$K \to (\pi\pi)_{I=2}$ Decay Amplitudes

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Future Directions in Lattice Gauge Theory LGT10,

CERN, July 19th - August 13th 2010

- In this talk, I will review the status of the computations of the $K \to (\pi\pi)_{I=2}$ decay amplitudes by the RBC-UKQCD collaboration.

  T.Blum, P.Boyle, D. Broemmel, J. Flynn, N. Garron, E. Goode, T. Izubuchi, M. Lightman, Qi Liu, R. Mawhinney, N. Christ, C. Sachrajda, A. Soni et al.

- Preliminary results from this study were presented by E.Goode and M.Lightman at
  - Lattice 2009: arXiv:0912.1667

    $\Delta I = 3/2,\ K \to \pi\pi$ Decays with Light, Non-Zero Momentum Pions
  - Lattice 2010:

    http://agenda.infn.it/contributionDisplay.py?contribId=11&sessionId=22&confId=2128

    $\Delta I = 3/2,\ K \to \pi\pi$ Matrix Elements with Nearly Physical Pion Masses

- Other aspects of the RBC/UKQCD kaon programme will be presented by Qi Liu (following talk) and Norman Christ.

Outline of Talk

1. Introduction
2. $K \to \pi\pi$ decay amplitudes from $K \to \pi$ Matrix Elements.
3. Direct evaluation of $K \to \pi\pi$ Decays – Infrared Issues
4. Calculation and Results
5. Conclusions

All numerical results are preliminary.

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All numerical results are preliminary.

- Of course we would like to evaluate all the $K\to\pi\pi$ matrix elements in lattice simulations and reconstruct A_0 and A_2 and understand the $\Delta I=1/2$ rule and the value of ε'/ε (see below).
- In the meantime however, we know $\mathrm{Re}\,A_0$ and $\mathrm{Re}\,A_2$ from experiment.
  In this talk I attempt to demonstrate that we can also compute $\mathrm{Re}\,A_2$.
- The experimental value of ε'/ε gives us one relation between $\mathrm{Im}\,A_0$ and $\mathrm{Im}\,A_2$, thus if we evaluate $\mathrm{Im}\,A_2$ then within the standard model we know $\mathrm{Im}\,A_0$ to some precision. Thanks to Andrzej Buras for stressing this to me.
  In this talk I attempt to demonstrate that we can indeed compute $\mathrm{Im}\,A_2$.
- I stress again that ultimately of course, we wish to do better than this.

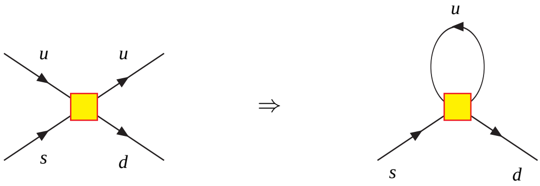
Qi Liu - Following Talk

2. $K \to \pi\pi$ decay amplitudes from $K \to \pi$ Matrix Elements

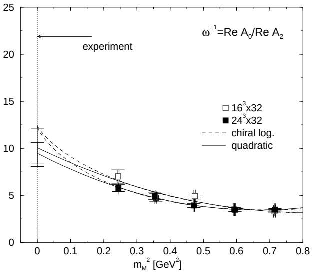
- At lowest order in the SU(3) chiral expansion one can obtain the $K \to \pi\pi$ decay amplitude by calculating $K \to \pi$ and $K \to$ vacuum matrix elements.
- In 2001, two collaborations published some very interesting (quenched) results on non-leptonic kaon decays in general and on the $\Delta I = 1/2$ rule and ε'/ε in particular:

Collaboration(s)	Re A_0/Re A_2	ε'/ε
RBC	25.3 ± 1.8	$-(4.0 \pm 2.3) \times 10^{-4}$
CP-PACS	9÷12	(-7 ÷ -2)$\times 10^{-4}$
Experiments	22.2	$(17.2 \pm 1.8) \times 10^{-4}$

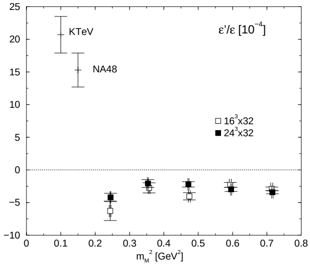
- This required the control of the *ultraviolet problem*, the subtraction of power divergences and renormalization of the operators – highly non-trivial.
  - Four-quark operators mix, for example, with two quark operators $\Rightarrow$ power divergences:



- Re A_0/Re A_2 as a function of the meson mass.



