Lattice Perturbation Theory: is still there space for novelty?

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Future directions in lattice gauge theory - LGT10, August 3rd 2010

(Mostly) in collaboration with M. Brambilla, L. Scorzato

Outline - Why such a title?

I have been involved in a non-standard (stochastic) approach to LPT for quite a long time ...

NSPT: any interesting future direction?

I thought quite a lot and then decided to talk about

- 3 loops Renormalization Costants (not only the finite ones!)
- the Dirac operator spectrum in PT (maybe also in different backgrounds)

Just for those who have never heard about it, I will start with a sketchy introduction to Numerical Stochastic Pertubation Theory.

The numerical tool (Numerical Stochastic Perturbation Theory)

In the Stochastic Quantization (Parisi, Wu 1981) framework (η white noise)

$$\frac{\partial}{\partial t}\phi_\eta(x,t) = -\frac{\delta S[\phi]}{\delta\phi_\eta(x,t)} + \eta(x,t).$$

$$\lim_{t\to\infty}\langle\phi(x_1,t)\dots\phi(x_n,t)\rangle_\eta = \langle\phi(x_1)\dots\phi(x_n)\rangle.$$

we expand the solution, put the expansion into Langevin eqn (we get a hierarchy of eqns) and compute observables order by order

$$\phi_\eta(x,t) = \phi_\eta^{(0)}(x,t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x,t)$$

$$O\left[\sum_n g^n \phi_\eta^{(n)}(x,t)\right] = \sum_n g^n O^{(n)}(x,t).$$

In the integral version of Langevin eqn the expansion would give raise to Stochastic PT (diagrams). NSPT (Di Renzo, Marchesini, Onofri 1994) puts instead the equations on a computer ...

Something like a "perturbative MonteCarlo"

In the case of Lattice Gauge Theories

$$U_{x\mu}(\tau;\eta) \to 1 + \sum_{k=1} \beta^{-k/2} U^{(k)}_{x\mu}(\tau;\eta)$$

which (by the way) could even be

$$U_{x\mu}(\tau;\eta) \to U_0 + \sum_{k=1} \beta^{-k/2} U^{(k)}_{x\mu}(\tau;\eta)$$

and a numerical scheme for the integration of Langevin eqn is the order by order version of

$$U_{x\mu}(\tau+1;\eta) = e^{-i\tau\nabla_{x,\mu}S_G[U] - i\sqrt{\tau}\eta_\mu} U_{x\mu}(\tau;\eta)$$

Fermions are not a problem ...

Including fermions means dealing with

$$e^{-S_G} \det M = e^{-S_{eff}} = e^{-(S_G - Tr \ln M)}$$

which for the Langevin equation implies

$$\begin{aligned} \nabla_{x,\mu} S_G \mapsto \nabla_{x,\mu} S_{eff} &= \nabla_{x,\mu} S_G - \nabla_{x,\mu} \mathrm{Tr}\ \ln M \\ &= \nabla_{x,\mu} S_G - \mathrm{Tr}\left((\nabla_{x,\mu} M) M^{-1} \right) \end{aligned}$$

Common practice: introducing a(nother) gaussian source, we can re-express

$$\langle \xi_i \xi_j \rangle_\xi = \delta_{ij} \qquad \nabla_{x,\mu} S_G - \mathrm{Re}\left({\xi_k}^\dagger (\nabla_{x,\mu} M)_{kl} (M^{-1})_{ln} \xi_n \right)$$

The Dirac operator is also given as a power expansion

$$M^{-1} = \sum_{k=0} \beta^{-k/2} {M^{-1}}^{(k)} = {M^{(0)}}^{-1} + \sum_{k>0} \beta^{-k/2} {M^{-1}}^{(k)}$$

The main building block $\psi^{(j)} \equiv {M^{-1}}^{(j)} \xi$ comes from a simple recursion

$$\begin{aligned}
\psi^{(0)} &= {M^{(0)}}^{-1} \xi \\
\psi^{(1)} &= -{M^{(0)}}^{-1} M^{(1)} \psi^{(0)} \\
\psi^{(2)} &= -{M^{(0)}}^{-1} \left[M^{(2)} \psi^{(0)} + M^{(1)} \psi^{(1)} \right] \\
\psi^{(3)} &= -{M^{(0)}}^{-1} \left[M^{(3)} \psi^{(0)} + M^{(2)} \psi^{(1)} + M^{(1)} \psi^{(2)} \right] \\
\cdots \\
\psi^{(n)} &= -{M^{(0)}}^{-1} \sum_{j=0}^{n-1} M^{(n-j)} \psi^{(j)} \\
\cdots
\end{aligned}$$

3 loops Renormalization Constants - Motivations

Renormalizations constants: to which extent PT vs non-PT is the real issue?

