Calorons with non-trivial holonomy

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in: Future directions in lattice gauge theory

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SU(2)

Harrington & Shepard, Phys. Rev. **D17** (1978) 2122.

Periodic (in time) array of (charge 1) instantons: Using 't Hooft ansatz

$$A_\mu(x) = \frac{i}{2}\,\bar{\eta}^a_{\mu\nu}\,\tau_a\,\partial_\nu \ln \phi(x) \qquad \frac{\Box\phi}{\phi}=0$$

$$\phi(x) = 1 + \sum_{n\in\mathbb{Z}} \frac{\rho^2}{(\vec{x}-\vec{a})^2 + (t-a_0-n\beta)^2}$$

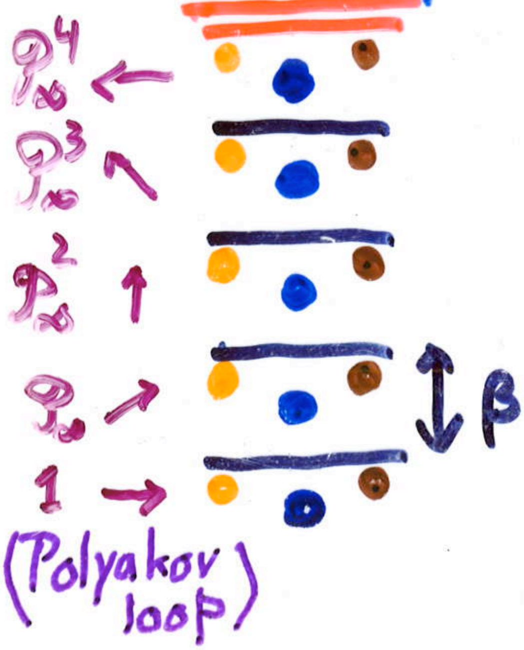
$$= 1 + \frac{\pi\rho^2}{r}\,\frac{\sinh(2\pi r)}{\cosh(2\pi r)-\cos(2\pi t)} \qquad r=|\vec{x}|$$

$(\beta=1)$
$(a_\mu=0)$

For large ρ this approaches a BPS monopole (in singular gauge)

P. Rossi, Nucl. Phys. B149 ('79) 170

Overlap: For large ρ scale is set by β (= distance between periodic copies)



What happens when periodic copies are relative gauge rotated?

$\beta = \rho = 1$

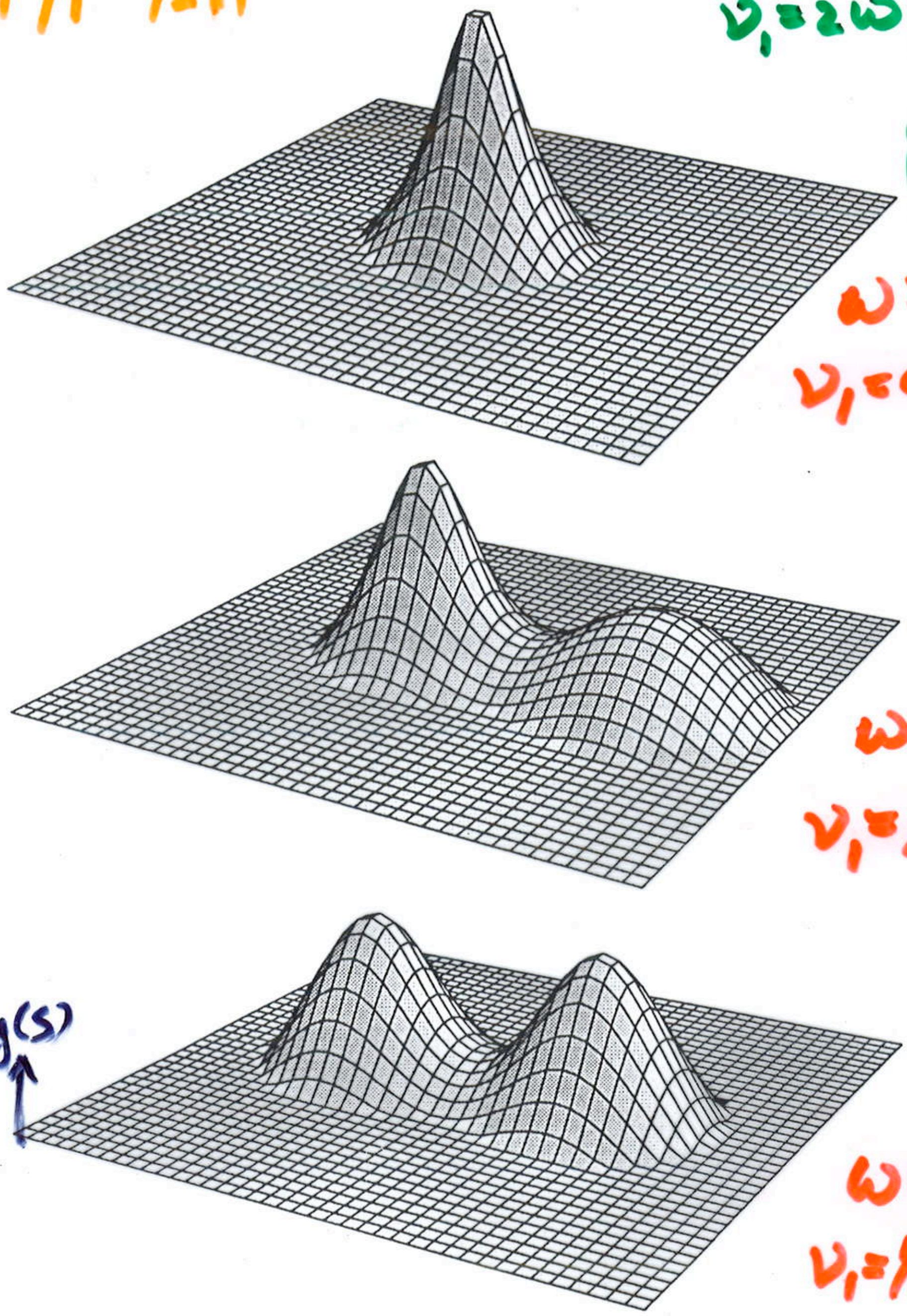
$\pi \rho^2 = |\vec{y}_1 - \vec{y}_2| \beta$