- ε'/ε as a function of the meson mass.



- The RBC and CP-PACS simulations were quenched, and relied on the validity of lowest order χPT in the region of approximately 400-800 MeV.
- Given the cancelations between different matrix elements (particularly O_6 and O_8) the negative value of ε'/ε is not such an embarrassment but

Must do better!

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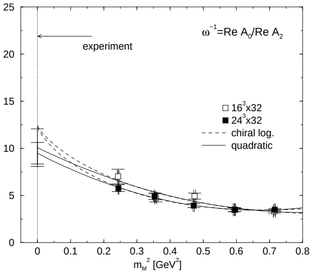
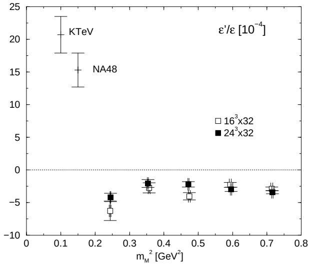
- Re A_0/Re A_2 as a function of the meson mass.



- ε'/ε as a function of the meson mass.



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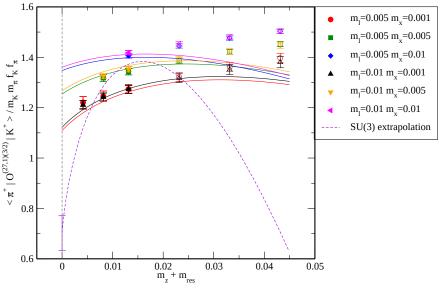
Must do better!

Unquenched Calculation

S.Li, Ph.D. thesis, RBC-UKQCD in preparation



$$O^{3/2}_{(27,1)} = (\bar{s}d)_L \left\{ (\bar{u}u)_L - (\bar{d}d)_L \right\} + (\bar{s}u)_L \, (\bar{u}d)_L$$

- RBC/(UKQCD) have repeated the calculation with the 24^3 DWF ensembles in the pion-mass range 240-415 MeV.
- For illustration consider the determination of α_{27}, the LO LEC for the (27,1) operator. Satisfactory fits were obtained, but again the corrections were found to be huge, casting serious doubt on the approach.
- Soft pion theorems are not sufficiently reliable $\Rightarrow$ need to compute $K \to \pi\pi$ matrix elements.

To arrive at this important conclusion required a truly major effort.

- J.Laiho et al. challenge this conclusion. Poster, Lattice conference

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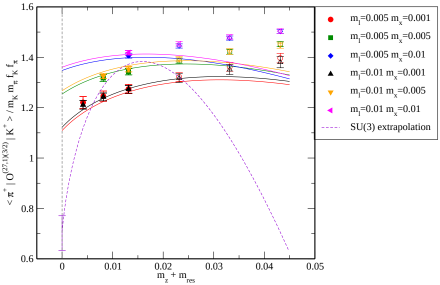

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3. Direct Calculations of $K \to \pi\pi$ Decay Amplitudes

- We need to be able to calculate $K \to \pi\pi$ matrix elements directly.
- The main theoretical ingredients of the *infrared* problem with two-pions in the s-wave are now understood.
- Two-pion quantization condition in a finite-volume

$$\delta(q^*) + \phi^P(q^*) = n\pi\,,$$

where $E^2 = 4(m_\pi^2 + q^{*2})$, δ is the s-wave $\pi\pi$ phase shift and ϕ^P is a kinematic function.

M.Lüscher, 1986, 1991, $\cdots$.

- The relation between the physical $K \to \pi\pi$ amplitude A and the finite-volume matrix element M

$$|A|^2 = 8\pi V^2 \frac{m_K E^2}{q^{*2}} \left\{\delta'(q^*) + \phi^{P\prime}(q^*)\right\} |M|^2\,,$$

where $\prime$ denotes differentiation w.r.t. q^* .

L.Lellouch and M.Lüscher, hep-lat/0003023; C.h.Kim, CTS and S.Sharpe, hep-lat/0507006; N.H.Christ, C.h.Kim and T.Yamazaki hep-lat/0507009

- Computation of $K \to (\pi\pi)_{I=2}$ matrix elements does not require the subtraction of power divergences or the evaluation of disconnected diagrams.
- In principle, we understand how to calculate the $\Delta I = 3/2$ $K \to \pi\pi$ matrix elements.
- Our aim is to calculate the matrix elements with as good a precision as we can.

