Renormalization of minimally doubled fermions

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Introduction

Minimally doubled fermions (2 flavors):
realize the minimal doubling allowed by the Nielsen-Ninomiya theorem

Preserve an exact chiral symmetry for a degenerate doublet of quarks

chiral symmetry protects mass renormalization

Remain at the same time also strictly local

fast for simulations

A cost-effective realization of chiral symmetry at nonzero lattice spacing

We can construct a conserved axial current, which has a simple expression

Compared with staggered fermions:

- same kind of $U(1) \otimes U(1)$ chiral symmetry
- 2 flavors instead of 4
  $\Rightarrow$ no uncontrolled extrapolation to 2 physical light flavors
- no complicated intertwining of spin and flavor

Introduction

Ideal for $N_f = 2$ simulations: no rooting needed!

Much cheaper and simpler than Ginsparg-Wilson fermions
(overlap, domain-wall, fixed-point)

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Minimally doubled fermions: *'new'* . . . but also *'old'*

Revival in the last 2 years – initiated by studies on graphenes by Creutz , in December 2007

Here we consider two realizations of minimally doubled fermions:
Boriçi-Creutz and Karsten-Wilczek fermions – and derive their Feynman rules

We then compute the self-energy of the quark and the renormalization of the Dirac bilinears

Mixings of a new kind arise, as a consequence of the breaking of the hypercubic symmetry $\rightarrow$ preferred direction in euclidean spacetime

One of the main aims of our work is the investigation of the mixing patterns that appear in radiative corrections

Introduction

We also construct the **conserved** vector and **axial** currents

They have simple expressions which involve only nearest-neighbors sites

One of the very few lattice discretizations in which one can give a simple expression (and ultralocal) for a conserved axial current

This conserved axial current is even ultralocal

These features could turn out to be very useful also in numerical simulations

We also compute the vacuum polarization of the gluon

Here the breaking of hypercubic symmetry does not generate any kind of power divergences

In principle these divergences could have arisen with a $1/a^2$ or $1/a$ dependency

All this is also an example of the usefulness of perturbation theory in helping to unfold theoretical aspects of (new) lattice formulations

Boriçi-Creutz fermions

Boriçi and Creutz: fermionic action with the free Dirac operator (in momentum space)

$$D(p) = i \sum_\mu (\gamma_\mu \sin p_\mu + \gamma'_\mu \cos p_\mu) - 2i\Gamma + m_0$$

where

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \qquad (\Gamma^2 = 1)$$

and

$$\gamma'_\mu = \Gamma \gamma_\mu \Gamma = \Gamma - \gamma_\mu$$

Useful relations:

$$\sum_\mu \gamma_\mu = \sum_\mu \gamma'_\mu = 2\Gamma, \quad \{\Gamma, \gamma_\mu\} = 1, \quad \{\Gamma, \gamma'_\mu\} = 1$$

The action vanishes at $p_1 = (0,0,0,0)$ and $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$

A linear combination of two (physically equivalent) naive fermions , corresponding to the first two terms in the action

$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$ selects a special direction $\rightarrow$ hypercubic breaking

Karsten-Wilczek fermions

Already in the Eighties: Karsten (1981) and then Wilczek (1987) proposed some particular kind of minimally doubled fermions

Unitary equivalent to each other, after phase redefinitions

Wilczek [PRL 59, 2397 (1987)] proposed a special choice of the function $P_\mu(p)$ which minimizes the numbers of doublers

The free Karsten-Wilczek Dirac operator

$$D(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i\gamma_4 \sum_{k=1}^{\mathbf{3}} (1 - \cos p_k)$$

has zeros at $p_1 = (0,0,0,0)$ and $p_2 = (0,0,0,\pi)$

Drawback: it destroys the equivalence of the four directions under discrete permutations

$\rightarrow$ breaking of the hypercubic symmetry

Hypercubic breaking

The actions of minimally doubled fermions have two zeros

$\Rightarrow$ there is always a special direction in euclidean space (given by the line that connects these two zeros)