There are obvious concerns with PT, but renormalization systematics is rich for both PT and non-PT!

1. truncation errors (PT)
2. (almost always) chiral extrapolations
3. (always) continuum extrapolation
4. (often) finite size effects
5. n_f ...

Keep in mind:

- No theoretical obstacle for the computation of log-divergent Z's (e.g. Z_s)
- In principle proof of multiplicative renormalization is PT

Di Renzo, Miccio, Scorzato, Torrero Eur.Phys.J.C51(2007)645;

Di Renzo, Ilgenfritz, Perlt, Schiller, Torrero Nucl.Phys.B831(2010)262.

1a We can go to high loop: NSPT + RI'-MOM scheme

We compute quark bilinears bracketted in fixed momentum states and amputate them to Γ functions

$$\int dx \, \langle p| \, \overline{\psi}(x)\Gamma\psi(x) \, |p\rangle \; = \; G_\Gamma(p) \qquad\qquad G_\Gamma(p) \;\; \rightarrow \;\; \Gamma_\Gamma(p)$$

We project on tree-level structures

$$O_\Gamma(p) = Tr\left(\hat{P}_{O_\Gamma}\Gamma_\Gamma(p)\right).$$

We define the field renormalization

$$Z_q(\mu, g) = -i\frac{1}{12}\frac{Tr(\not{p}S^{-1}(p))}{p^2}$$

and finally define renormalization constants

$$Z_{O_\Gamma}(\mu, g) Z_q^{-1}(\mu, g) O_\Gamma(p)|_{p^2=\mu^2} = 1$$

Much is known in this scheme! 3 loops (J. Gracey)

1b We can take anomalous dimensions for free (an example)

$$Z_q(\hat\mu) = 1 + \sum_{n>0} d_n \alpha_0^n + F(\hat\mu) \qquad d_n = \sum_{i=0}^{n} d_n^{(i)} L^i$$

$$\psi_0 = Z_q(\mu)^{-1/2}\psi_R(\mu) \qquad Z_q(\mu)^{-1/2} = 1 + \sum_{n>0} c_n \alpha^n.$$

After differentiating with respect to $\log\mu$

$$0 = \sum_{n>0}\sum_{i=1}^{n}\left[ic_n^{(i)}L^{i-1}\alpha^n + nc_n^{(i)}L^i\alpha^{n-1}2\beta\right]\psi_R + Z_q^{-1/2}\gamma_q\psi_R$$

we can collect orders in α and logs to get the $c_n^{(i)}$:

$$\begin{aligned} c_1^{(1)} &= \gamma_q^{(1)} \\ c_2^{(1)} &= \gamma_q^{(2)} + c_1^{(0)}\left(\gamma_q^{(1)} + 2\beta_0\right) \\ &\ldots \end{aligned}$$

In Landau gauge $\gamma_q^{(1)} = 0$ and $Z_q(\hat\mu)$ is

$$\begin{aligned} Z_q(\hat\mu) &= 1 + Z_q^{(1)}\alpha_0 + \left[Z_q^{(2)} - 2\gamma_q^{(2)}L\right]\alpha_0^2 + \\ &+ \left[Z_q^{(3)} - \left(4\gamma_q^{(2)}K_1 + 2\gamma_q^{(3)} + 2\gamma_q^{(2)}Z_q^{(1)}\right)L + 4\beta_0\gamma_q^{(2)}L^2\right]\alpha_0^3 \end{aligned}$$

2 No chiral extrapolation: we stay at zero mass

In the (Wilson) quark self-energy there is a counterterm (critical mass)

