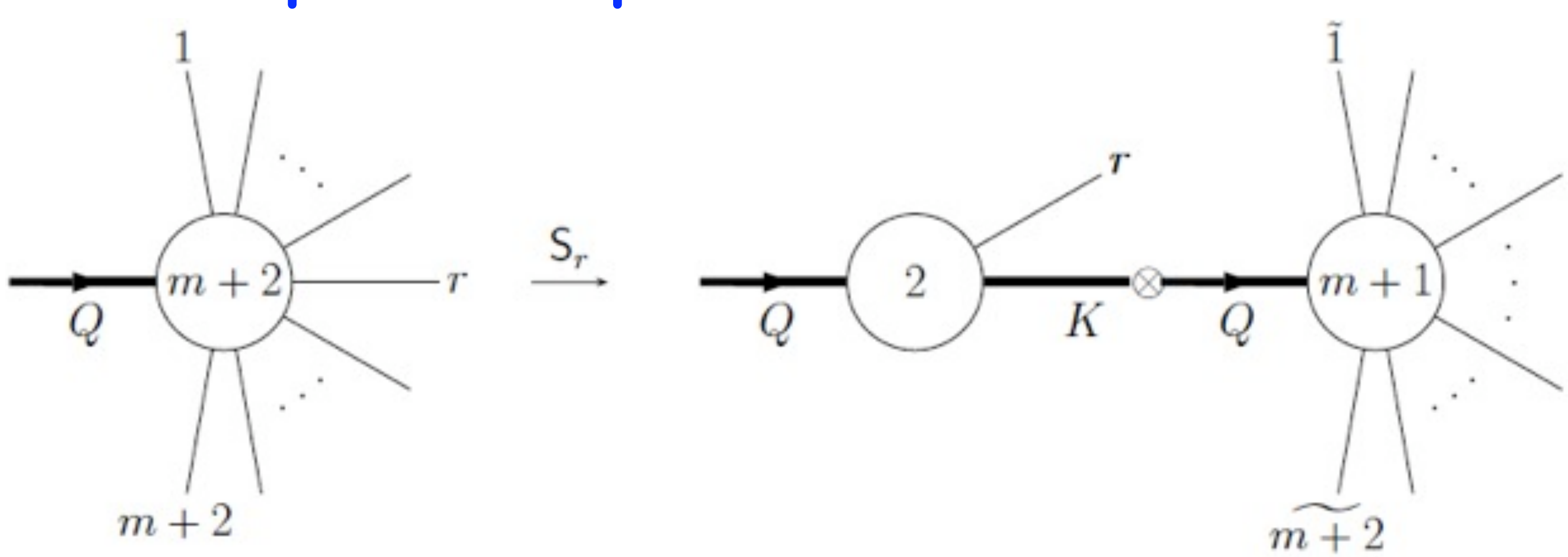

Momentum mappings

define subtractions

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Integrating the counterterms

G. Somogyi, ZT arXiv:0807.0509

U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi, ZT arXiv:0807.0514

P. Bolzoni, S. Moch, G. Somogyi, ZT arXiv:0905.4390

Integrated counterterms

two types of singly-unresolved

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$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} \boxed{+ \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right) \boxed{+ \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1}} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

Collinear integrals

convolution of the integral of AP-splitting function over ordinary phase space

$$\int_0^{\alpha_0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\int \mathrm{d}\phi_2(p_i,p_r;p_{(ir)})\frac{1}{s_{ir}^{1+\kappa\epsilon}}P^{(\kappa)}_{f_i f_r}(z_i,z_r;\epsilon)\,, \qquad \kappa=0,\,1$$

$$\mathrm{d}\phi_2(p_i,p_r;p_{(ir)}) = \frac{s_{ir}^{-\epsilon}}{8\pi}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\mathrm{d}s_{ir}\,\mathrm{d}v\,\delta\big(s_{ir}-Q^2\alpha\big(\alpha+(1-\alpha)x\big)\big)$$

$$\times\;[v\,(1-v)]^{-\epsilon}\,\Theta(1-v)\Theta(v)$$

Collinear integrals

convolution of the integral of AP-splitting function over ordinary phase space

$$\int_0^{\alpha_0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\int \mathrm{d}\phi_2(p_i,p_r;p_{(ir)})\frac{1}{s_{ir}^{1+\kappa\epsilon}}P^{(\kappa)}_{f_if_r}(z_i,z_r;\epsilon)\,, \qquad \kappa = 0,\,1$$

$$\frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}}\,g_I^{(\pm)}(z_r)\,, \qquad z_r = \frac{\alpha Q^2 + (1-\alpha)v s_{\widetilde{ir}Q}}{2\alpha Q^2 + (1-\alpha)s_{\widetilde{ir}Q}}$$

δ	Function	$g_I^{(\pm)}(z)$
0	g_A	1
∓ 1	$g_B^{(\pm)}$	$(1-z)^{\pm\epsilon}$
0	$g_C^{(\pm)}$	$(1-z)^{\pm\epsilon}\,{}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,z)$
± 1	$g_D^{(\pm)}$	${}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,1-z)$