Thus, these actions cannot maintain a full hypercubic symmetry

They are symmetric only under the **subgroup** of the hypercubic group which preserves (up to a sign) a **fixed direction**

For the Boriçi-Creutz action this is a major hypercube diagonal, while for other minimally doubled actions it may not be a diagonal – for example for the Karsten-Wilczek action is the x_4 axis

Although the distance between the 2 Fermi points is the same ($p_2^2 - p_1^2 = \pi^2$), these two realization of minimally doubled fermions are not equivalent

The breaking of the hypercubic symmetry implies the appearance of mixings with operators of different dimensionality, like $\overline{\psi}\Gamma\psi$ or $\overline{\psi}\Gamma D^2\psi$

For minimally doubled fermions a mixing with dimension-3 operators cannot be avoided (*Bedaque, Buchoff, Tiburzi and Walker-Loud*)

Propagators, vertices, . . .

Quark propagator for Boriçi-Creutz fermions:

$$S(p) = a\,\frac{-i\sum_\mu\left[\gamma_\mu \sin ap_\mu - 2\,\gamma'_\mu\ \sin^2 ap_\mu/2\right] + am_0}{4\sum_\mu\left[\sin^2 ap_\mu/2 + \sin ap_\mu\left(\sin^2 ap_\mu/2 - \frac{1}{2}\sum_\nu \sin^2 ap_\nu/2\right)\right] + (am_0)^2}$$

The second pole at $ap = (\pi/2, \pi/2, \pi/2, \pi/2)$ describes (as expected) a particle of opposite chirality to the one at $ap = (0, 0, 0, 0)$

Quark propagator for Karsten-Wilczek fermions (2nd pole at $ap = (0, 0, 0, \pi)$) :

$$S(p) = a\,\frac{-i\sum_{\mu=1}^{4}\gamma_\mu \sin ap_\mu - 2i\,\gamma_4\sum_{k=1}^{3}\sin^2\frac{ap_k}{2} + am_0}{\sum_{\mu=1}^{4}\sin^2 ap_\mu + 4\sin ap_4\sum_{k=1}^{3}\sin^2\frac{ap_k}{2} + 4\left(\sum_{k=1}^{3}\sin^2\frac{ap_k}{2}\right)^2 + (am_0)^2}$$

Quark-quark-gluon and quark-quark-gluon-gluon vertices (Boriçi-Creutz):

$$V_1(p_1, p_2) = -ig_0\left(\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} - \gamma'_\mu \sin\frac{a(p_1+p_2)_\mu}{2}\right)$$

$$V_2(p_1, p_2) = \frac{1}{2}iag_0^2\left(\gamma_\mu \sin\frac{a(p_1+p_2)_\mu}{2} + \gamma'_\mu \cos\frac{a(p_1+p_2)_\mu}{2}\right)$$

. . .

Counterterms

Each of these two bare actions does not contain all possible operators allowed by the respective symmetries (broken hypercubic group)

Radiative corrections generate new contributions whose form is not matched by any term in the original bare actions

Counterterms are then necessary for a consistent renormalized theory

This consistency requirement will uniquely determine their coefficients

Our task: add to the bare actions all possible counterterms allowed by the remaining symmetries (after hypercubic symmetry has been broken)

They are lattice artefacts peculiar to minimally doubled fermions

In the following we will consider the massless case $m_0 = 0$

Chiral symmetry strongly restricts the number of possible counterterms

For Boriçi-Creutz fermions, operators are allowed where summations over just single indices are present (in addition to the standard Einstein summation over two indices)

Then objects like $\sum_\mu \gamma_\mu = \Gamma$ appear

Counterterms

We find that there can be only one dimension-4 counterterm: $\overline{\psi}\,\Gamma \sum_\mu D_\mu \psi$