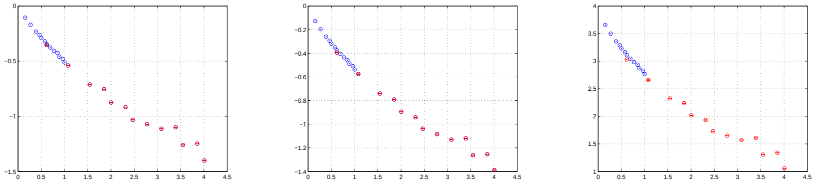
$$\begin{aligned} a\Gamma_2(\hat{p}, \hat{m}_{cr}, \beta^{-1}) &= aS(\hat{p}, \hat{m}_{cr}, \beta^{-1})^{-1} \\ &= i\hat{\not{p}} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) \end{aligned}$$

$$\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_V(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_o(\hat{p}, \hat{m}_{cr}, \beta^{-1})$$

which we plug in order by order.

Data are for $n_f = 2$ TLSymanzick/Wilson

(32^4 and 16^4 lattices; 24^4 and 12^4 are also almost done)



As an example, let's go back to the quark self-energy and look for the field renormalization.

(In our notation $\hat{p} = pa$)

$$\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_V(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_o(\hat{p}, \hat{m}_{cr}, \beta^{-1})$$

Let's H4-Taylor expand it

$$\hat{\Sigma}_V = i \sum_\mu \gamma_\mu \hat{p}_\mu \left(\hat{\Sigma}_V^{(0)} + \hat{p}_\mu^2 \hat{\Sigma}_V^{(1)} + \hat{p}_\mu^4 \hat{\Sigma}_V^{(2)} + \dots \right)$$

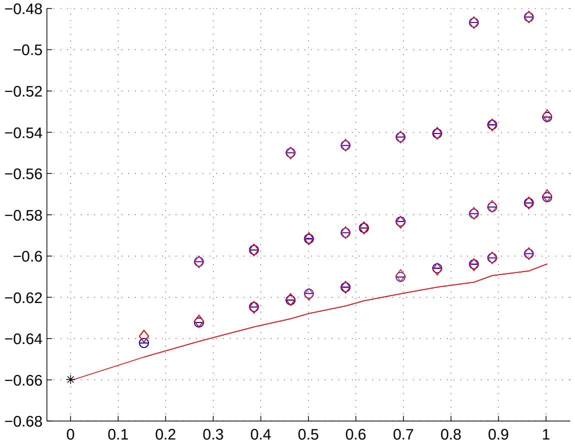
$\Sigma^{(n)}$ are also H4-Taylor expanded (once possible log's have been subtracted)

$$\hat{\Sigma}_V^{(n)} = \alpha_1^{(n)} 1 + \alpha_2^{(n)} \sum_\nu \hat{p}_\nu^2 + \alpha_3^{(n)} \sum_\nu \hat{p}_\nu^4 + \alpha_4^{(n)} \sum_{\nu \neq \rho} \hat{p}_\nu^2 \hat{p}_\rho^2 + \mathcal{O}(a^6)$$

The only term surviving the $a \to 0$ limit is $\alpha_1^{(0)}$.

Continuum limit at work

Here is the field renormalization at 1 loop (from the self-energy) on a 32^4 lattice

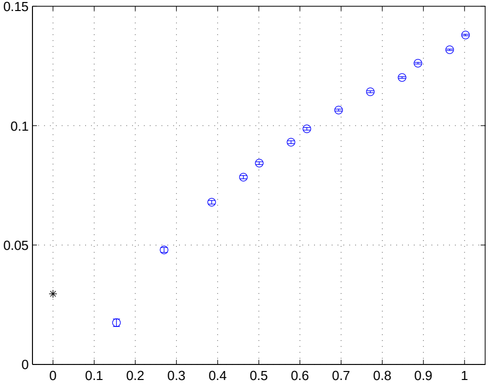


Something can still go wrong

So, let's look for the quark mass renormalization (Z_s actually; 32^4)

In this case a log has been subtracted (see $Z_{O_\Gamma}(\mu, g) Z_q^{-1}(\mu, g) O_\Gamma(p)|_{p^2=\mu^2} = 1$)

$$Z_q^{(1)} - Z_s^{(1)} = O_s^{(1)} - \gamma_s^{(1)} L$$

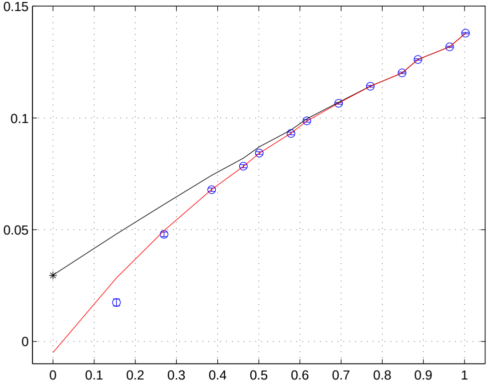


At this stage, don't trust IR...

