Topological Gravity on the Lattice

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Gravity and gauge theory

Gravity	Gauge theory
$g_{\mu\nu}$	A_μ
$\int \sqrt{g}R(g)$	$\int F(A)^2$
General coordinate inv.	Local gauge inv.
spacetime symmetry	internal symmetry
non-renormalizable $G_N \sim l_p^2$	renormalizable $[g] = 0$

Appear to be very different beasts

Not so fast ...

- Remarkably - can forge much closer connection
- Construct class of gauge theory in which internal symmetries correspond to spacetime symmetries
  - Metric tensor related to new gauge fields associated to local translations
  - In classical limit Yang-Mills curvature identified with Riemannian curvature
  - General coordinate invariance arises as gauge symmetry.

Prime example: (2+1) gravity as a Chern-Simons theory

Chern-Simons theory

In three dimensions can write alternative to Yang-Mills

$$S_{\rm CS} = k \int_M d^3x \; \epsilon^{\mu\nu\lambda} {\rm Tr} \left(A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu \left[A_\nu A_\lambda \right] \right)$$

- Gauge invariant: under $\delta A_\mu = D_\mu \phi$ (ϕ in $su(N)$) find $\delta S_{\rm CS} = \text{boundary term}$ (Bianchi identity). (caveat: large transformations can quantize k)
- Renormalizable. In fact finite !
- $S_{\rm CS}$ independent of background metric – topological

Classical equations: $F_{\mu\nu} = 0$.

Almost trivial ... non-trivial solutions correspond to Polyakov lines

$$W(\gamma) = {\rm Tr} \mathcal{P} \int_\gamma e^{A_\mu dx^\mu}$$

3D Gravity as CS theory

Starting from CS action:

$$S_{\rm CS} = k \int d^3x \epsilon^{\mu\nu\lambda} \hat{\rm Tr} \left(A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu \left[A_\nu, A_\lambda \right] \right)$$

Choose gauge group $SO(1,3)$ (Euclidean model):

$$A_\mu = \sum_{A<B} A_\mu^{AB} \frac{1}{4} \left[\gamma^A, \gamma^B \right] \quad A, B = 1 \ldots 4$$

and use modified trace

$$\hat{\rm Tr}(X) = {\rm Tr}(\gamma_5 X)$$

This has effect of contracting indices using

$$Tr(\gamma_5 \gamma_A \gamma_B \gamma_C \gamma_D) = \epsilon_{ABCD}$$

Continuing

Decompose fields into Lorentz and translational components:

$$A_\mu = \sum_{a<b} \omega_\mu^{ab}\gamma^{ab} + \frac{1}{l}e_\mu^a\gamma^{4a} \quad a,b = 1\ldots 3$$

$$F_{\mu\nu}^{ab} = \sum_{a<b}\left(R_{\mu\nu}^{ab} + \frac{1}{l^2}e_{[\mu}^a\, e_{\nu]}^b\right)$$

$$F_{\mu\nu}^a = \frac{1}{l}\sum_a D_{[\mu}\, e_{\nu]}^a$$

Plugging into CS action:

$$S_{\mathrm{EH}} = \frac{k}{l}\int d^3x\ \epsilon^{\mu\nu\lambda}\epsilon_{abc}\left(e_\mu^a R_{\nu\lambda}^{bc} + \frac{1}{l^2}e_\mu^a e_\nu^b e_\lambda^c\right)$$

Tetrad-Palatini formulation of 3D GR!

Why is this GR ?

- Interpret e^a_μ as dreibein with $e^a_\mu e^a_\nu = g_{\mu\nu}$ (invariant under local $SO(3)$ Lorentz symmetry)
- Corresponding gauge field is spin connection ω_μ.
- First order formulation of GR – (ω, e) independent variables.
- Classical equations of motion set $F^{AB}_{\mu\nu} = 0$ or

$$\left(R^{ab}_{\mu\nu} + \frac{1}{l^2} e^a_{[\mu} e^b_{\nu]}\right) = 0$$

$$T^a_{\mu\nu} = \frac{1}{l} D_{[\mu} e^a_{\nu]} = 0$$

- Torsion free condition $T = 0$ yields $\omega = \omega(e)$
- Then curvature equation yields constant curvature space $\mathcal{H}^3$. $R = e^\mu_a e^\nu_b R^{ab}_{\mu\nu}(\omega(e)) = -\frac{1}{l^2}$ where $(e^a_\mu)^{-1} = e^\mu_a$

Caveats

- Correspondence with metric formulation requires $T^a_{\mu\nu} = 0$ and e^a_μ be invertible. Question of whether true globally depends on topology background space.
- CS path integral measure unique but in general includes such noninvertible configs. Topological phase of GR when $e = 0$?
- 3D CS very simple .. action quadratic so perturbation theory possible – finiteness.
- What determines the scale l ? Natural to associate with cosmological radius of curvature. Then $\frac{k}{l} = \frac{1}{l_p}$.