Possible discretization: form similar to the hopping term in the action

$$c_4(g_0)\ \frac{1}{2a} \sum_\mu \Big(\overline{\psi}(x)\,\Gamma\, U_\mu(x)\,\psi(x+a\widehat{\mu}) - \overline{\psi}(x+a\widehat{\mu})\,\Gamma\, U_\mu^\dagger(x)\,\psi(x) \Big)$$

There is also one counterterm of dimension three: $\frac{ic_3(g_0)}{a}\ \overline{\psi}(x)\,\Gamma\,\psi(x)$

This is already present in the bare action, but with a fixed coefficient , $-2/a$

The appearance of this counterterm means that in the general renormalized action the coefficient of this operator must be kept general

For Karsten-Wilczek fermions we find an analogous situation

Here objects are allowed in which we constrain any index to be equal to 4

Only gauge-invariant counterterm of dimension four: $\overline{\psi}\,\gamma_4 D_4\,\psi$

A suitable discretization:

$$d_4(g_0)\ \frac{1}{2a} \Big(\overline{\psi}(x)\,\gamma_4\, U_4(x)\,\psi(x+a\widehat{4}) - \overline{\psi}(x+a\widehat{4})\,\gamma_4\, U_4^\dagger(x)\,\psi(x) \Big)$$

Counterterms

There is also one counterterm of dimension three, $\boxed{\frac{id_3(g_0)}{a}\ \overline{\psi}(x)\,\gamma_4\,\psi(x)}$

(already present in the bare Karsten-Wilczek action, with a fixed coefficient)

In perturbation theory the coefficients of all these counterterms are functions of the coupling which start at order g_0^2

They give rise at one loop to additional contributions to fermion lines

The rules for the corrections to fermion propagators, needed for our one-loop calculations, can be easily derived

For external lines, they are given in momentum space respectively by

$$-ic_4(g_0)\ \Gamma \sum_\nu p_\nu, \quad -\frac{ic_3(g_0)}{a}\,\Gamma$$

for Boriçi-Creutz fermions, and by

$$-id_4(g_0)\ \gamma_4\, p_4, \quad -\frac{id_3(g_0)}{a}\,\gamma_4$$

for Karsten-Wilczek fermions

Counterterms

We will determine all these coefficients (at one loop) by requiring that the renormalized self-energy assumes its standard form

Counterterm interaction vertices are generated as well

These vertex insertions are at least of order g_0^3, and thus they cannot contribute to the one-loop amplitudes that we study here

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The form of the counterterms remains the same at all orders of perturbation theory

Only the values of their coefficients change according to the loop order

The same counterterms appear at the nonperturbative level, and will be required for a consistent numerical simulation of these fermions

We also want to emphasize that counterterms not only provide additional Feynman rules for the calculation of loop amplitudes

They can modify Ward identities and hence, in particular, contribute additional terms to the conserved currents

Self-energy

The quark self-energy (without counterterms) of a Boriçi-Creutz fermion is given at one loop by

$$\Sigma(p,m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + c_1(g_0)\cdot i\,\Gamma\sum_\mu p_\mu + c_2(g_0)\cdot i\,\frac{\Gamma}{a}$$

with

$$\Sigma_1(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2+6.80663+(1-\alpha)\Big(-\log a^2p^2+4.792010\Big)\right]+O(g_0^4)$$

$$\Sigma_2(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2-29.48729+(1-\alpha)\Big(-\log a^2p^2+5.792010\Big)\right]+O(g_0^4)$$

$$c_1(g_0) = 1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F+O(g_0^4)$$

$$c_2(g_0) = 29.54170\cdot\frac{g_0^2}{16\pi^2}\,C_F+O(g_0^4)$$

The full inverse propagator at one loop can be written (*without counterterms*) as

$$\Sigma^{-1}(p,m_0) = \Big(1-\Sigma_1\Big)\cdot\left\{i\not{p}+m_0\Big(1-\Sigma_2+\Sigma_1\Big)-\frac{ic_1}{2}\sum_\mu\gamma_\mu\sum_\nu p_\nu-\frac{ic_2}{a}\,\Gamma\right\}$$