$P_\infty = \exp(2\pi i \vec{\omega} \cdot \vec{\tau})$

$\omega \equiv |\vec{\omega}|$

$\nu_1 = 2\omega, \ \nu_2 = 1 - 2\omega = 2\bar{\omega}$

($\nu_i \hat{=}$ action fraction)



$\omega = 0$

$\nu_1 = 0, \ \nu_2 = 1$

SU(2)

$\omega = 1/8$

$\nu_1 = 1/4, \ \nu_2 = 3/4$

lg(s)

$\omega = 1/4$

$\nu_1 = 1/2, \ \nu_2 = 1/2$

Figure 1: Profiles for calorons at $\omega = 0$, 0.125, 0.25 (from top to bottom) with $\rho = 1$. The axis connecting the lumps, separated by a distance π (for $\omega \neq 0$), corresponds to the direction of $\hat{\omega}$. The other direction indicates the distance to this axis, making use of the axial symmetry of the solutions. Vertically is plotted the action density, at the time of its maximal value, on equal logarithmic scales for the three profiles. The profiles were cut off at an action density below $1/e$. The mass ratio of the two lumps is approximately $\omega/\bar{\omega}$, i.e. zero (no second lump), a third and one (equal masses), for the respective values of ω.

$P_\infty = \exp(2\pi i \omega \tau_3)$

$\omega = 1/8$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{3}{4}$

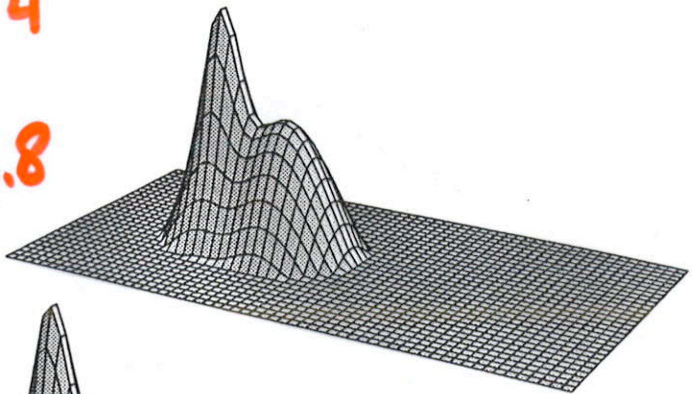
$\beta = 1$

(equivalently: $\beta \leftrightarrow \frac{1}{\pi\rho^2}$)

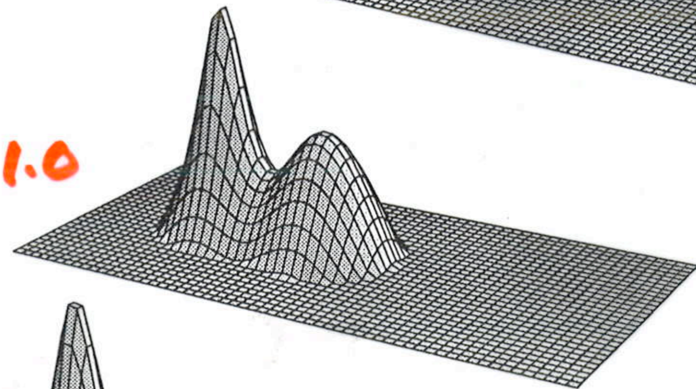
SU(2)