- The operators whose matrix elements have to be calculated are:

$$\begin{aligned}
O^{3/2}_{(27,1)} &= (\bar{s}^i d^i)_L \left\{(\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L\right\} + (\bar{s}^i u^i)_L\,(\bar{u}^j d^j)_L \\
O^{3/2}_{7} &= (\bar{s}^i d^i)_L \left\{(\bar{u}^j u^j)_R - (\bar{d}^j d^j)_R\right\} + (\bar{s}^i u^i)_L\,(\bar{u}^j d^j)_R \\
O^{3/2}_{8} &= (\bar{s}^i d^j)_L \left\{(\bar{u}^j u^i)_R - (\bar{d}^j d^i)_R\right\} + (\bar{s}^i u^j)_L\,(\bar{u}^j d^i)_R
\end{aligned}$$

- It is convenient to use the Wigner-Eckart Theorem: (Notation - $O^{\Delta I}_{\Delta I_z}$)

$$_{I=2}\langle \pi^+(p_1)\pi^0(p_2)\,|O^{3/2}_{1/2}|K^+\rangle = \frac{3}{2}\langle \pi^+(p_1)\pi^+(p_2)\,|O^{3/2}_{3/2}|K^+\rangle\,,$$

where
  - $O^{3/2}_{3/2}$ has the flavour structure $(\bar{s}d)\,(\bar{u}d)$.
  - $O^{3/2}_{1/2}$ has the flavour structure $(\bar{s}d)\,(\,(\bar{u}u)-(\bar{d}d)\,)+(\bar{s}u)\,(\bar{u}d)$.

- We can then use antiperiodic boundary conditions for the u-quark say, so that the $\pi\pi$ ground-state is $\langle\pi^+(\pi/L)\pi^+(-\pi/L)|$. C-h Kim, Ph.D. Thesis
  - Do not have to isolate an excited state.
  - Size (L) needed for physical $K\to\pi\pi$ decay halved (6 fm $\to$ 3 fm).

C.h. Kim and CTS, arXiv:1003.3191

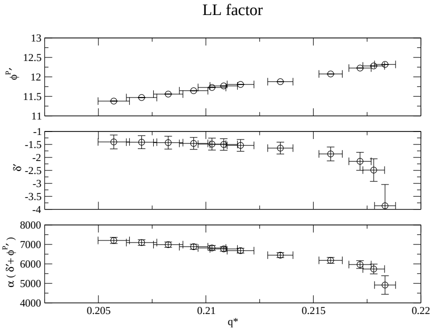
- Use the Wigner-Eckart Theorem to relate the physical $K \to \pi^+\pi^0$ matrix element to that for $K \to \pi^+\pi^+$

$$ {}_{I=2}\langle \pi^+(p_1)\pi^0(p_2)\,|O^{3/2}_{1/2}|K^+\rangle = \frac{3}{2}\langle \pi^+(p_1)\pi^+(p_2)\,|O^{3/2}_{3/2}|K^+\rangle\,, $$

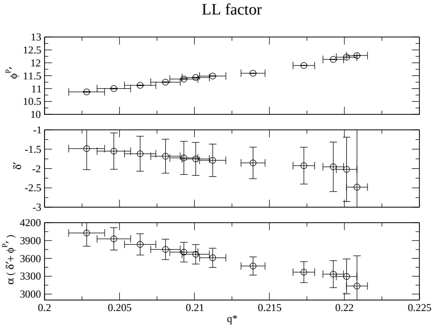
- Calculate the $K \to \pi^+\pi^+$ matrix element with the u-quark with twisted boundary conditions with twisting angle θ.
- Perform a Fourier transform of one of the pion interpolating operators with additional momentum $-2\pi/L$.
  The ground state now corresponds to one pion with momentum θ/L and the other with momentum $(\theta-2\pi)/L$.
- The corresponding $\pi\pi$ s-wave phase-shift can then be obtained by the Lüscher formula as a function of $\theta \Rightarrow$ this allows for the derivative of the phase-shift to be evaluated directly at the masses being simulated.
- We have carried this procedure out in an exploratory calculation. Fig
- Unfortunately this technique does not work for $K \to (\pi\pi)_{I=0}$ decays.