Self-energy

We can only cast the renormalized propagator in the standard form

$$\Sigma(p, m_0) = \frac{Z_2}{i\not{p} + Z_m\, m_0}$$

with the wave-function and quark mass renormalization given by

$$Z_2 = \Big(1 - \Sigma_1\Big)^{-1}, \qquad Z_m = 1 - \Big(\Sigma_2 - \Sigma_1\Big)$$

if we use counterterms to cancel the Lorentz non-invariant factors (c_1 and c_2)

The term proportional to $c_1(g_0)$ can be eliminated by using the counterterm of the form

$$\overline{\psi} \sum_\mu \gamma_\mu \sum_\nu D_\nu\, \psi$$

The term proportional to $c_2(g_0)$ can be eliminated by the counterterm

$$\frac{1}{a}\, \overline{\psi}\, \Gamma\, \psi$$

For Boriçi-Creutz fermions we then determine at one loop

$$c_3(g_0) = 29.54170 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

$$c_4(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

Self-energy

Full inverse propagator (without counterterms) for Karsten-Wilczek fermions:

$$\Sigma^{-1}(p,m_0) = \Big(1-\Sigma_1\Big)\cdot\Big(i\not{p} + m_0\Big(1-\Sigma_2+\Sigma_1\Big) - id_1\,\gamma_4 p_4 - \frac{id_2}{a}\,\gamma_4\Big)$$

with

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2 + 9.24089 + (1-\alpha)\Big(-\log a^2p^2 + 4.792010\Big)\right]$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2 - 24.36875 + (1-\alpha)\Big(-\log a^2p^2 + 5.792010\Big)\right]$$

$$d_1(g_0) = -0.12554\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4), \qquad d_2(g_0) = -29.53230\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

Similarly to before, by adding to the Karsten-Wilczek action counterterms of the form

$$\overline{\psi}\,\gamma_4\,D_4\,\psi, \qquad \frac{1}{a}\,\overline{\psi}\,\gamma_4\,\psi$$

the renormalized propagator can be written in the standard form

Then, at one loop

$$d_3(g_0) = -29.53230\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4), \qquad d_4(g_0) = -0.12554\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

Local bilinears

No new mixings for the scalar (pseudoscalar) density and the tensor current

The vertex diagram of the vector current for Boriçi-Creutz fermions gives

$$\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2p^2 + 9.54612 + (1-\alpha)\Big(\log a^2p^2 - 4.792010 \Big) \right] + c_1^v(g_0)\,\Gamma$$

with the coefficient of the mixing given by

$$c_1^v(g_0) = -0.10037 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

(axial current: the numbers are the same)

Vector current of Karsten-Wilczek fermions:

$$\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2p^2 + 10.44610 - \delta_{\mu 4} \cdot 2.88914 + (1-\alpha)\Big(\log a^2p^2 - 4.792010 \Big) \right]$$

The spatial and temporal components of the vector (as well the axial) current receive different radiative corrections!

Cross-mixings between the spatial and temporal components appear to be absent – each of these components still renormalizes multiplicatively

Conserved vector and axial currents

Z_V and Z_A (of the local currents) are not equal to one

The local vector and axial currents are not conserved

We need to consider the chiral Ward identities in order to work with currents which are protected from renormalization

We have constructed the conserved vector and axial currents, and verified that at one loop their renormalization constants are equal to one

We act on the Boriçi-Creutz action in position space

$$S = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_\mu \Big[\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) - \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x) \Big] + \overline{\psi}(x)\left(m_0 - \frac{2i\Gamma}{a}\right)\psi(x) \Bigg]$$

with the vector transformation

$$\delta_V \psi = i\alpha\,\psi, \quad \delta_V \overline{\psi} = -i\alpha\,\overline{\psi}$$

or the axial transformation

$$\delta_A \psi = i\alpha\,\gamma_5 \psi, \quad \delta_A \overline{\psi} = i\alpha\,\overline{\psi}\gamma_5$$