Calorons: when separ. becomes monopoles

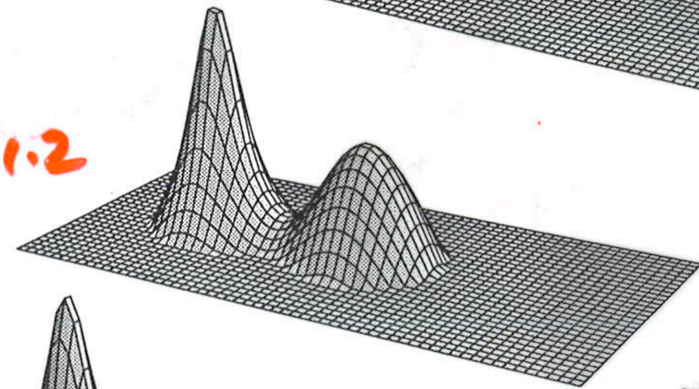
$\rho = 0.8$



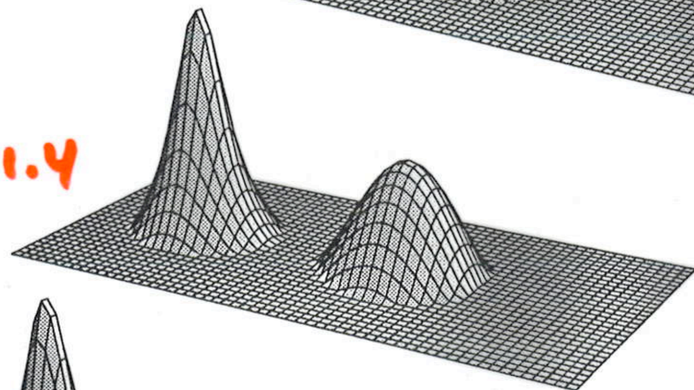
$\rho = 1.0$

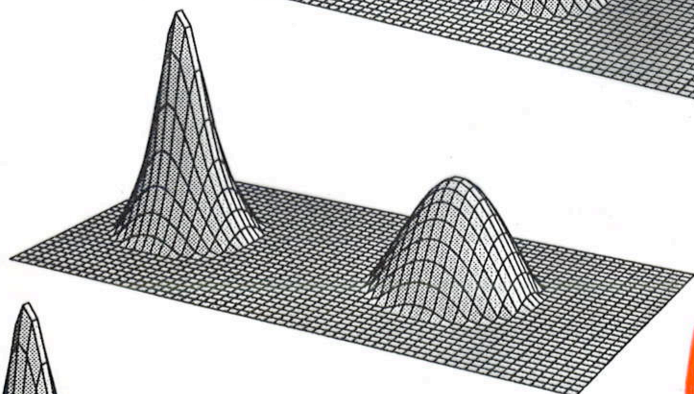


$\rho = 1.2$

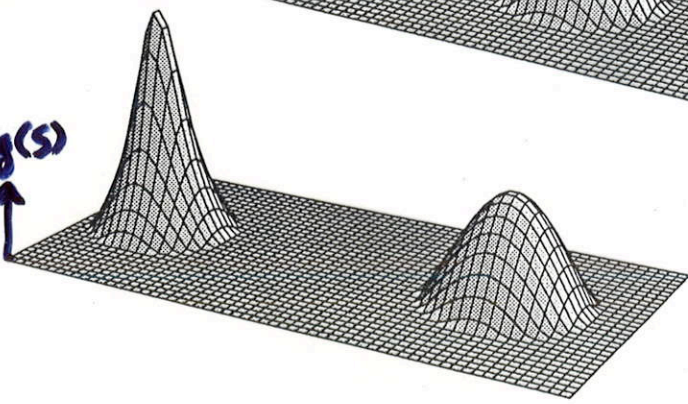


$\rho = 1.4$





$\rho = 1.6$



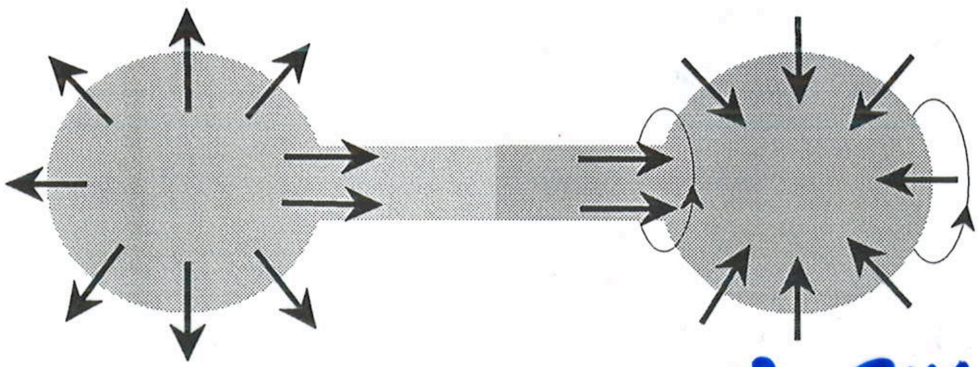
$\rho = 1.8$

$\omega = 1/8$; cutoff $1/e^2$

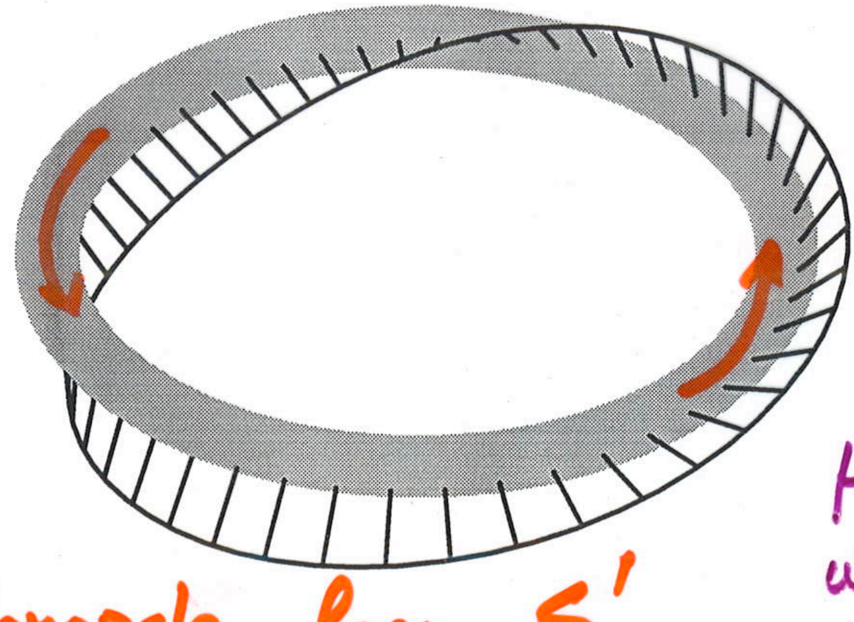
$\rho = 0.8, 1, 1.2, 1.4, 1.6, 1.8$

C. Taubes ('82)

See: Cargèse '83 – Plenum '84, p. 563



$S^2 = SU(2)/U(1)$



Hopf map
winding # 1

Monopole loop S^1

"Frame" $SU(2)/U(1) = S^2$

$\Longrightarrow S^1 \times_{\text{twisted}} S^2 = S^3$

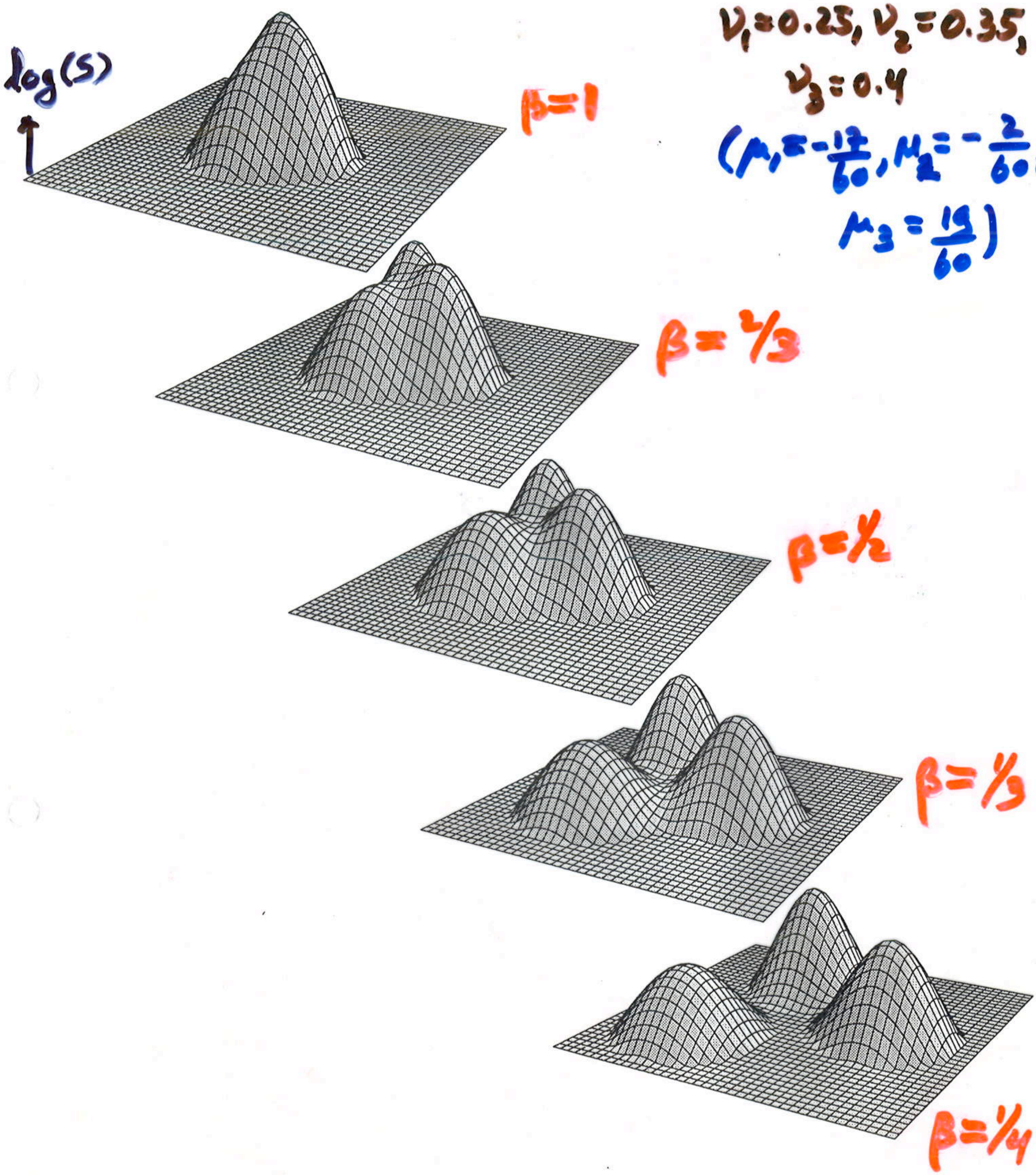
Made more precise by O. Jahn*

*] O. Jahn, J. Phys. A33 (2000) 2997

SU(3)

$\nu_1 = 0.25,\ \nu_2 = 0.35,\ \nu_3 = 0.4$

$\left(\mu_1 = -\frac{17}{60},\ \mu_2 = -\frac{2}{60},\ \mu_3 = \frac{19}{60}\right)$



SU(3)

T = 1, 1.5, 2, 3, 4 | cutoff 1/e

SU(n)

$\beta = 1$, top. ch. $= 1$

$$\mathrm{tr}\, F_{\mu\nu}^2(x) = \partial_\mu^2 \partial_\nu^2 \log \psi(x)$$