Exploratory Evaluation of the Lellouch-Lüscher Factor

C.h. Kim and CTS, arXiv:1003.3191



$m_q = 0.004$



$m_q = 0.002$

C.h. Kim and CTS, arXiv:1003.3191

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- Calculate the $K \to \pi^+\pi^+$ matrix element with the d-quark with twisted boundary conditions with twisting angle θ.
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- We have carried this procedure out in an exploratory calculation. Fig
- Unfortunately this technique does not work for $K \to (\pi\pi)_{I=0}$ decays.

- We argued that SU(2) ChPT can be used for $K_{\ell 3}$ form factors at $q^2 = 0$ (and other q^2 where the kaon is not soft) . J.Flynn, CTS, arXiV:0809.1229.

$$\begin{aligned} f^0(0) = f^+(0) &= F_+\left(1 - \frac{3}{4}\frac{m_\pi^2}{16\pi^2 f^2}\log\left(\frac{m_\pi^2}{\mu^2}\right) + c_+ m_\pi^2\right) \\ f^-(0) &= F_-\left(1 - \frac{3}{4}\frac{m_\pi^2}{16\pi^2 f^2}\log\left(\frac{m_\pi^2}{\mu^2}\right) + c_- m_\pi^2\right). \end{aligned}$$

- It is possible to calculate the chiral logarithm because this comes from a soft internal loop.
- This idea has been extended to $K \to \pi\pi$ decays, J.Bijnens and A Celis, arXiV:0906.0302

$$A_2 = A_2^{\rm LO}\left(1 - \frac{15}{4}\frac{m_\pi^2}{16\pi^2 f^2}\log\left(\frac{m_\pi^2}{\mu^2}\right)\right) + \lambda_2\, m_\pi^2$$

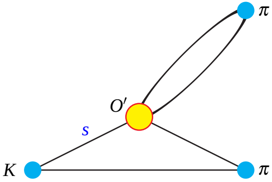
$$A_0 = A_0^{\rm LO}\left(1 - \frac{3}{4}\frac{m_\pi^2}{16\pi^2 f^2}\log\left(\frac{m_\pi^2}{\mu^2}\right)\right) + \lambda_0\, m_\pi^2,$$

(and to $B \to \pi$ and $D \to \pi$ semileptonic decays. J.Bijnens and I Jemos, arXiV:1006.1197)

It would be useful to know the results at NNLO.

- It would be reassuring to confirm that it is possible to develop an effective theory in which hard and soft pions are separated.

RBC-UKQCD, M.Lightman and E.Goode, Lattice 2010



- The RBC/UKQCD strategy at this stage is to perform the simulations on a large lattice, $L \simeq 4.5$ fm, with light pions ($32^3 \times 64 \times 32$)

$$m_\pi \simeq 145\,\text{MeV} \qquad \text{Unitary } m_\pi \simeq 180\,\text{MeV}\,.$$

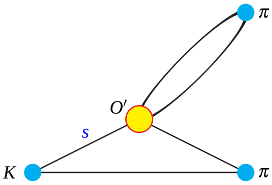
- The price is that the lattice is coarse, $a^{-1} \simeq 1.4$ GeV.
- With DWF, m_{res} increases as the coupling becomes stronger $\Rightarrow$ change the gauge action (from Iwasaki) by multiplying by the *Auxilliary Determinant* .

D.Renfrew, T.Blum, N.Christ, R.Mawhinney and P.Vranas, arXiv:0902.2587

R. Mawhinney, Lattice 2010

- This is tuned to suppress m_{res} but to maintain topology changing.

Preliminary $\Delta I = 3/2$ Matrix Elements – Cont.

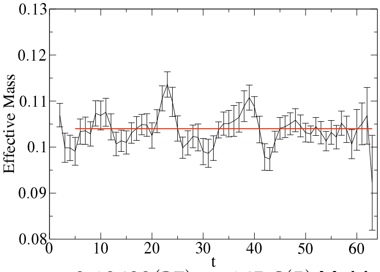


- The masses and momenta are as follows:

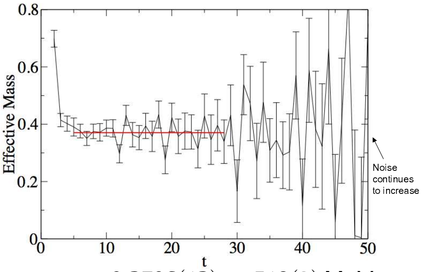
Quantity	This Calculation	Physical
m_π	145.6(5) MeV	139.6 MeV
m_K	519(2) MeV	493.7 MeV
$E_{\pi\pi}(p_\pi \simeq 0)$	294(1) MeV	
$E_{\pi\pi}(p_\pi \simeq \sqrt{2}\pi/L)$	516(9) MeV	
$E_{\pi\pi}(p_\pi \simeq \sqrt{2}\pi/L) - m_K$	-2.7(8.3) MeV	

- The results presented here were obtained with 47 configurations (plan to continue to 100) .