Conserved vector and axial currents

We then obtain the conserved vector current for Boriçi-Creutz fermions as

$$V_\mu^{cons}(x) = \frac{1}{2}\Big[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,U_\mu^\dagger(x)\,\psi(x)\Big]$$

while the axial current (conserved in the case $m_0 = 0$) is

$$A_\mu^{cons}(x) = \frac{1}{2}\Big[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,\gamma_5\,U_\mu^\dagger(x)\,\psi(x)\Big]$$

We have computed the renormalization of these point-split currents

The sum of vertex, sails and operator tadpole gives *(in the vector case)*

$$\frac{g_0^2}{16\pi^2}\,C_F\,\gamma_\mu\left[-\log a^2p^2 - 6.80664 + (1-\alpha)\Big(\log a^2p^2 - 4.79202\Big)\right] \quad + c_1^{cv}(g_0)\,\Gamma$$

where the coefficient of the mixing is $c_1^{cv}(g_0) = -1.52766 \cdot \frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$

The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

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The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

But what about the mixing term, proportional to Γ ?

We should take into account the counterterms . . .

Conserved vector and axial currents

The counterterm $\overline{\psi}(x)\,\frac{i\Gamma}{a}\,\psi(x)$ does **not** modify the Ward identities

On the contrary, the counterterm

$$\frac{c_4(g_0)}{4}\sum_\mu\sum_\nu\Big(\overline{\psi}(x)\,\gamma_\nu\,U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\,\gamma_\nu\,U^\dagger_\mu(x)\,\psi(x)\Big)$$

generates new terms in the Ward identities and then in the conserved currents

The additional term in the conserved vector current so generated reads

$$\frac{c_4(g_0)}{4}\left[\overline{\psi}(x)\Big(\sum_\nu\gamma_\nu\Big)U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\Big(\sum_\nu\gamma_\nu\Big)U^\dagger_\mu(x)\,\psi(x)\right]$$

Its 1-loop contribution is easy to compute (c_4 *is already of order* g_0^2 !): $\underline{c_4(g_0)\,\Gamma}$

The value of c_4 is known from the self-energy $\Rightarrow$ $c_4(g_0)\,\Gamma \;=\; -c_1^{cv}(g_0)\,\Gamma$

Only this value of c_4 exactly cancels the Γ mixing term present in the 1-loop conserved current without counterterms

Thus, we obtain that the renormalization constant of these point-split currents is one – which confirms that they are conserved currents

Everything is consistent. . .

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$S = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu(x) \, \psi(x + a\widehat{\mu})$$

$$- \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu^\dagger(x) \, \psi(x) \Big] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \Bigg]$$

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$S = a^4 \sum_x \Big[\frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) U_\mu(x)\, \psi(x+a\widehat{\mu})$$

$$-\overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) U_\mu^\dagger(x)\, \psi(x) \Big] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \Big]$$

For Karsten-Wilczek fermions, application of the chiral Ward identities gives for the conserved axial current

$$A_\mu^{\rm c}(x) = \frac{1}{2} \Big(\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu(x)\, \psi(x+a\widehat{\mu})$$

$$+\overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu^\dagger(x)\, \psi(x) \Big)$$

$$+\frac{d_4(g_0)}{2} \Big(\overline{\psi}(x)\, \gamma_4\gamma_5\, U_4(x)\, \psi(x+a\widehat{4}) + \overline{\psi}(x+a\widehat{4})\, \gamma_4\gamma_5\, U_4^\dagger(x)\, \psi(x) \Big)$$

Once more, is a simple expression which involve only nearest-neighbour sites

We checked explicitly that its renormalization constant is one

Vacuum polarization

For Boriçi-Creutz fermions (without gluonic counterterm): ($\mathrm{Tr}\,(\mathrm{t^a t^b}) = \mathrm{C}_2\,\delta^{\mathrm{ab}}$)