IR can't be trusted (in particular when an anomalous dimension is around).

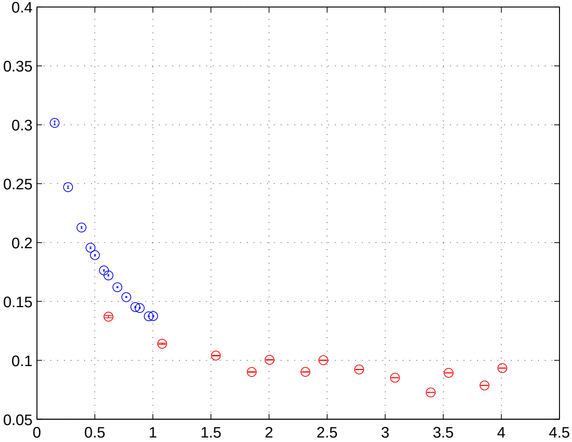
Landau gauge for field renormalization did not need a subtraction, while now (remember)

$$Z_q^{(1)} - Z_s^{(1)} = O_s^{(1)} - \gamma_s^{(1)} L$$



It's a finite size effect!

... as can be seen by inspecting O_s (the un-log-subtracted observable) on different lattice sizes (32^4 and 16^4)



4 Taming finite size: get to $L \to \infty$

On dimensional grounds we expect (take once again $\Sigma^{(n)}$) *pL effects*

$$\begin{aligned} \hat{\Sigma}_V^{(n)}(\hat{p}, pL) &= \hat{\Sigma}_V^{(n)}(\hat{p}, \infty) + \left(\hat{\Sigma}_V^{(n)}(\hat{p}, pL) - \hat{\Sigma}_V^{(n)}(\hat{p}, \infty)\right) \\ &= \hat{\Sigma}_V^{(n)}(\hat{p}, \infty) + \Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL) \end{aligned}$$

so that a better expansion to fit is

$$\begin{aligned} \hat{\Sigma}_V^{(n)}(\hat{p}, pL) &= \alpha_1^{(n)} 1 + \alpha_2^{(n)} \sum_\nu \hat{p}_\nu^2 + \alpha_3^{(n)} \sum_\nu \hat{p}_\nu^4 + \\ &\quad + \alpha_4^{(n)} \left(\sum_\nu \hat{p}_\nu^2\right)^2 + \Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL) + \ldots \end{aligned}$$

In first approximation

$$\Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL) \sim \Delta\hat{\Sigma}_V^{(n)}(pL)$$

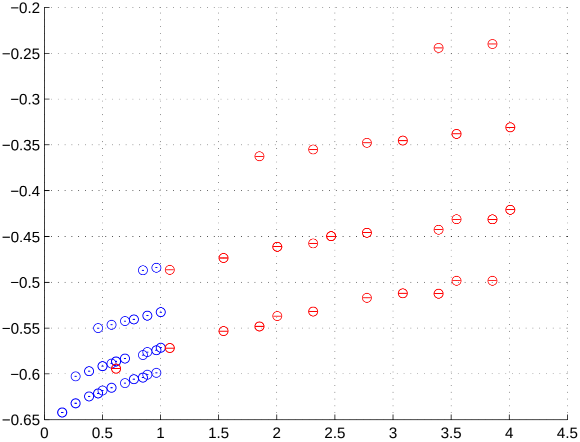
But

$$p_\mu L = \frac{2\pi n_\mu}{L} L = 2\pi n_\mu !$$

i.e. same correction on different lattice sizes for the same $\{n_1, n_2, n_3, n_4\}$.