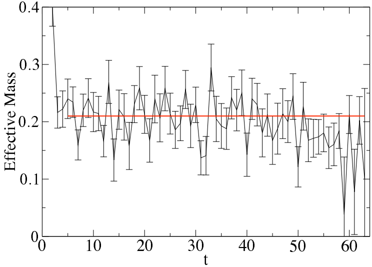
Effective Masses



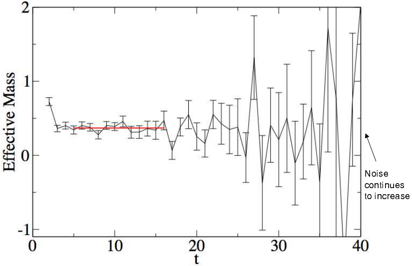
$m_\pi = 0.10400(37) \Rightarrow 145.6(5)$ MeV



$m_K = 0.3706(13) \Rightarrow 519(2)$ MeV

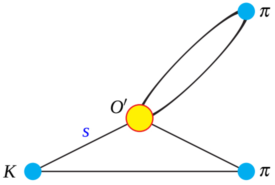


$E_{\pi\pi} = 0.2100(10) \Rightarrow 294(1)$ MeV



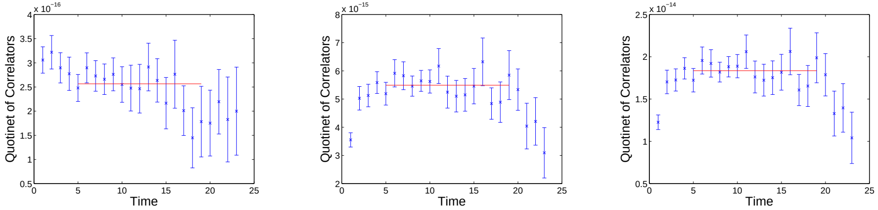
$E_{\pi\pi} = 0.3687(61) \Rightarrow 516(9)$ MeV

Effective Masses – Cont.



Source	Re(A_2) (10^{-8} GeV)
$t_K = 20$	1.52(12)
$t_K = 24$	1.52(10)
$t_K = 28$	1.71(13)
$t_K = 32$	1.35(22)
Weighted Average	1.56(7)
Experiment	1.5

Stat. error only



$O'^{3/2}_{(27,1)} = (\bar{s}d)_L\,(\bar{u}d)_L$ $\quad O'^{3/2}_{7} = (\bar{s}d)_L\,(\bar{u}d)_R$ $\quad O'^{3/2}_{8} = (\bar{s}^i d^j)_L\,(\bar{u}^j d^i)_R$

Sample plateaus for the matrix elements at matched kinematics ($p_\pi = \sqrt{2}p_{\min}$).

- We have preliminary results for the $\Delta I = 3/2\ K \to \pi\pi$ decay amplitude on 32^3 lattices with 2+1 flavours of DWF and the Iwasaki-DSDR gauge action.
- $m_\pi = 145.6(5)$ MeV, $m_K = 519(2)$ MeV, $E_{\pi\pi} = 516(9)$ MeV .
- Re $A_2 = 1.56(7)(25) \times 10^{-8}$ GeV .
  Error is dominated by lattice artefacts, a^{-3} on a coarse lattice.
- Im $A_2 = -9.6(4)(24) \times 10^{-13}$ GeV .
  In addition to lattice artefacts, we are in the process of performing the NPR for the EWP operators $O_{7,8}$ The result above is obtained by taking $Z_{ij} = 0.9(0.18)\delta ij$.
- Im A_2/Re $A_2 = -6.2(0.3)(1.3)\,10^{-5}$.
- We are confirming that these calculations are possible.
- Calculations of the $\Delta I = 1/2$ amplitudes are much more challenging - Qi Liu next talk.