Soft integrals

convolution of the integral of the eikonal factors over ordinary phase space

$$\mathcal{J} \propto -\int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1}\frac{Q^2}{2\pi}\int \mathrm{d}\phi_2(p_r,K;Q)\left(\frac{s_{ik}}{s_{ir}s_{kr}}\right)^{1+\kappa\epsilon}$$

$$\mathrm{d}\phi_2(p_r,K;Q) = \frac{(Q^2)^{-\epsilon}}{16\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathrm{d}\varepsilon_r\,\varepsilon_r^{1-2\epsilon}\delta(y-\varepsilon_r)$$
$$\times\,\mathrm{d}(\cos\vartheta)\,\mathrm{d}(\cos\varphi)(\sin\vartheta)^{-2\epsilon}(\sin\varphi)^{-1-2\epsilon}$$

Basic forms of integrals

Integration of the counterterms over the unresolved phase space is difficult

collinear-type:

$$\mathcal{I} \propto x \int_0^{\alpha_0} \mathrm{d}\alpha\, \alpha^{-1-(1+\kappa)\epsilon} (1-\alpha)^{2d_0-1} [\alpha + (1-\alpha)x]^{-1-(1+\kappa)\epsilon}$$

$$\times \int_0^1 \mathrm{d}v [v\,(1-v)]^{-\epsilon} \left(\frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x}\right)^{k+\delta\epsilon} g\left(\frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x}\right)$$

soft-type:

$$\mathcal{J} \propto -\int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left(\frac{s_{ik}}{s_{ir}s_{kr}}\right)^{1+\kappa\epsilon}$$

$$\mathcal{K} \propto \int_0^{y_0} \mathrm{d}y\,(1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) 2 \left(\frac{1}{s_{ir}} \frac{z_i}{z_r}\right)^{1+\kappa\epsilon}$$

Integrated counterterms

two types of iterated singly-unresolved

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right) \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} \boxed{+ \int_2} \left(\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} \boxed{- \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \Big)} + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \Big(\boxed{\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \Big)^{\mathrm{A}_1}} \right] \right\} J_m$$

Integrating iterated counterterms

One of 25 subtraction terms: collinear-double collinear subtraction

$$\mathcal{C}_{kt}\mathcal{C}^{(0)}_{ir;kt} = (8\pi\alpha_s\mu^{2\epsilon})^2 \frac{1}{s_{kt}}\frac{1}{\widehat{s}_{ir}} \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\})|P^{(0)}_{f_k f_t}(z_{t,k};\epsilon)P^{(0)}_{f_i f_r}(\widehat{z}_{r,i};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\})\rangle$$
$$\times\, (1-\alpha_{kt})^{2d_0-2m(1-\epsilon)}(1-\widehat{\alpha}_{kt})^{2d_0-2m(1-\epsilon)}\Theta(\alpha_0-\alpha_{kt})\Theta(\alpha_0-\widehat{\alpha}_{ir})$$

obtained by an iterated mapping

$$\{p\}_{m+2} \xrightarrow{\mathsf{C}_{kt}} \{\hat{p}\}_{m+1} \xrightarrow{\mathsf{C}_{\hat{i}\hat{r}}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$

Then we define the function $\mathrm{C}_{kt}\mathrm{C}^{(0)}_{ir;kt}(\widetilde{x}_{kt},\widetilde{x}_{ir},\epsilon,\alpha_0,d_0)$ by

$$\int [\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]\mathcal{C}_{kt}\mathcal{C}^{(0)}_{ir;kt} \equiv \left[\frac{\alpha_s}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \mathrm{C}_{kt}\mathrm{C}^{(0)}_{ir;kt}\boldsymbol{T}^2_{kt}\boldsymbol{T}^2_{ir}|\mathcal{M}^{(0)}_m(\{\tilde{p}\})|^2$$

Integrating iterated counterterms

Use explicit parametrization of $\left[\mathrm{d}\widehat{p}_{1,m}\right]$ and $\left[\mathrm{d}p_{1,m+1}\right]$ to write

$\mathrm{C}_{kt}\mathrm{C}^{(0)}_{ir;kt}(\widetilde{x}_{kt}, \widetilde{x}_{ir}, \epsilon, \alpha_0, d_0)$ as a linear combination of basic integrals

$$
\begin{aligned}
\mathcal{I}^{(4)}_{\mathcal{C}}(x_k, x_i; \epsilon, \alpha_0, d_0, k, l) &= x_k x_i \\
&\times \int_0^{\,{}^{0}} \mathrm{d}\beta \, (1-\beta)^{2d_0-2+2} \; \beta^{-1-} \; [\beta + (1-\beta)x_i]^{-1-} \\
&\times \int_0^{\,{}^{0}} \mathrm{d}\alpha \, (1-\alpha)^{2d_0-1} \alpha^{-1-} \; [\alpha + (1-\alpha)(1-\beta)x_k]^{-1-} \\
&\times \int_0^1 \mathrm{d}u \, u^{-} \; (1-u)^{-} \; \left(\frac{\beta + (1-\beta)x_i u}{2\beta + (1-\beta)x_i}\right)^l \\
&\times \int_0^1 \mathrm{d}v \, v^{-} \; (1-v)^{-} \; \left(\frac{\alpha + (1-\alpha)(1-\beta)x_k v}{2\alpha + (1-\alpha)(1-\beta)x_k}\right)^k , \qquad k, l = -1, 0, 1, 2
\end{aligned}
$$