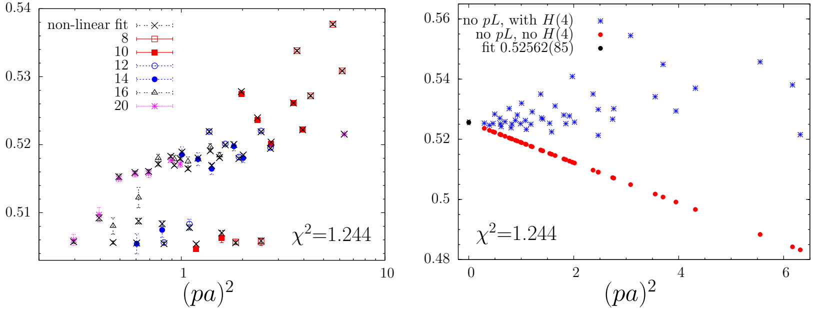
Let's gain some insight

Go back to 1 loop field renormalization, both on 32^4 and 16^4

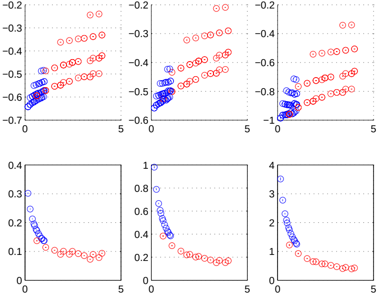


The method has already been successful

It has been working pretty well in the case of the gluon and ghost propagators (F. Di Renzo in collaboration with M. Ilgenfritz, H. Perlt, A. Schiller, C. Torrero)



First goal is the quark mass renormalization constant.



We have

- Wilson/Wilson (various n_f) to 3/4 loops
- $n_f = 2$ TLSymanzick/Wilson to 3 loop
- Started $n_f = 4$ Iwasaki/Wilson to 3 loop

We have also set up NSPT for staggered actions (very first steps).

We hope we can reduce systematics and bridge various determinations of m_q

The Dirac spectrum in PT - Motivations

Spontaneous chiral symmetry breaking:

a small quark mass leads to a macroscopic reallignement of the QCD vacuum.

Since

$$Z = \langle \prod_f \det(\mathcal{D} + m_f) \rangle = \langle \prod_f \prod_n (i\lambda_n + m_f) \rangle$$

one would think a small quark mass should be dominated by larger eigenvalues, but this is not the case if there is an accumulation of Dirac eigenvalues near zero.

Having defined the eigenvalues density

$$\rho(\lambda) = \langle \sum_n \delta(\lambda - \lambda_n) \rangle$$

one can gain insight from the Banks-Casher (1980) formula.

The chiral condensate (which is the order parameter for the chiral transition) is related to low modes of the Dirac spectrum

$$\langle \bar{\psi}\psi \rangle = \lim_{\epsilon \to 0} \lim_{m \to 0} \lim_{V \to \infty} \frac{\pi \rho(\epsilon)}{V}$$

Where do these eigenvalues come from?

1. Free case is the deep perturbative regime, which is the chirally symmetric regime: without interactions low modes are not there.
2. Every quantum interaction produces repulsion among eigenvalues, so that they could come from the bulk.

... and PT is in a tantalizing situation:

1. It sits deep in the chirally symmetric phase (and we are looking for something taking place in the other phase!).
2. (level splitting) PT naturally accounts for repulsion among eigenvalues.

Brambilla, Di Renzo PoS LAT2009(2009)209.

A textbook computation...

In NSPT (like in any computer simulation) you never handle fermions. The Dirac operator is evaluated in the background of the (generated) gauge field. As any other operarator, the NSPT Dirac operator is given as an expansion. This means that we want to solve the typical eigenvalue/eigenvector problem

$$M = M_0 + N = M_0 + \sum_i g^i N_i \qquad M\,|\alpha\rangle = \epsilon\,|\alpha\rangle$$

where

$$\epsilon = \epsilon_0 + g\,\epsilon_1 + g^2\,\epsilon_2 + \ldots \quad |\alpha\rangle = |\alpha_0\rangle + g\,|\alpha_1\rangle + g^2\,|\alpha_2\rangle + \ldots$$

The free field solution is (highly) degenerate. A convenient notation to look for the solution is the following (we explicitly consider components inside and outside the free field eigenspace)

$$|\alpha\rangle = |\alpha_0\rangle + P'_{in}|\alpha\rangle + P_{out}|\alpha\rangle$$

in terms of which

$$0 = (\epsilon - M_0 - N)|\alpha_0\rangle + (\epsilon - M_0 - N)P'_{in}|\alpha\rangle + P_{out}(\epsilon - M_0 - N)|\alpha\rangle$$