$$\psi(x) = \frac{1}{2}\mathrm{tr}\left(A_n A_{n-1} \cdots A_1\right) - \cos(2\pi x_0)$$

$$A_m = \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_m & s_m \\ s_m & c_m \end{pmatrix} \frac{1}{r_m}$$

$$r_m = |\vec{x} - \vec{y}_m|$$

$$c_m = \cosh(2\pi \boxed{\nu_m} r_m)$$

$$s_m = \sinh(2\pi \boxed{\nu_m} r_m)$$

$$\boxed{\begin{array}{c} SU(2): \\ |\vec{y}_1 - \vec{y}_2| = \pi\rho^2 \\ \hline \mu_1 = -\omega,\ \mu_2 = \omega \end{array}}$$

$$P(\vec{x}) = P\exp \int_0^\beta dx_0\, A_0(\vec{x}; x_0)$$

$$\xrightarrow[|\vec{x}| \to \infty]{} P_\infty = \exp[2\pi i\, \mathrm{diag}(\mu_1 \cdots \mu_n)]$$

$$\sum \mu_i = 0 \qquad \mu_1 \le \mu_2 \cdots \le \mu_n \le \mu_{n+1} \equiv \mu_1 + 1$$

$$\nu_m \equiv \mu_{m+1} - \mu_m \longrightarrow \text{Mass} = 8\pi^2 \nu_m / \beta$$

$$\boxed{\sum_m \nu_m = 1 \iff S = 8\pi^2}$$

Fermion zero-mode $\quad SU(n)$

$$\not{D}\Psi = \bar{\sigma}_\mu(\partial_\mu + A_\mu)\Psi \; (= 0)$$

$$\bar{\sigma}_\mu = (1_2 ; -i\vec{\tau}) \qquad \Psi(t+\beta, \vec{x}) = e^{-2\pi i z}\Psi(t;\vec{x}) \qquad \boxed{\beta = 1}$$