Integrating iterated counterterms

Use explicit parametrization of $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$ to write

$\mathrm{C}_{kt}\mathrm{C}^{(0)}_{ir;kt}(\widetilde{x}_{kt}, \widetilde{x}_{ir}, \epsilon, \alpha_0, d_0)$ as a linear combination of basic integrals

$$\mathcal{I}^{(4)}_{\mathcal{C}}(x_k, x_i; \epsilon, \alpha_0, d_0, k, l) = x_k x_i$$

$$\times \int_0^{0} \mathrm{d}\beta\, (1-\beta)^{2d_0-2+2}$$

$s_{ir}(\beta, x_i)^{-1-\epsilon}$

$$\times \int_0^{0} \mathrm{d}\alpha\, (1-\alpha)^{2d_0-1} \alpha^{-1-} \, [\alpha + (1-\alpha)(1-\beta)x_k]^{-1-}$$

$$\times \int_0^1 \mathrm{d}u\, u^{-} \, (1-u)^{-} \, \left(\frac{\beta + (1-\beta)x_i u}{2\beta + (1-\beta)x_i}\right)^l$$

$$\times \int_0^1 \mathrm{d}v\, v^{-} \, (1-v)^{-} \, \left(\frac{\alpha + (1-\alpha)(1-\beta)x_k v}{2\alpha + (1-\alpha)(1-\beta)x_k}\right)^k , \qquad k, l = -1, 0, 1, 2$$

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&\times \int_0^{0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1} \; s_{kt}(\alpha, \beta, x_k)^{-1-\epsilon} \\
&\times \int_0^1 \mathrm{d}u\, u^{-}\;(1-u)^{-}\;\left(\frac{\beta+(1-\beta)x_i u}{2\beta+(1-\beta)x_i}\right)^l \\
&\times \int_0^1 \mathrm{d}v\, v^{-}\;(1-v)^{-}\;\left(\frac{\alpha+(1-\alpha)(1-\beta)x_k v}{2\alpha+(1-\alpha)(1-\beta)x_k}\right)^k, \qquad k, l = -1, 0, 1, 2
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Integrating iterated counterterms

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$$
\begin{aligned}
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&\times \int_0^{0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1}\; s_{kt}(\alpha, \beta, x_k)^{-1-\epsilon} \\
&\times \int_0^1 \mathrm{d}u\, u^{-}\,(1-u)^{-}\,\left(Z_{r;i}(\beta, x_i, u)\right)^{l} \\
&\times \int_0^1 \mathrm{d}v\, v^{-}\,(1-v)^{-}\,\left(\frac{\alpha+(1-\alpha)(1-\beta)x_k v}{2\alpha+(1-\alpha)(1-\beta)x_k}\right)^{k}, \qquad k, l = -1, 0, 1, 2
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$$

Integrating iterated counterterms

Use explicit parametrization of $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$ to write

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$$\mathcal{I}^{(4)}_{\mathcal{C}}(x_k, x_i; \epsilon, \alpha_0, d_0, k, l) = x_k x_i$$

$$\times \int_0^{0} \mathrm{d}\beta\,(1-\beta)^{2d_0-2+2} \, s_{ir}(\beta, x_i)^{-1-\epsilon}$$

$$\times \int_0^{0} \mathrm{d}\alpha\,(1-\alpha)^{2d_0-1} \, s_{kt}(\alpha, \beta, x_k)^{-1-\epsilon}$$

$$\times \int_0^1 \mathrm{d}u\, u^{-}\,(1-u)^{-} \left(Z_{r;i}(\beta, x_i, u)\right)^{l}$$

$$\times \int_0^1 \mathrm{d}v\, v^{-}\,(1-v)^{-} \left(Z_{k;t}(\alpha, \beta, x_i, v)\right)^{k}, \qquad k, l = -1, 0, 1, 2$$

Three methods

to compute the integrals:

- IBP's to reduce to master integrals + solution of MI's by differential equations
- MB representations to extract poles structure + summation of nested series
- SD

Three methods

Method	Analytical	Numerical
IBP	✓ Singly-unresolved integrals - Bottleneck is the proliferation of denominators	✓ Evaluating analytical expressions - No numbers without full analytical results
MB	✓ Iterated singly unresolved integrals - Bottleneck is the evaluation of sums	✓ Direct numerical evalution of MB integrals possible ✓ Fast and accurate
SD	✓ Easy to automate - Only in principle, except for leading pole	✓ Straightforward - In general slower & less accurate than MB

Analytical vs. numerical

Analytical vs. numerical

Matter of principle:

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- Analytical results show that the integrated counterterms are smooth functions of the kinematic variables

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Hence:

Analytical vs. numerical

Matter of principle:

- Cancellation of poles requires the coefficients of poles in integrated counterterms in analytical form
- Analytical forms are fast and accurate compared to numerical ones

However:

- Analytical results show that the integrated counterterms are smooth functions of the kinematic variables

Hence:

- Finite terms of integrated counterterms can be given in form of interpolating tables or approximating functions. Thus numerical form – computed once with required precision – is sufficient.