Now we only need to apply the three projectors

$$
\begin{aligned}
\epsilon_n &= \sum_{k=0}^{n} \langle \alpha_0 | N_{n-k} | \alpha_k \rangle \\
P_{out}|\alpha\rangle &= (\epsilon - M_0 - P_{out}N)^{-1} \left(P_{out}N|\alpha_0\rangle + P_{out}NP'_{in}|\alpha\rangle \right) \\
P'_{in}|\alpha\rangle &= (\epsilon - \epsilon_0 - P'_{in}N)^{-1} \left(P'_{in}N|\alpha_0\rangle + P'_{in}NP_{out}|\alpha\rangle \right)
\end{aligned}
$$

- Construction is iterative.
- Eigenvalues repel each other.

This is the correct framework is degeneracy is lifted at first order. This is not always the case (and a third, fourth, ... projector is needed).

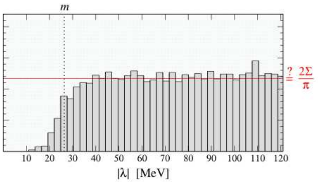
Can we inspect the reshuffling of eigenvalues due to this repulsion?

We will compute $D^\dagger D$ for Wilson fermions.

A couple of pictures to keep in mind

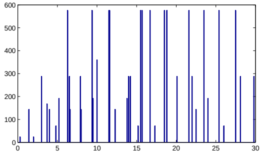
A typical non-perturbative determination of the Dirac spectrum (M. Luscher) (eigenvalue density is an histogram)

Spectrum of the hermitian Wilson–Dirac operator $\gamma_5 D_m$ on a 48×24^3 lattice



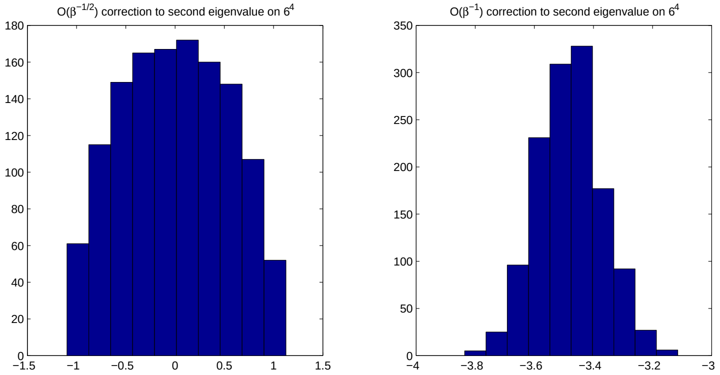
M.L. '07 [JHEP 0707 (2007) 081]

... and this is the free field spectrum (again, $D^\dagger D$ for Wilson fermions)



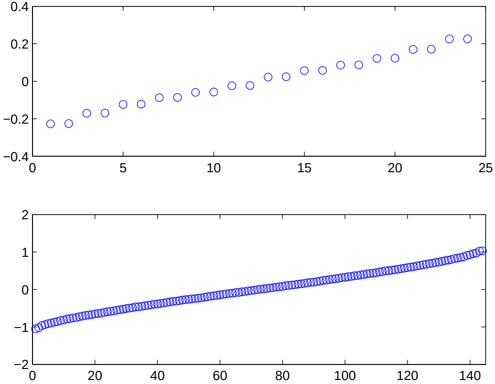
Some raw results

Let's inspect examples of our results: a bunch of measurements for first (trivial) and second (one loop) order corrections to free field in the second lowest lying eigenspace on a 6^4 lattice. Eigenspace is degenerate (the dimension of this eigenspace is 144); on top of this degeneracy the histograms entail the multeplicity which comes from the number of measurements.



O(g) results

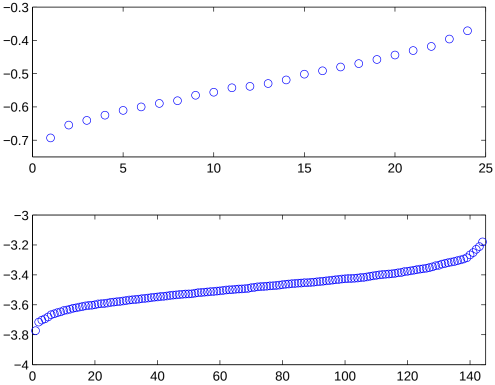
First (odd) order corrections to the first and second lowest lying eigenvalues.
Notice the different shapes (in one case degeneracy not fully lifted)



Odd orders broaden the free field eigenvalues.