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 23.6793\right)\right]$$

$$-\left((p_\mu + p_\nu)\sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu}\left(\sum_\lambda p_\lambda\right)^2\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 0.9094$$

For Karsten-Wilczek fermions (without gluonic counterterm):

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 19.99468\right)\right]$$

$$-\left(p_\mu p_\nu\,(\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu}\,\left(p^2\,\delta_{\mu 4}\delta_{\nu 4} + p_4^2\right)\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 12.69766$$

There are new terms, compared with a standard situation like Wilson fermions

Although each of these actions breaks hypercubic symmetry in its appropriate and peculiar way, these new terms still satisfy the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$

Very important: there are no power-divergences ($1/a^2$ or $1/a$) in our results for the vacuum polarization!

Gluonic counterterms

We need counterterms also for the pure gauge part of the actions of minimally doubled fermions

Although at the bare level the breaking of hypercubic symmetry is a feature of the fermionic actions only, in the renormalized theory it propagates (*via the interactions between quarks and gluons*) also to the pure gauge sector

These counterterms must be of the $\mathrm{tr}\, FF$ form, but with nonconventional choices of the indices, reflecting the breaking of the hypercubic symmetry

Only purely gluonic counterterm possible for the Boriçi-Creutz action:

$$c_P(g_0) \sum_{\lambda\rho\tau} \mathrm{tr}\, F_{\lambda\rho}(x)\, F_{\rho\tau}(x)$$

At one loop this counterterm is relevant only for gluon propagators

Denoting the fixed external indices at both ends with μ and ν, all possible lattice discretizations of this counterterm give in momentum space the same Feynman rule:

$$-c_P(g_0) \left[(p_\mu + p_\nu) \sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu} \Big(\sum_\lambda p_\lambda \Big)^2 \right]$$

The presence of this counterterm is essential for the correct renormalization of the vacuum polarization

Gluonic counterterms

It is not hard to imagine that in the case of Karsten-Wilczek fermions the temporal plaquettes will be renormalized differently from the other plaquettes

Indeed, the counterterm to be introduced contains an asymmetry between these two kinds of plaquettes, and can be written in continuum form as

$$d_P(g_0) \sum_{\rho\lambda} \mathrm{tr}\, F_{\rho\lambda}(x)\, F_{\rho\lambda}(x)\, \delta_{\rho 4}$$

This is the only purely gluonic counterterm needed for this action, since introducing also a $\delta_{\lambda 4}$ in the above expression will produce a vanishing object

It is immediate to write a lattice discretization for it, using the plaquette:

$$d_P(g_0)\, \frac{\beta}{2} \sum_{\rho\lambda} \Big(1 - \frac{1}{N_C}\, \mathrm{tr}\, P_{4\lambda}(x)\Big)$$

The Feynman rule for this counterterm reads

$$-d_P(g_0)\, \Big[p_\mu p_\nu\, (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu}\, \Big(p^2\, \delta_{\mu 4}\delta_{\nu 4} + p_4^2\Big)\Big]$$

and again is needed in the vacuum polarization

Gluonic counterterms

The cancellation of the hypercubic breaking terms of the vacuum polarization determines

$$c_P(g_0) = -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

$$d_P(g_0) = -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

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All counterterms remain of the same form at all orders of perturbation theory

Only the values of their coefficients depend on the number of loops

The same counterterms appear at the nonperturbative level, and will be required for a consistent simulation of these fermions

We would now like to see how the one-loop calculations presented so far can shed light on numerical simulations of minimally doubled fermions

These simulations will have to employ the complete renormalized actions (in position space), including the counterterms

We can write the renormalized actions as follows:

On the simulations

For Boriçi-Creutz fermions

$$
\begin{aligned}
S^f_{BC} \; = \; & a^4 \sum_x \Bigg\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x)\, (\gamma_\mu + c_4(\beta)\,\Gamma + i\gamma'_\mu)\, U_\mu(x)\, \psi(x + a\widehat{\mu}) \\
& - \overline{\psi}(x + a\widehat{\mu})\, (\gamma_\mu + c_4(\beta)\,\Gamma - i\gamma'_\mu)\, U^\dagger_\mu(x)\, \psi(x) \Big] \\
& + \overline{\psi}(x) \left(m_0 + \widetilde{c}_3(\beta)\, \frac{i\,\Gamma}{a} \right) \psi(x) \\
& + \beta \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\, P_{\mu\nu} \right) + c_P(\beta) \sum_{\mu\nu\rho} \mathrm{tr}\, F^{lat}_{\mu\rho}(x)\, F^{lat}_{\rho\nu}(x) \Bigg\}
\end{aligned}
$$

We have redefined the coefficient of the dimension-3 counterterm, using $\widetilde{c}_3(\beta) = -2 + c_3(\beta)$ (which does not vanish at tree level)

F^{lat} is some lattice discretization of the field-strength tensor

On the simulations

The renormalized action for Karsten-Wilczek fermions reads

$$S^f_{KW} = a^4 \sum_x \Big\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x)\, (\gamma_\mu (1 + d_4(\beta)\,\delta_{\mu 4}) - i\gamma_4\,(1-\delta_{\mu 4}))\, U_\mu(x)\, \psi(x + a\widehat{\mu})$$

$$-\overline{\psi}(x + a\widehat{\mu})\, (\gamma_\mu (1 + d_4(\beta)\,\delta_{\mu 4}) + i\gamma_4\,(1-\delta_{\mu 4}))\, U^\dagger_\mu(x)\, \psi(x) \Big]$$

$$+\overline{\psi}(x) \left(m_0 + \widetilde{d}_3(\beta)\, \frac{i\,\gamma_4}{a} \right) \psi(x)$$

$$+\beta \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{tr}\, P_{\mu\nu} \right) \left(1 + d_P(\beta)\,\delta_{\mu 4} \right) \Big\}$$

where $\widetilde{d}_3(\beta) = 3 + d_3(\beta)$ has a non-zero value at tree level

In perturbation theory the coefficients of the counterterms have the expansions

$$\widetilde{c}_3(g_0) = -2 + c_3^{(1)} g_0^2 + c_3^{(2)} g_0^4 + \ldots; \qquad \widetilde{d}_3(g_0) = 3 + d_3^{(1)} g_0^2 + d_3^{(2)} g_0^4 + \ldots$$

$$c_4(g_0) = c_4^{(1)} g_0^2 + c_4^{(2)} g_0^4 + \ldots; \qquad d_4(g_0) = d_4^{(1)} g_0^2 + d_4^{(2)} g_0^4 + \ldots$$

$$c_P(g_0) = c_P^{(1)} g_0^2 + c_P^{(2)} g_0^4 + \ldots; \qquad d_P(g_0) = d_P^{(1)} g_0^2 + d_P^{(2)} g_0^4 + \ldots$$

On the simulations

In perturbation theory the four-dimensional counterterm to the fermionic action is necessary for the proper construction of the conserved currents

Its coefficient, as determined from the one-loop self-energy, has exactly the right value for which the conserved currents remain unrenormalized

One possible nonperturbative determination of c_4 (and d_4): require that the electric charge is one , using the (unrenormalized) conserved currents

Another effect of radiative corrections is to move the poles of the quark propagator away from their tree-level positions

It is the task of the dimension-3 counterterm, for the appropriate value of the coefficient c_3 (or d_3), to bring the two poles back to their original locations

These shifts can introduce oscillations in some hadronic correlation functions *(similarly to staggered fermions)*

One possible way to determine c_3 (d_3): tune it in appropriate correlation functions until these oscillations are removed

No sign problem for the Monte Carlo generation of configurations: the gauge action is real, and the eigenvalues of the Dirac operator come in complex conjugate pairs $\longrightarrow$ fermion determinant always non-negative

On the simulations

The purely gluonic counterterm for Boriçi-Creutz fermions introduces in the renormalized action operators of the kind $E \cdot B$, $E_1 E_2$, $B_2 B_3$ *(and similar)*