Results

Integrated counterterms

singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon) = \frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}^{(0)}_{1,i}(y_{iQ};\epsilon)\,\boldsymbol{T}_i^2 + \sum_{k\neq i} \mathrm{S}^{(0)ik}_1(Y_{ik,Q};\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\right]$$

$$y_{iQ} \equiv x_i = \frac{2p_i\cdot Q}{Q^2} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ}y_{kQ}}$$

Ensures common collinear limit for S_1^{il} and S_1^{rl} if $p_i || p_r$

(essential for iteration & colour coherence: $T_i \cdot T_l + T_r \cdot T_l = T_{(ir)} \cdot T_l$)

Integrated counterterms

singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

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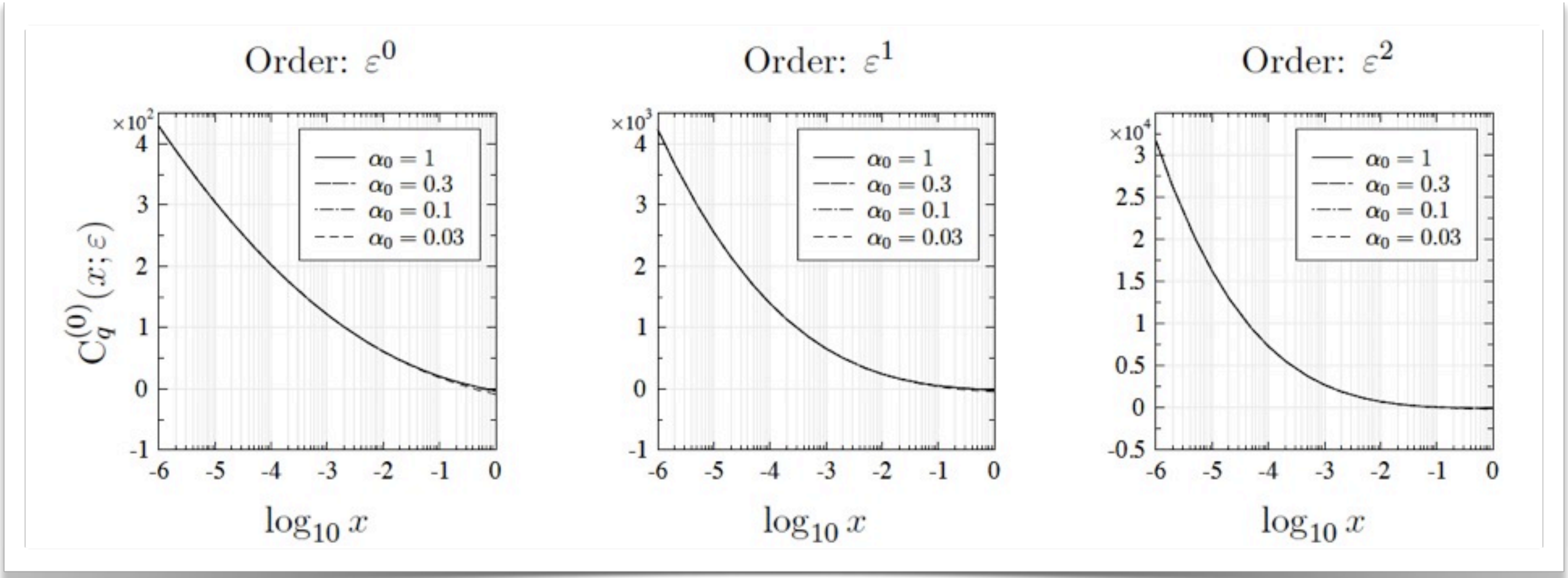

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singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