$z = \frac{1}{2}$ anti-periodic | $z = 0$ periodic

$$|\Psi(x)|^2 = -\frac{1}{(2\pi)^2}\partial_\mu^2 \hat{f}_x(z,z)$$

for $z \in [\mu_m, \mu_{m+1}]$

$$\hat{f}_x(z,z) = \frac{\pi \langle v_m(z) | A_{m-1} \cdots A_1 A_n \cdots A_m | w_m(z) \rangle}{r_m \psi(x)}$$

$$v_m^1(z) = -w_m^2(z) = \sinh(2\pi(z-\mu_m)r_m)$$

$$v_m^2(z) = w_m^1(z) = \cosh(2\pi(z-\mu_m)r_m)$$

When $|\vec{y}_m - \vec{y}_{m+1}| \to \infty$ for all m

$$\hat{f}_x(z,z) \longrightarrow \frac{\sinh(2\pi(z-\mu_m)r_m)\sinh(2\pi(z-\mu_{m+1})r_m)}{(2\pi)^{-1} r_m \sinh(2\pi \nu_m r_m)}$$

$\nu_m \to \mu_{m+1} - \mu_m$

For $SU(2)$ this gives:

$$\hat{f}_x(z,z) \longrightarrow \frac{\pi \tanh(\pi \nu_m r_m)}{r_m}$$

$z = 0 \to m = 1 \quad ; \quad z = \frac{1}{2} \to m = 2$

$\beta = 1,\ \rho = 1.2$

$\mu_1 = -\omega = -\mu_2 = -1/8$

anti-periodic

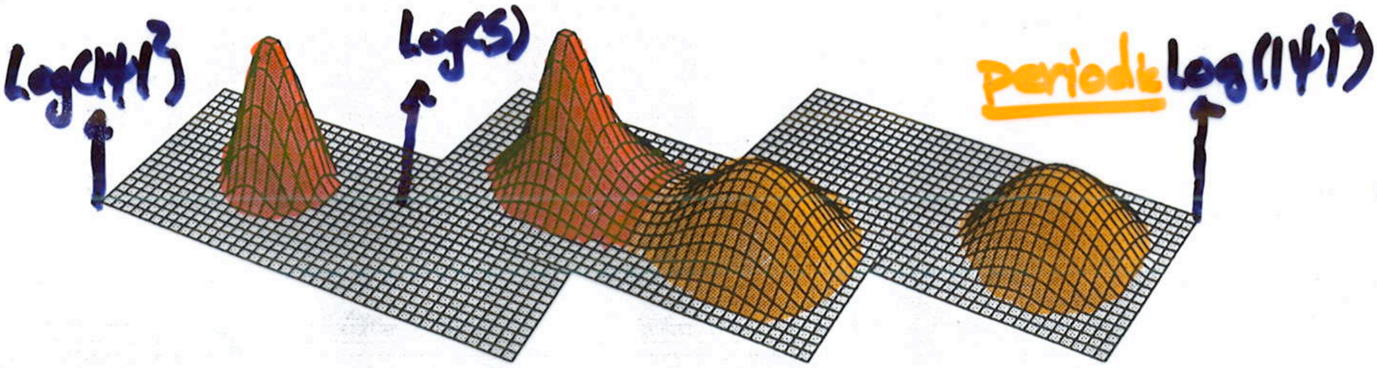


Figure 1: For the two figures on the sides we plot on the same scale the logarithm of the zero-mode densities (cutoff below $1/e^5$) for $\omega = 1/8$ (left Ψ^- / right Ψ^+) and $\omega = 3/8$ (right Ψ^- / left Ψ^+), with $\beta = 1$ and $\rho = 1.2$. In the middle figure we show for the same parameters (both choices of ω give the same action density) the logarithm of the action density (cutoff below $1/2e^2$).

$\nu_1 = 2\omega,\ \nu_2 = 1 - 2\omega$

$\mu_1 = -\omega,\ \mu_2 = \omega$

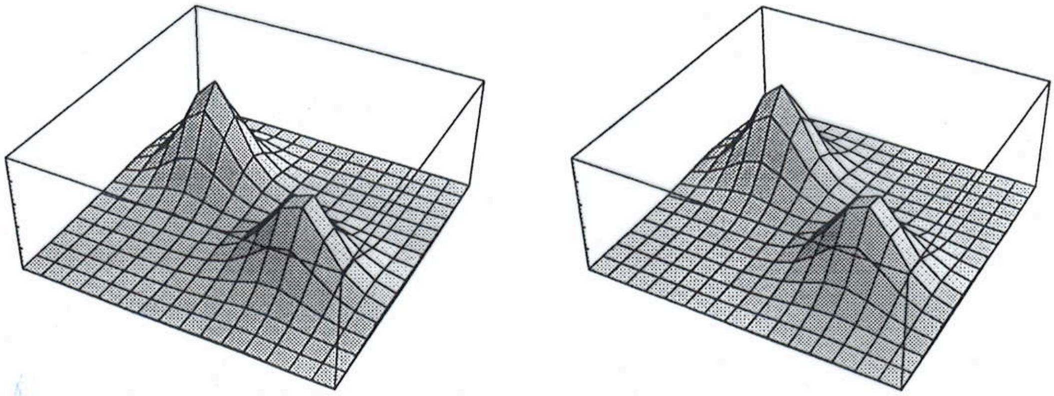


Figure 3: Zero-mode density profiles for the two zero-modes of the lattice caloron (left) on a 4×16^3 lattice for $\vec{k} = (1,1,1)$, created with improved cooling ($\varepsilon = 0$). The profiles fit well to the two zero-modes for the infinite volume analytic caloron solution (shown on the right at $y = t = 0$) with $\omega = \frac{1}{4}$ and constituents at $\vec{y}_1 = (2.50, 0.12, 0.95)$ and $\vec{y}_2 = (1.38, -0.24, 2.67)$, in units where $\beta = l_t = 1$ (or $a = \frac{1}{4}$) and the left most lattice point corresponding to $x = z = 0$. The plots give the added densities of the two zero-modes.

(with M. García Pérez, A. Gonzalez-Arroyo and C. Pena)

hep-th/9905016/9905138

(*)

adjoint!

