QCD at Finite Density and the Phase of the Fermion Determinant

Jacobus Verbaarschot

jacobus.verbaarschot@stonybrook.edu

Stony Brook University

Acknowledgments

Collaborators: Kim Splittorff (NBI)

Lorenzo Ravagli (Bank, Paris)

James Osborn (Argonne National Laboratory)

Maria-Paula Lombarda (Frascati)

Poul Damgaard (NBIA)

Financial Support: Stony Brook University

US Department of Energy

References

K. Splittorff and J. J. M. Verbaarschot, Phase of the fermion determinant at nonzero chemical potential, Phys. Rev. Lett. **98**, 031601 (2007) [arXiv:hep-lat/0609076].

L. Ravagli and J. J. M. Verbaarschot, QCD in One Dimension at Nonzero Chemical Potential, Phys. Rev. D **76**, 054506 (2007) [arXiv:0704.1111 [hep-th]].

J.C. Osborn, K. Splittorff and J.J.M. Verbaarschot, Phase Diagram of the Dirac Spectrum at Nonzero Chemical Potential, Phys. Rev. D **78**, 105006 (2008) [arXiv:0807.4584 [hep-lat]].

M.-P. Lombardo, K. Splittorff and J.J.M. Verbaarschot, Distributions of the Phase Angle of the Fermion Determinant in QCD, Phys. Rev. D **80**, 054509 (2009) [arXiv:0904.2122 [hep-lat]].

M.-P. Lombardo, K. Splittorff and J.J.M. Verbaarschot, The Fluctuations of the Quark Number and the Chiral Condensate, Phys. Rev. D **81**, 045012 (2010) [arXiv:0910.5482 [hep-lat]].

M.-P. Lombardo, K. Splittorff and J.J.M. Verbaarschot, Teflon Plated Observables (2010).

Contents

I. Motivation

Questions

QCD Phase Diagram

Issues and Questions

- QCD at nonzero chemical potential has a sign problem and an overlap problem.
- Can we quantify the sign problem and overlap problem, and determine it dependence on the parameters of the phase diagram?
- Are there regions of phase space or observables for which these problems become manageable?
- Will it ever be possible to access interesting physics related to the existence of a Fermi surface by lattice QCD methods?
- Is the sign problem a fundamental problem rather than a technical problem? ‘

QCD Partition Function

The QCD partition at temperature $1/\beta$ and quark chemical potential μ is given by

$$Z_{\rm QCD}(\mu,\beta) \quad = \quad \sum_k e^{-\beta(E_k - \mu N_k)},$$

where the sum is over all states with energy E_k and quark number N_k .

- Because of charge conjugation symmetry, $Z_{\rm QCD}(\mu,\beta)$ is an even function of μ .
- $Z_{\rm QCD}(\mu,\beta)$ is expected to have a well-defined high-temperature expansion in powers of μ^2/T^2 .
- Interesting effects related to the formation of a Fermi-sphere cannot be obtained from this expansion.
- This partition function can be rewritten as a Euclidean quantum field theory

$$Z_{\rm QCD} \quad = \quad \langle \prod_k^{N_f} \det(D + m_k + \mu_k \gamma_0) \rangle_{\rm YM}.$$