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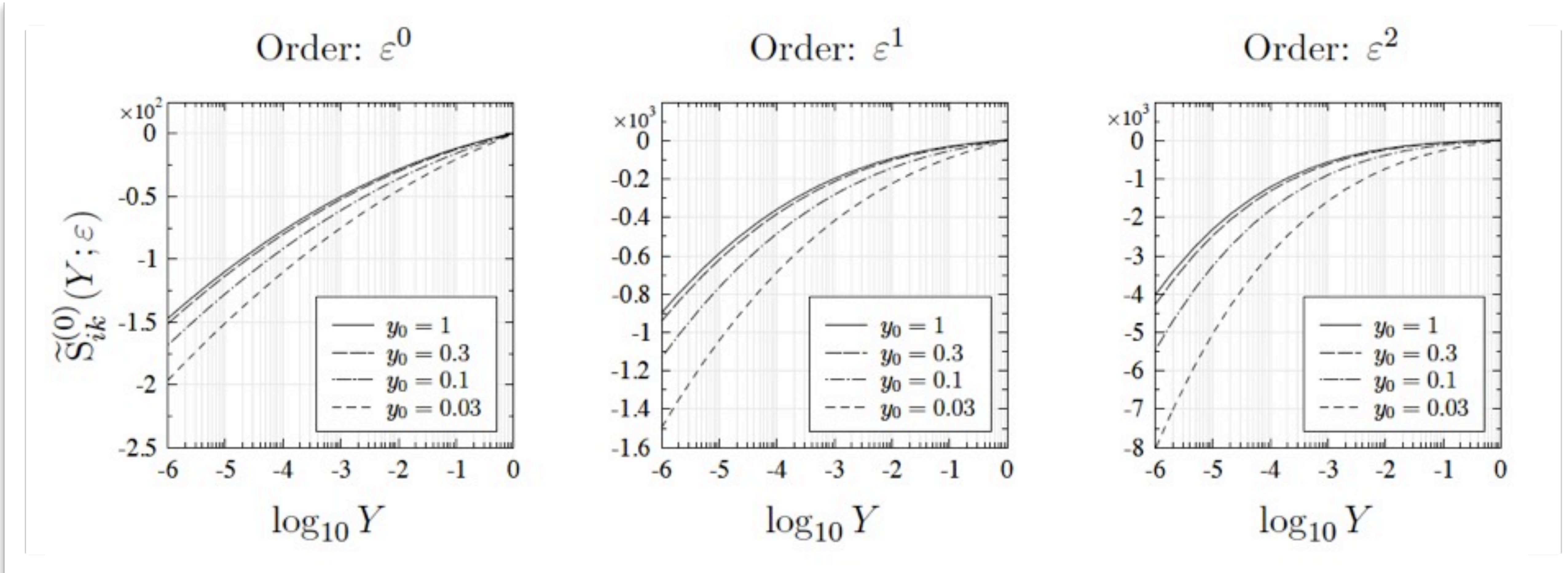


Integrated counterterms

singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

$$\boldsymbol{I}^{(0)}_1(\{p\}_{m+1};\epsilon) = \frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}^{(0)}_{1,i}(y_{iQ};\epsilon)\, \boldsymbol{T}^2_i + \sum_{k\neq i} \mathrm{S}^{(0)ik}_1(Y_{ik,Q};\epsilon)\, \boldsymbol{T}_i \cdot \boldsymbol{T}_k\right]$$

$$\int_1 \mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} = \mathrm{d}\sigma^{\mathrm{V}}_{m} \otimes \boldsymbol{I}^{(0)}_1(\{p\}_{m};\epsilon) + \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes \boldsymbol{I}^{(1)}_1(\{p\}_{m};\epsilon)$$

$$\boldsymbol{I}^{(1)}_1(\{p\}_{m};\epsilon) \propto \sum_i \Big[\mathrm{C}^{(1)}_{1,i}(y_{iQ};\epsilon)\, \boldsymbol{T}^2_i + \sum_{k\neq i} \mathrm{S}^{(1)ik}_1(Y_{ik,Q};\epsilon)\, \boldsymbol{T}_i \cdot \boldsymbol{T}_k$$
$$+ \sum_{k\neq i} \sum_{l\neq i,k} \mathrm{S}^{(1)ikl}_1(Y_{ik,Q}, Y_{il,Q}, Y_{kl,Q};\epsilon) \sum_{a,b,c} f_{abc} T^a_i T^b_k T^c_l \Big]$$

Regularized RR and RV contributions

can now be computed by numerical

Monte Carlo integrations

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right) \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

Example: 3-jet event shapes

✓ Constructed $d\sigma_5$ and $d\sigma_4$ for e+e- → 3 jets (regularized RR and RV)

✓ Checked numerically that (for J = C or 1 - T)

- in all singly- and doubly-unresolved limits

$$\frac{\mathrm{d}\sigma_5^{\mathrm{RR},\mathrm{A}_2} J_3 + \mathrm{d}\sigma_5^{\mathrm{RR},\mathrm{A}_1} J_4 - \mathrm{d}\sigma_5^{\mathrm{RR},\mathrm{A}_{12}} J_3}{\mathrm{d}\sigma_5^{\mathrm{RR}}} \to 1$$