For SUSY-YM each constituent monopole carries 2 gluino zero-modes. This saturates $\langle \lambda\lambda \rangle$ and resolves an old problem in computing the Witten index (N.M. Davies, T.J. Hollowood, V.V. Khoze, M.P. Mattis, hep-th/9905015.) ← NPB 559 (1999) 123

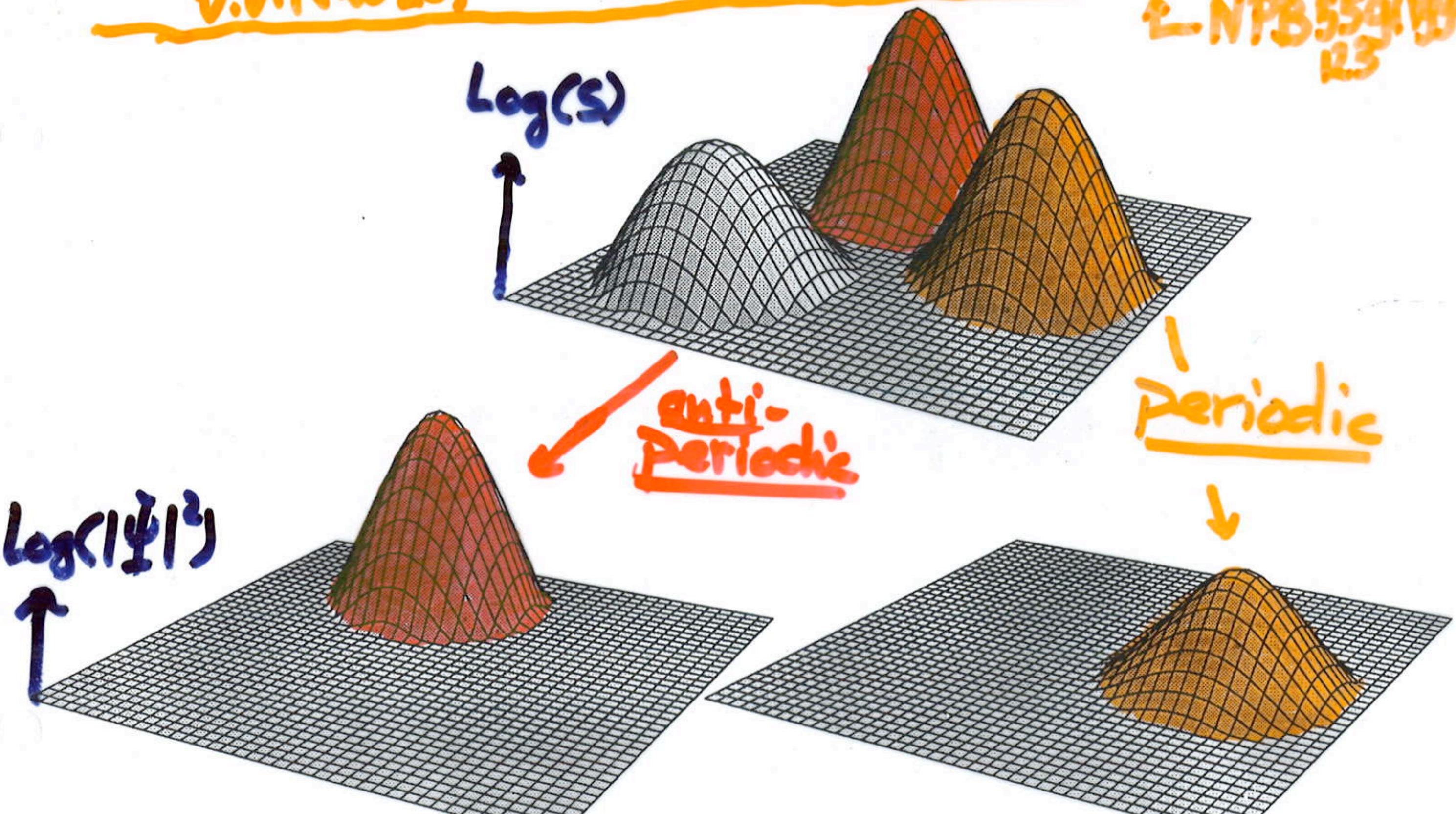


Figure 1: The the action densities (top) for the $SU(3)$ caloron, cut off at $1/(2e)$, on a logarithmic scale, with $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$ for $t = 0$ in the plane defined by $\vec{y}_1 = (-2, -2, 0)$, $\vec{y}_2 = (0, 2, 0)$ and $\vec{y}_3 = (2, -1, 0)$, for $\beta = 1$, with masses $8\pi^2\nu_i$, $(\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4)$. On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales, cut off below $1/e^5$.