1 loop results

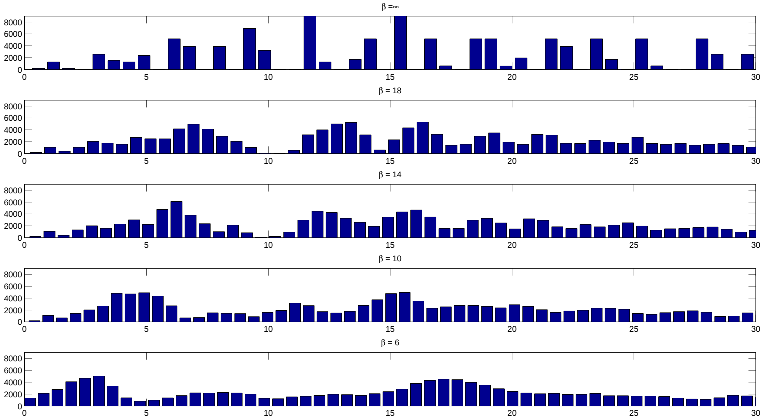
First loop corrections to the first and second lowest lying eigenvalues. This time corrections are not centered in zero.



Loop orders move the free field eigenvalues.

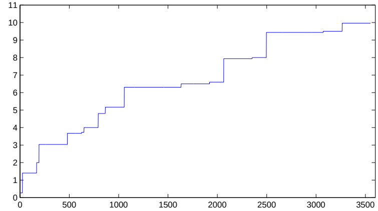
Let's go quick and dirty

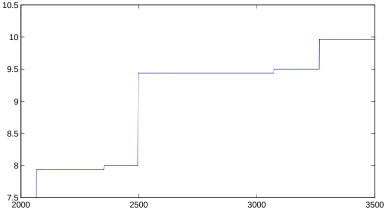
We now can not resist and mimic standard non-perturbative computations of spectra. Given the perturbative corrections, we sum the series at various values of β and histogram results (in the figure, one loop)



Where do the big effects take place?

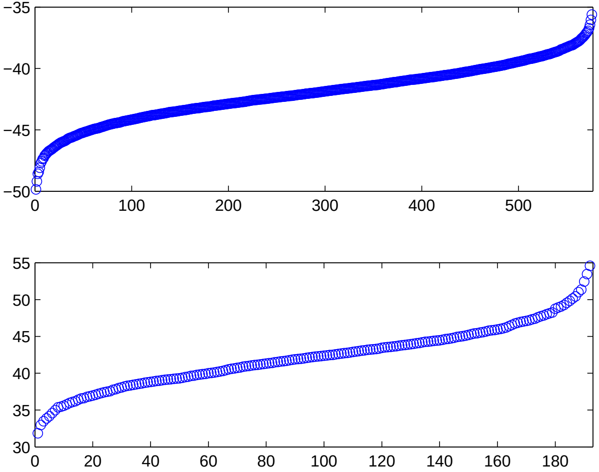
It's actually a bulk effect! Some levels are very close to each other in the free field spectrum





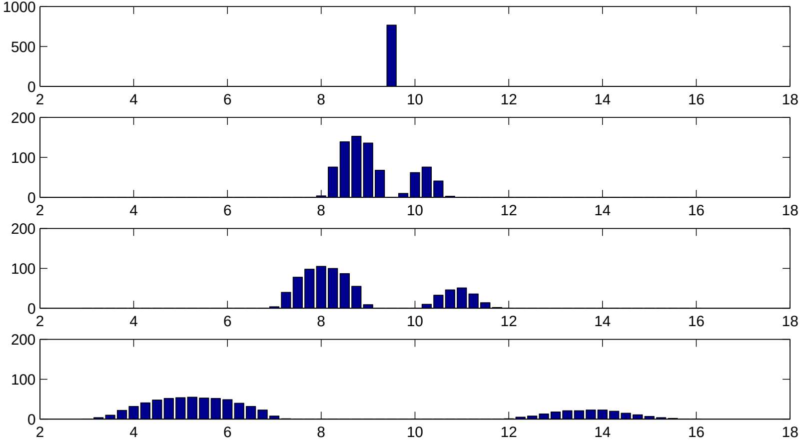
Where do the big effects take place?