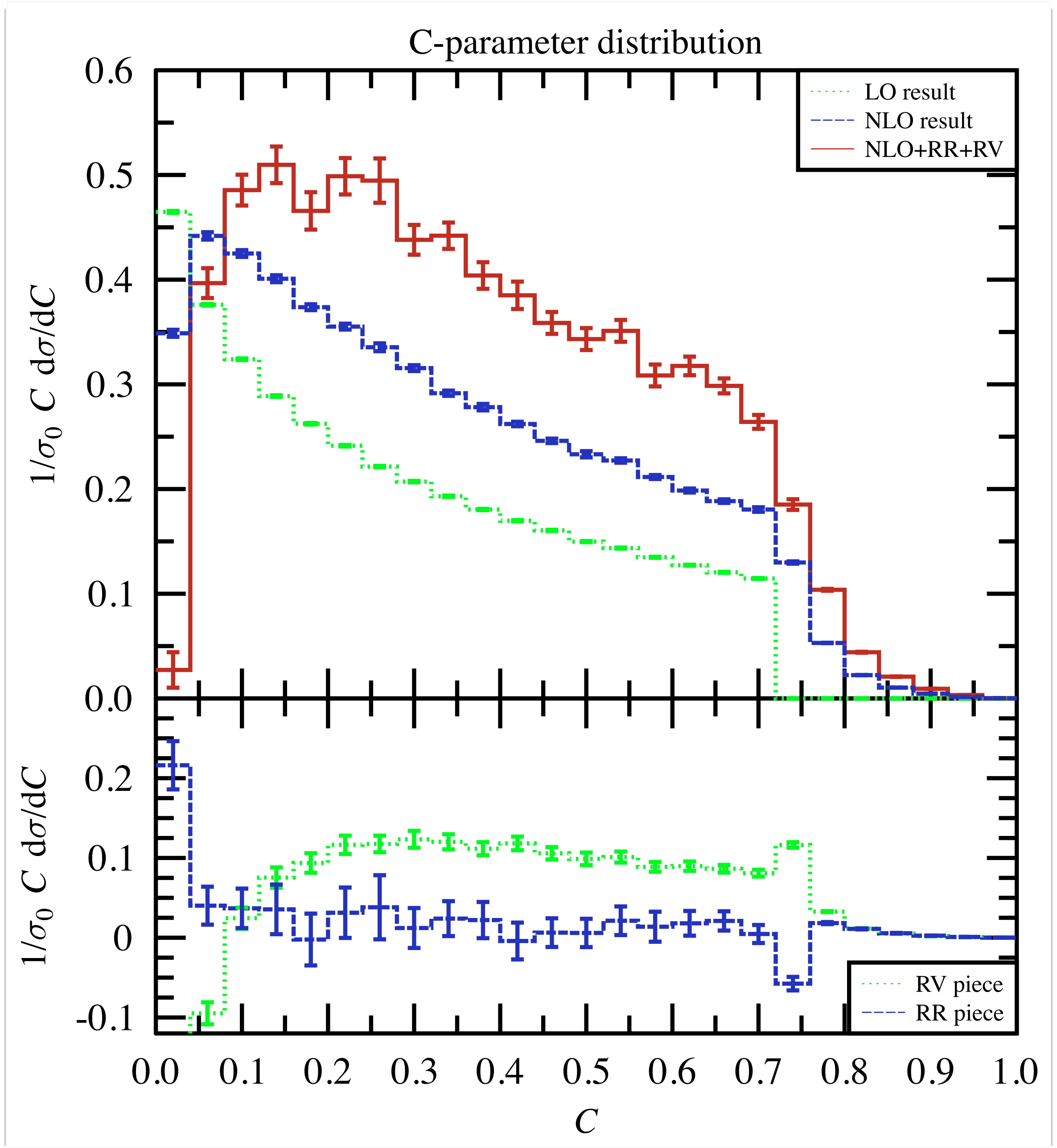
- in all singly unresolved limits

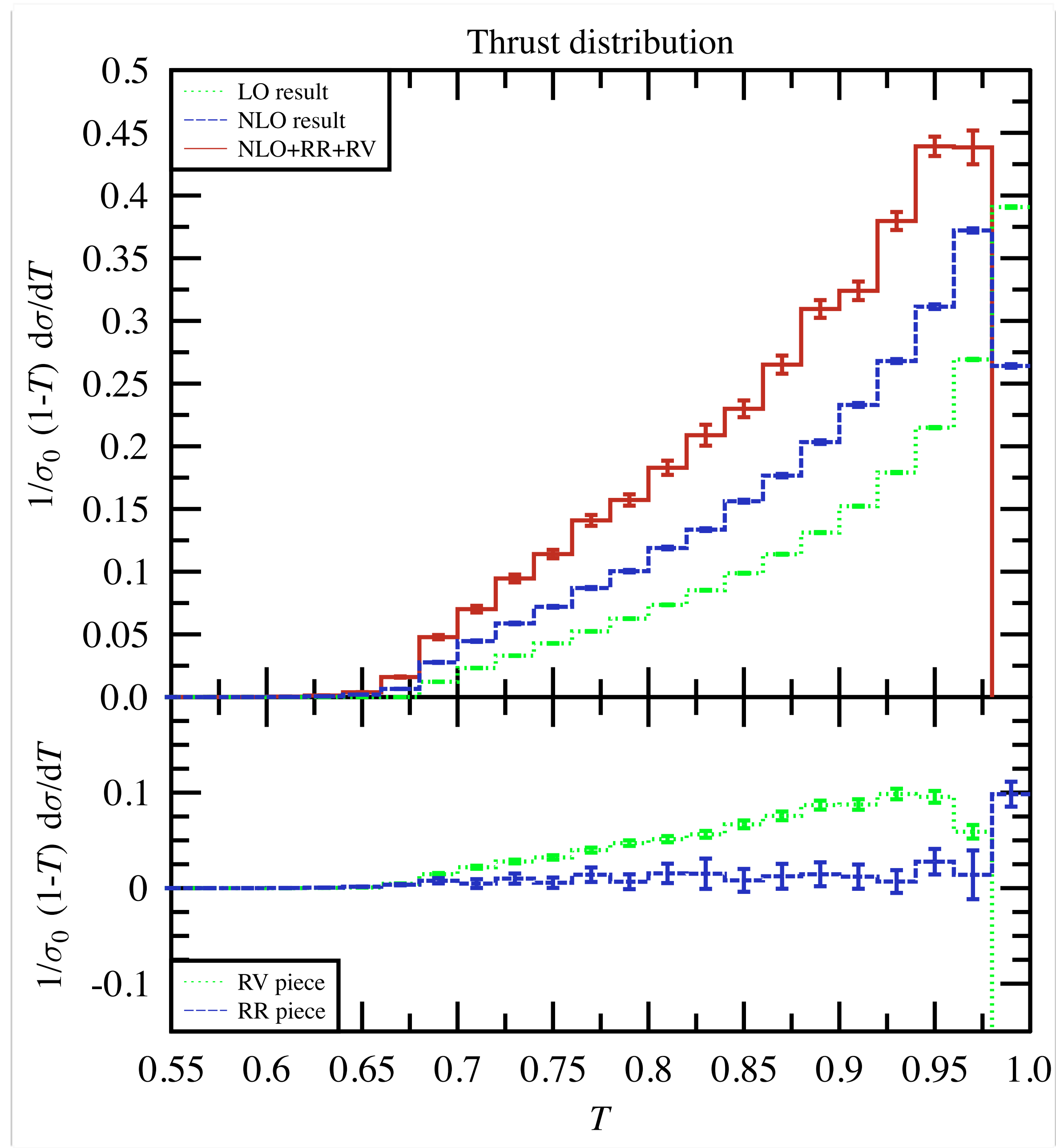
$$\frac{\mathrm{d}\sigma_4^{\mathrm{RV},\mathrm{A}_1} J_3 - \int_1 \mathrm{d}\sigma_5^{\mathrm{RR},\mathrm{A}_1} J_4 - \left(\int_1 \mathrm{d}\sigma_5^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1} J_3}{\mathrm{d}\sigma_4^{\mathrm{RV}}} \to 1$$