(with T. Kraan and M. Chernodub)
NPB (Proc. Suppl) 83-84 (2000) 556

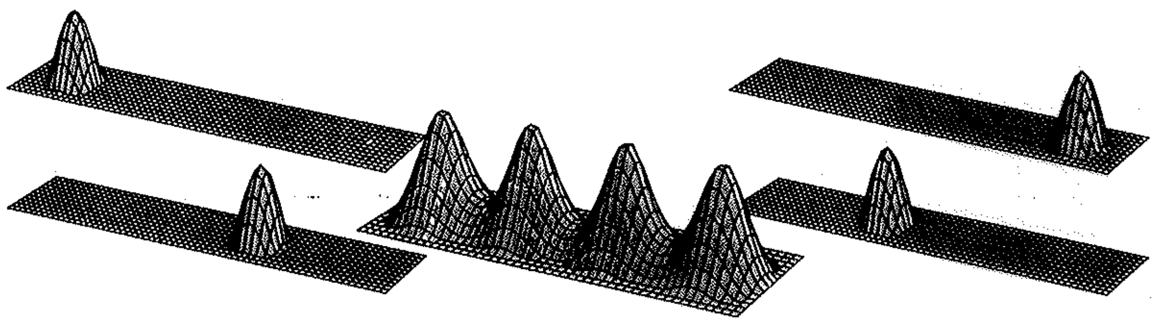
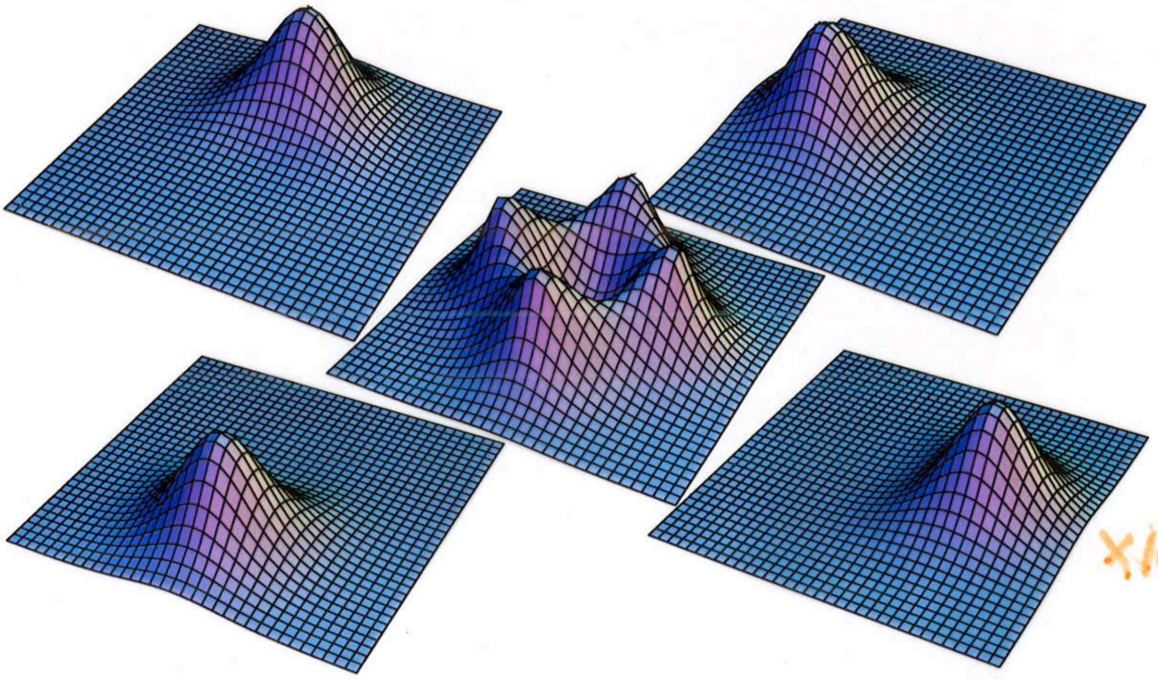
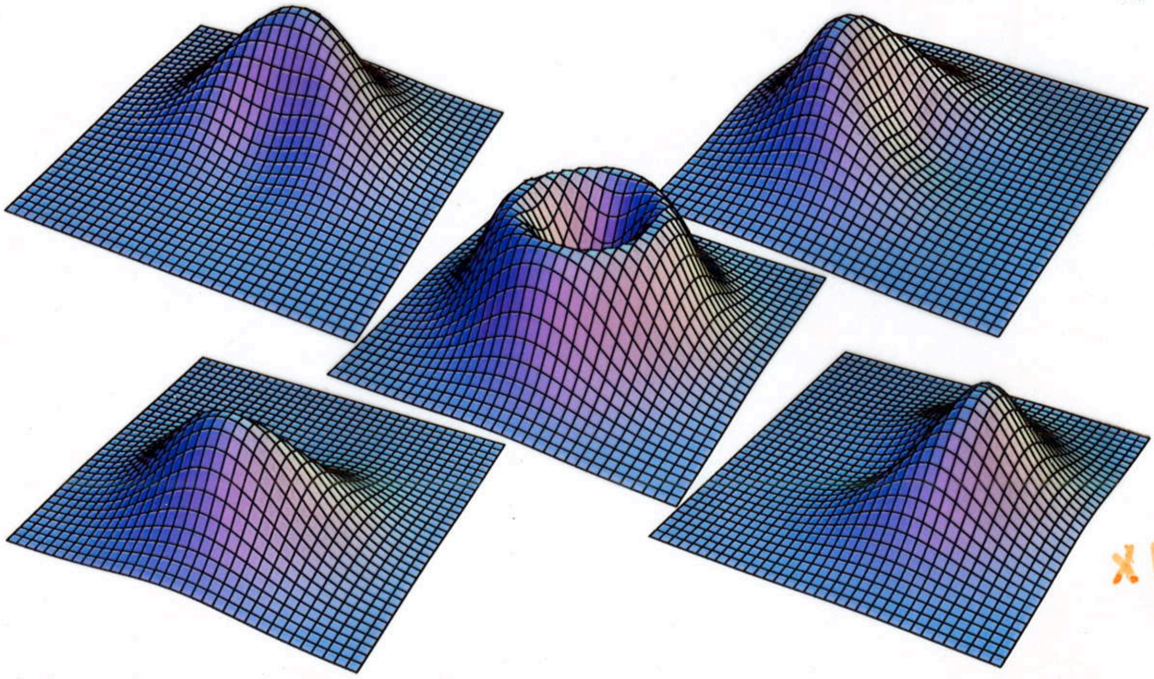


Figure 3: Zero-mode densities for a typical charge 2, $SU(2)$ axially symmetric solution. For comparison the action density (cmp. Fig. 2 of Ref. [11]) is shown in the middle. All are on a logarithmic scale, cutoff below e^{-3}. On the left is shown the two periodic zero-modes ($z = 0$) and on the right the two anti-periodic zero-modes ($z = 1/2$).



$\times 10\pi^2$

Figure 3: The action density in the plane of the constituents at $t = 0$ and the densitities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions for an SU(2) charge 2 caloron in the so-called "crossed" configuration with $\mathbf{k} = 0.997$, $D = 8.753$ and $\operatorname{tr}\mathcal{P}_\infty = 0$.



$\times 16\pi^2$

Figure 4: As above, but for $\mathbf{k} = 0.962$ and $D = 3.894$.

(non-static)

Diakonov and Petrov, PRD 76 (2007) 056001 [arXiv:0704.3181 [hep-th]], have written a hyperkähler metric which approximates the metric for arbitrary # of calorons.