In a Lorentz invariant theory, instead, only the terms E^2 and B^2 are allowed

Fixing the coefficient c_P could then be done by measuring $\langle E \cdot B \rangle$, $\langle E_1 E_2 \rangle$, $\cdots$, and tuning c_P in such a way that one (or more) of these expectation values is restored to its proper value pertinent to a Lorentz invariant theory, i.e. zero

These effects could turn out to be rather small , given that in the tree-level action only the fermionic part breaks the hypercubic symmetry

It could also be that other derived quantities are more sensitive to this coefficient, and more suitable for its nonperturbative determination

In general one can look for Ward identities in which violations of the standard Lorentz invariant form, as functions of c_P, occur

For Karsten-Wilczek fermions the purely gluonic counterterm introduces an asymmetry between the plaquettes with a temporal index and the other ones

One can then fix d_P by computing a plaquette or Wilson loop lying entirely in two spatial directions, and then equating its result to an ordinary plaquette or Wilson loop which also extends in the time direction

On the simulations

At the end only Monte Carlo simulations can reveal the actual amount of symmetry breaking

This could turn out to be large or small according to the observable considered

One important such quantity is the mass splitting of the charged pions relative to the neutral pion

One must be a bit careful : there is only a $U(1) \otimes U(1)$ chiral symmetry

Consequence: π^0 is massless, as the unique Goldstone boson (for $m_0 \rightarrow 0$), but π^+ and π^- are massive

Furthermore, the magnitude of these symmetry-breaking effects could turn out to be substantially different for Boriçi-Creutz compared to Karsten-Wilczek fermions

Thus, one of these two actions could in this way be raised to become the preferred one for numerical simulations

On the improvement

$$D^f_{\mathrm{Wilson}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) - ar\sum_{\mu=1}^{4}\nabla^*_\mu\nabla_\mu\right\}$$

$$D^f_{\mathrm{BC}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) + ia\sum_{\mu=1}^{4}\gamma'_\mu\,\nabla^*_\mu\nabla_\mu\right\}$$

$$D^f_{\mathrm{KW}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) - ia\gamma_4\sum_{k=1}^{3}\nabla^*_k\nabla_k\right\}$$

where $\nabla_\mu\psi(x) = \frac{1}{a}\left[U_\mu(x)\,\psi\,(x+a\widehat{\mu}) - \psi(x)\right]$ is the nearest-neighbor forward covariant derivative, and ∇^*_μ the corresponding backward one

All these three formulations contain a dimension-5 operator in the bare action $\rightarrow$ we expect leading lattice artefacts to be of order a

Additional dimension-5 operators occur not only in the quark sector (*e.g.*, $\overline{\psi}\,\Gamma\sum_{\mu\nu}D_\mu D_\nu\psi$), but also in the pure gauge part (*e.g.*, $\sum_{\mu\nu\lambda}F_{\mu\nu}D_\lambda F_{\mu\nu}$)

When Lorentz invariance is broken, the statement that only operators with even dimension can appear in the pure gauge action is no longer true

Conclusions

- Boriçi-Creutz and Karsten-Wilczek fermions are described by a fully consistent renormalized quantum field theory
- Three counterterms need to be added to the bare actions
- All their coefficients can be calculated in perturbation theory – or nonperturbatively from Monte Carlo simulations
- After these subtractions are consistently taken into account, the power divergence in the self-energy is eliminated
- No other power divergences occur for all quantities that we calculated
- Scalar, pseudoscalar and tensor operators show no new mixings at all
- Local vector and axial currents mix with new operators which are not invariant under the hypercubic group
- The vacuum polarization does not present new divergences
- Leading lattice artefacts seem to be of order a

- Conserved vector and axial currents can be defined, and they involve only nearest-neighbors sites
  - they do not have mixings, and their renormalization constant is one
  - one of the very few cases where one can define a simple conserved axial current (also ultralocal)