➡ the counterterms are fully local

Regularized RR and RV contributions

can now be computed by numerical Monte Carlo integrations





Rest of integrated counterterms

after summing over unresolved flavours

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} \right)^{\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2} \left\{ \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon), \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon) \right\} + \boldsymbol{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \right]$$

Rest of integrated counterterms

after summing over unresolved flavours

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2} \left\{ \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon), \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon) \right\} + \boldsymbol{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \right]$$

✓

Rest of integrated counterterms

after summing over unresolved flavours

$$\int_1 \Big(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \Big)^{\mathrm{A_1}} = \mathrm{d}\sigma^{\mathrm{B}}_m \otimes \Big[\frac{1}{2} \Big\{ \boldsymbol{I}^{(0)}_1(\{p\}_m;\epsilon), \boldsymbol{I}^{(0)}_1(\{p\}_m;\epsilon) \Big\} + \boldsymbol{I}^{R\times(0)}_1(\{p\}_m;\epsilon) \Big]$$

✓ ✓

$$\boldsymbol{I}^{R\times(0)}_1(\{p\}_{m+1};\epsilon) \propto \sum_i \Big[\mathrm{C}^{R\times(0)}_i(y_{iQ};\epsilon)\, \boldsymbol{T}^2_i + \sum_{k\neq i} \mathrm{S}^{R\times(0),ik}(Y_{ik,Q};\epsilon)\, \boldsymbol{T}_i\cdot\boldsymbol{T}_k \Big]$$

Rest of integrated counterterms

after summing over unresolved flavours

$$\int_1 \Big(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} \Big)^{\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \Big[\frac{1}{2} \Big\{ \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon), \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon) \Big\} + \boldsymbol{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \Big]$$

✓ ✓

$$\boldsymbol{I}_1^{R\times(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \Big[\mathrm{C}_i^{R\times(0)}(y_{iQ};\epsilon)\, \boldsymbol{T}_i^2 + \sum_{k\neq i} \mathrm{S}^{R\times(0),ik}(Y_{ik,Q};\epsilon)\, \boldsymbol{T}_i \cdot \boldsymbol{T}_k \Big]$$

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

✓

Rest of integrated counterterms

after summing over unresolved flavours

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} \right)^{\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2} \left\{ \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon), \boldsymbol{I}_1^{(0)}(\{p\}_m;\epsilon) \right\} + \boldsymbol{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \right]$$

✓ ✓

$$\boldsymbol{I}_1^{R\times(0)}(\{p\}_{m+1};\epsilon) \propto \sum_i \left[\mathrm{C}_i^{R\times(0)}(y_{iQ};\epsilon)\, \boldsymbol{T}_i^2 + \sum_{k\neq i} \mathrm{S}^{R\times(0),ik}(Y_{ik,Q};\epsilon)\, \boldsymbol{T}_i \cdot \boldsymbol{T}_k \right]$$

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

✓

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}_2^{(0)}(\{\ \ \}_m;\epsilon)$$

?