They generalize the metric for 1 Caloron (Lee, Weinberg, Yi; Kraan; Diakonov, Gromov — Gibbons-Manton form) to an approximation of the like-dyon case. It deviates from the like-dyon metric of Atiyah and Hitchin when one comes close together — it vanishes already when $r = 1/2\pi\nu_m$ (for $T=1$).

Nevertheless, they claim this already gives confinement; $T \lesssim T_c$ the minimum of the potential is $\nu_1 = \nu_2 = \dots = \nu_N = 1/N$ and the trace of the Polyakov loop is 0.

(the corrections due to the small fluctuations are still to be incorporated)

See also D. Diakonov,
arXiv:0807.0902 [hep-th] (Cracow Lect.)
and D. Diakonov and V. Petrov,
arXiv:0809.2063 [hep-th], and
D. Diakonov, arXiv: 0906.2456 [hep-ph]
(ITEP/Schladming Lect.)

There recently was some critisism:
"Cautionary remarks on the moduli
space metric for multi-dyon simulations",
by F. Bruckmann, e.a., arXiv: 0903.3075 [hep-ph],

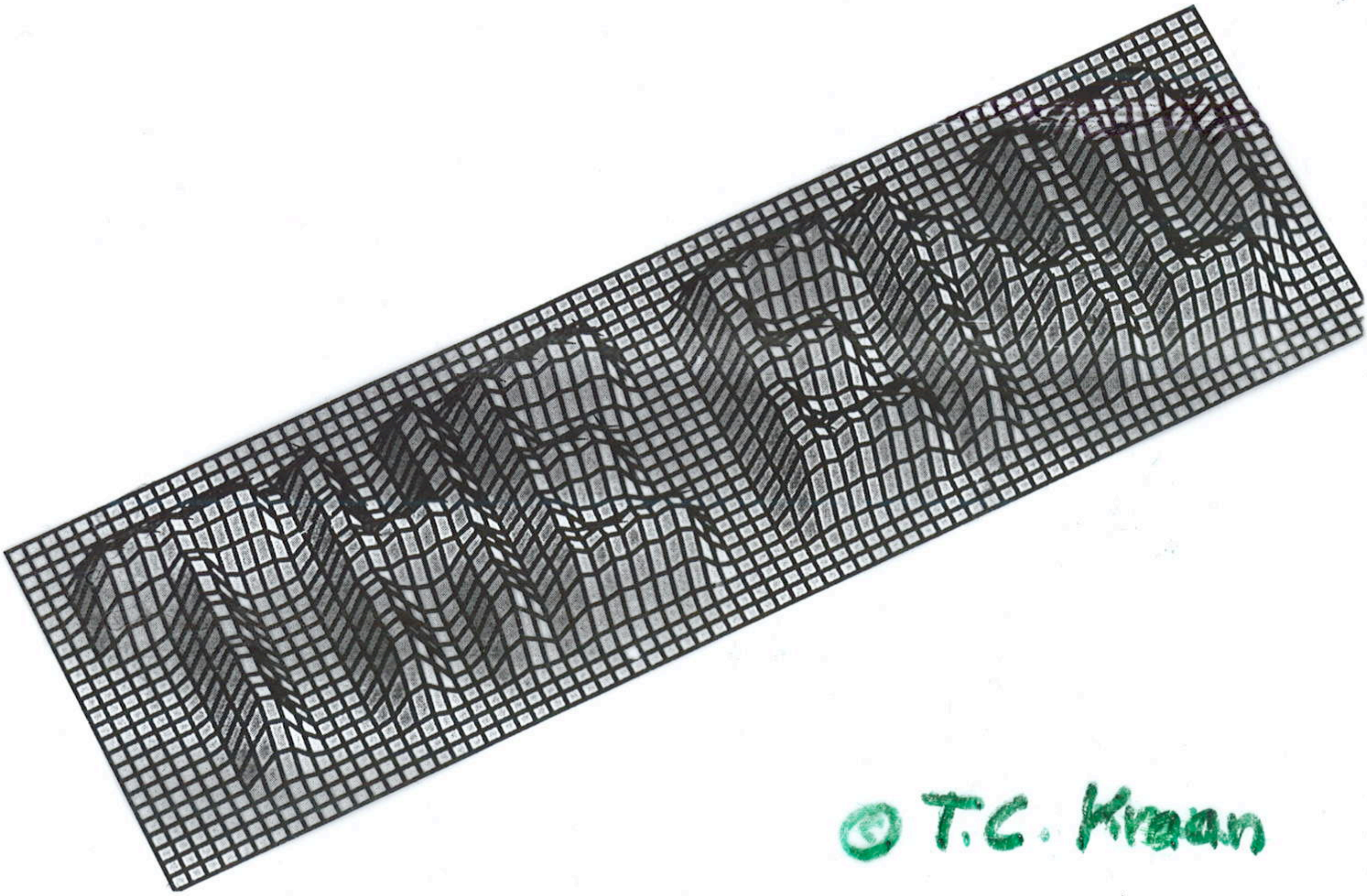
(*)

M. García Pérez and A. González-Arroyo, (susy)
JHEP 0611(2006)091,
M. García Pérez, A. González-Arroyo and
A. Sastre, Phys. Lett. B668(2008)340
+ arXiv:0905.0645 [hep-th]
(= JHEP 0906(2009)065)

The adjoint fermionic zero-modes are now give in analytical form(*)

(+) M. Ünsal, Phys. Rev. D80 (2009) 065001
[arXiv: 0709.3269]

Ünsal has published a paper(+) concerning the mechanism of confinement in QCD-like theories, f.e. SU(2) with $1 \leq n_f \leq 4$ adjoint Majorana fermions.
He argues that there on BPS and KK monopoles (precisely the constituents of the calorons) which have zero-modes under the adjoint fermions. They than make BPS-RR bound states (instead of BPS-KK)



An Atipical SU(73)
Caloron solution.