What about diffeomorphisms ?

- The gravity theory is invariant under general coordinate transformations - where are these in CS theory ?
- Latter possesses local $SO(3)$ Lorentz symmetry plus local translations (e_μ gauge field)
- Remarkably can show that on shell the gauge symmetries are equivalent to coordinate transformations!

$$\delta_\xi e^a_\mu = D_\mu(\xi^\nu A_\nu)^a - T^a_{\mu\nu}\xi_\nu$$

- Thus on flat $SO(1,3)$ background gauge invariance yields coordinate invariance!

An Alternative

- CS are hard (impossible ?) to put on the lattice.
- Argue that an alternative theory exists which shares the same moduli space and topological observables.
- This theory can be discretized and studied using eg computer simulation.
- This theory is three dimensional $\mathcal{N} = 4$ (twisted) super YM theory.

(Twisted) supersymmetric lattice theory

- Fields: hypercubic complex link fields $\mathcal{U}_\mu(x), \mu = 1, 2, 3$ plus 8 twisted fermions $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\rho})$
- χ, θ live on face and body diagonals.
- $SU(2)$ gauge symmetry. But $\mathcal{U}_\mu$ in $SL(2, C) \sim SO(1, 3)$

Action:

$$S_1 = \mathcal{Q} \sum_{\mathbf{x}} \left(\chi^{\mu\nu} \mathcal{F}_{\mu\nu} + \eta \left[\overline{\mathcal{D}}^{(-)\mu} \mathcal{U}_\mu \right] + \frac{1}{2} \eta d \right)$$

$$S_2 = \sum_{\mathbf{x}} \theta_{\mu\nu\lambda} \overline{\mathcal{D}}^{(+)\lambda} \chi^{\mu\nu}$$

Twisted supersymmetry

Scalar supercharge $\mathcal{Q}$.

$$\begin{aligned}
\mathcal{Q}\mathcal{U}_\mu &= \psi_\mu \\
\mathcal{Q}\overline{\mathcal{U}}^\mu &= 0 \\
\mathcal{Q}\psi_\mu &= 0 \\
\mathcal{Q}\chi^{\mu\nu} &= \overline{\mathcal{F}}^{\mu\nu} \\
\mathcal{Q}\eta &= d \\
\mathcal{Q}d &= 0 \\
\mathcal{Q}\theta_{\mu\nu\lambda} &= 0
\end{aligned}$$

All fields in adjoint SU(2) and transform as links.

$$\mathcal{F}_{\mu\nu} = \mathcal{U}_\mu(x)\mathcal{U}_\nu(x+\mu) - \mathcal{U}_\nu(x)\mathcal{U}_\mu(x+\nu)$$

3d Gravity ?

Moduli space YM model:

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= 0 \\ \overline{\mathcal{D}}^{(-)\mu}\mathcal{U}_\mu &= 0 \quad \mathrm{mod\ SU(2)\ GT} \end{aligned}$$

Moduli space CS theory:

$$\mathcal{F}_{\mu\nu} = 0 \quad \mathrm{mod\ SL(2,C)\ GT}$$

Same! (Marcus)

Furthermore:

- Since $\mathcal{Q}\overline{\mathcal{U}}_\mu = 0$ Polyakov lines in YM will be topological - independent of metric and coupling constant in continuum.
- These are same operators as appeared in CS gravity!

Conjecture

Plausible:

Twisted SUSY theory gives alternative description of CS gravity

Bonus:

- YM action real, positive semi-definite.
- Only compact gauge symmetry. Measure well defined
- Lattice formulation possible preserving $\mathcal{Q}$-invariance (and no doubles, G.I etc)
- It may hence offer an alternative non-perturbative formulation of 3d gravity.

Conclusions

- Gauge theoretic approaches to gravity offer alternative ways to think about (discrete) quantum gravity
- 2+1 Chern Simons formulation is best understood example.
- We argue that this theory is equivalent to the topological sector of a twisted supersymmetric YM theory.
- The latter admits a gauge invariant and supersymmetric lattice regularization preserving the topological sector.

Thoughts for the future

- CS formulations of gravity exist in any odd dim.
- Eg in 5D
  - $S = \int_{M^5} \hat{\mathrm{Tr}}\left(A \wedge F \wedge F + \ldots\right)$
  - $A = (\omega, e)$ takes values in $SO(5,1)$
  - Related to some twisted 5D theory ? AdS/CFT ?
- Compactify 5D CS on interval - 4D EH gravity in Palatini form
- Discretize 4D? Possible. $S = \int_{M^4} \hat{\mathrm{Tr}}\left(\gamma_5 F \wedge F\right)$ Simulations - tricky - unbounded action !
- Supergravity: CS SUGRAs possible: $A = A^{ab} J^{ab} + \psi^\alpha Q^\alpha$ Possess local SUSY. Route to lattice supergravity ?