Rest of integrated counterterms

I_{12} and I_2 have the same colour and flavour decomposition

$$\boldsymbol{I}_{12}^{(0)}(\{p\}_m;\epsilon) \propto \Bigg\{ \sum_i \left[\mathrm{C}_{12,f_i}^{(0)} \boldsymbol{T}_i^2 + \sum_k \mathrm{C}_{12,f_i f_k}^{(0)} \boldsymbol{T}_k^2 \right] \boldsymbol{T}_i^2$$
$$+ \sum_{j,l} \left[\mathrm{S}_{12}^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{12,f_i}^{(0),(j,l)} \boldsymbol{T}_i^2 \right] \boldsymbol{T}_j \boldsymbol{T}_l$$
$$+ \sum_{i,k,j,l} \mathrm{S}_{12}^{(0),(i,k)(j,l)} \{ \boldsymbol{T}_i \boldsymbol{T}_k, \boldsymbol{T}_j \boldsymbol{T}_l \} \Bigg\}$$

The coefficients depend on ϵ (poles starting at $O(\epsilon^{-4})$), kinematics and PS cut parameters

Insertion operator I_{12}

Illustration: $e^+e^- \to 2$ jets

Born squared matrix element: $|\mathcal{M}_2^{(0)}(1_q, 2_{\bar{q}})|^2$

Colour and kinematics are trivial:

$$\boldsymbol{T}_1^2 = \boldsymbol{T}_2^2 = -\boldsymbol{T}_1\boldsymbol{T}_2 = C_\mathrm{F}\,, \qquad y_{12} = \frac{2p_1 \quad p_2}{Q^2} = 1$$

Insertion operator from iterated subtraction:

$$\boldsymbol{I}_{12}^{(0)}(p_1, p_2; \epsilon) =$$

$$= \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \left\{\frac{2C_\mathrm{F}(3C_\mathrm{F} - C_\mathrm{A})}{\epsilon^4} + \frac{C_\mathrm{F}}{6}\left[20C_\mathrm{A} + 81C_\mathrm{F} - 4T_\mathrm{R}n_\mathrm{f}\right.\right.$$

$$\left.\left. + 12(3C_\mathrm{A} - 2C_\mathrm{F})\Sigma(y_0, D_0') + 12(2C_\mathrm{A} - C_\mathrm{F})\Sigma(y_0, D_0' - 1)\right]\frac{1}{\epsilon^3} + \mathrm{O}(\varepsilon^{-2})\right\}$$

Higher order expansion coefficients are cumbersome

Insertion operator I_{12}

Illustration: $e^+e^- \to 3$ jets

Born squared matrix element: $|\mathcal{M}_3^{(0)}(1_q, 2_{\bar{q}}, 3_g)|^2$

Colour is still trivial:

$$\boldsymbol{T}_1^2 = \boldsymbol{T}_2^2 = C_{\mathrm{F}}\,, \quad \boldsymbol{T}_3^2 = C_{\mathrm{A}}\,, \quad \boldsymbol{T}_1\boldsymbol{T}_2 = \frac{C_{\mathrm{A}} - 2C_{\mathrm{F}}}{2}\,, \quad \boldsymbol{T}_1\boldsymbol{T}_3 = \boldsymbol{T}_2\boldsymbol{T}_3 = -\frac{C_{\mathrm{A}}}{2}$$

Insertion operator from iterated subtraction:

$$\begin{aligned}
&\boldsymbol{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \\
&= \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \left\{\frac{C_{\mathrm{A}}^2 + 2C_{\mathrm{A}}C_{\mathrm{F}} + 6C_{\mathrm{F}}^2}{\epsilon^4} + \left[\frac{11C_{\mathrm{A}}^2}{2} + \frac{50C_{\mathrm{A}}C_{\mathrm{F}}}{3} + 12C_{\mathrm{F}}^2\right.\right. \\
&- \frac{C_{\mathrm{A}}T_{\mathrm{R}}n_{\mathrm{f}}}{3} - \frac{C_{\mathrm{A}}^2 T_{\mathrm{R}}n_{\mathrm{f}}}{C_{\mathrm{F}}} - 4C_{\mathrm{F}}T_{\mathrm{R}}n_{\mathrm{f}} + \left(\frac{5C_{\mathrm{A}}^2}{2} - C_{\mathrm{A}}C_{\mathrm{F}} - 8C_{\mathrm{F}}^2\right)\ln y_{12} \\
&- \frac{C_{\mathrm{A}}(5C_{\mathrm{A}} + 8C_{\mathrm{F}})}{2}(\ln y_{13} + \ln y_{23}) + (C_{\mathrm{A}}^2 + 6C_{\mathrm{A}}2C_{\mathrm{F}} - 4C_{\mathrm{F}}^2)\Sigma(y_0, D_0') \\
&\left.\left. + 4C_{\mathrm{F}}(C_{\mathrm{A}} - C_{\mathrm{F}})\Sigma(y_0, D_0' - 1)\right]\frac{1}{\epsilon^3} + \mathrm{O}(\varepsilon^{-2})\right\}
\end{aligned}$$

Higher order expansion coefficients are cumbersome

Present status

Integration of the doubly-unresolved counterterms in progress (most difficult)

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right) \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

Conclusions

Conclusions

- ✓ We have set up a general subtraction scheme for computing NNLO jet cross sections, for processes with no coloured particles in the initial state
- ✓ We have investigated various methods to integrate the counterterms
- ✓ We used the MB method to perform the integration of all but doubly-unresolved counterterms. The SD method was used to provide independent checks
- ✴ The integration of the doubly-unresolved counterterm is feasible with our methods, and is work in progress