It's actually a bulk effect! Some levels are very close to each other in the free field spectrum and they strongly repel each other once interaction is on!



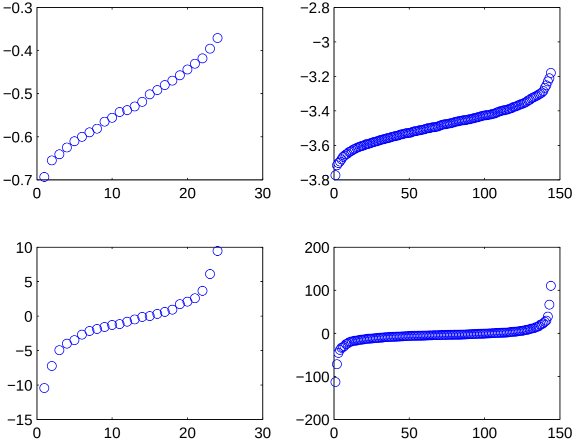
Where do the big effects take place?

Let's focus on the separation of two levels as the coupling increases ($\beta = \infty, 60, 30, 10$)



2 loop can go crazy!

Two loop effects can be drammatic: we compare first and second loop for the first two eigenalues



Some (self)criticism (1)

Notice that

$$\langle \sum_n \delta(\lambda - \lambda_n) \rangle \to \sum_n \left(\delta(\lambda - \lambda_n^{(0)}) + \sum_{i_n} \delta'(\lambda - \lambda_n^{(0)}) \langle \lambda_{i_n}^{(1)} \rangle + \dots \right)$$

This is fine for the computation of any observable, but returns a trivial result when plugged into the definition of the average number of eigenvalues (of $D^\dagger D$) within a given threshold

$$\nu(M, m) = \int_{-\Lambda}^{\Lambda} d\lambda \, \rho(\lambda, m), \qquad M^2 = m^2 + \Lambda^2$$

The latter is used (Luscher, Giusti 2009)) to define spectral sums

$$\sigma_k(\mu, m) = \langle \mathrm{Tr}\{(D_m^\dagger D_m + \mu^2)^{-k}\} \rangle = \int_0^\infty dM \, \nu(M, m) \, \frac{2kM}{(M^2 + \mu^2)^{k+1}}$$

which can be mapped to composite operators in Twisted Mass QCD, whose renormalization properties are natural. These can be used to show that

$$\nu_R(M_R, m_R) = \nu(M, m)$$

Some (self)criticism (2) ... and some optimism on top of that

Also, remember

$$
\begin{aligned}
P_{out}|\alpha\rangle &= (\epsilon - M_0 - P_{out}N)^{-1}\left(P_{out}N|\alpha_0\rangle + P_{out}NP'_{in}|\alpha\rangle\right) \\
P'_{in}|\alpha\rangle &= (\epsilon - \epsilon_0 - P'_{in}N)^{-1}\left(P'_{in}N|\alpha_0\rangle + P'_{in}NP_{out}|\alpha\rangle\right)
\end{aligned}
$$

Nearly degenerate levels build a big effect!

This is a big issue: when is degeneracy really lifted?

In the end, we definitely need to

1. understand renormalization issues (this is bare PT!);
2. better assess the degeneracy-lifting issues.

... not to mention assessment of finite V and finite a ...

Still there is some intuition to gain

The free field spectrum looks totally unstable as soon as the gauge interaction is switched on.

Project: extend to non-trivial background!

Conclusions

We can compute RI'-MOM log-divergent Renormalization Constants to 3 loops keeping all the systematics under a very good control.

1. We have results for (log-divergent) Wilson/Wilson and for TLSymanzick/Wilson Z's
2. Other regularizations on their (maybe not so short) way
3. The real issue is not PT vs Non-PT; the real issue is how to best control systematics.

We have computed the spectrum of the Dirac operator in bare PT by NSPT (1 and 2 loop)

1. Repulsion among eigenvalues is a strong effect!
2. Many items to be better understood ... but the overall picture is nevertheless intriguing.
3. We have the project to extend to non-trivial background (different $Z(3)$ vacua already started).