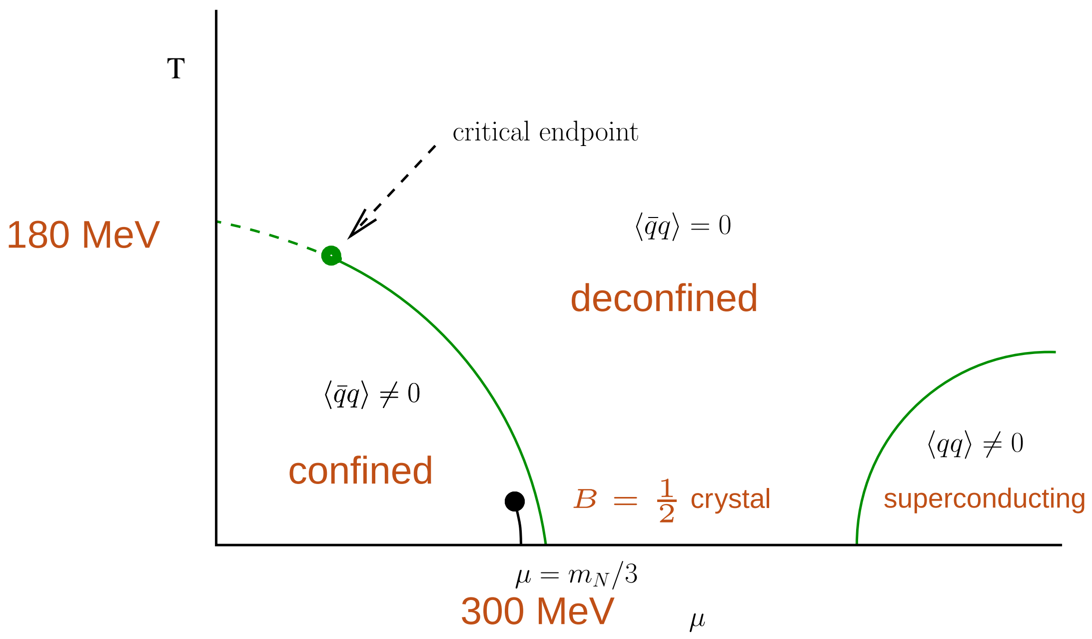
Dirac operator

quark mass matrix

imaginary vector potential

QCD Phase Diagram

The high temperature expansion of the free energy can be obtained by a Taylor expansion (Allton-et-al-2003, Gavai-Gupta-2003), reweighting (Fodor-Katz-2002) or from an extrapolation from imaginary μ (de Forcrand-Philipsen-2002, D'Elia-Lombardo-2002).



Schematic QCD phase diagram.

To get access to physics related to the formation of a Fermi surface one has to confront the sign problem.

III. Sign Problem

Average Phase Factor

Distribution of the Phase

Sign Problem for $\mu \neq 0$

Because the Dirac operator at nonzero μ is nonhermitean, the fermion determinant is complex

$$\det(D + \mu\gamma_0 + m) = e^{i\theta} |\det(D + \mu\gamma_0 + m)|.$$

The $fundamental$ problem is that the average phase factor may vanish in the thermodynamic limit, so that Monte-Carlo simulations are not possible (sign problem).

The severity of the sign problem can be measured by the ratio

$$\langle e^{2i\theta} \rangle_{1+1^*} \equiv \frac{\langle \det^2(D + m + \mu\gamma_0) \rangle}{\langle |\det(D + m + \mu\gamma_0)|^2 \rangle} \sim e^{-V(F_{N_f=2} - F_{pq})}.$$

full QCD
partition function

phase quenched
partition function

The phase quenched QCD partition function can be written as

$$Z_{|\mathrm{QCD}|} = \langle |\det(D + m + \mu\gamma_0)|^2 \rangle = \langle \det(D + m + \mu\gamma_0) \det(D + m - \mu\gamma_0) \rangle.$$

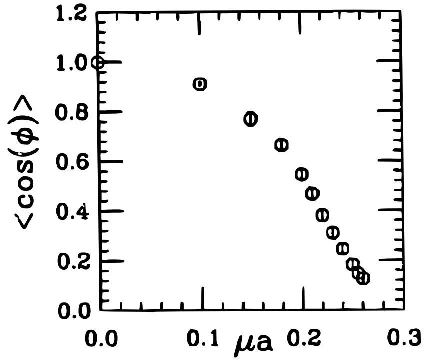
Because of this it can also be interpreted as QCD at isospin chemical potential $\mu_I = \mu$.

Alford-Kapustin-Wilczek-1999

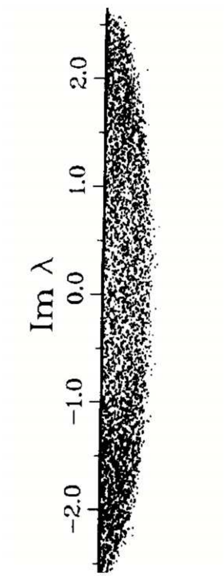
Phase Factor and Dirac Eigenvalues

$$\det(D + m + \mu\gamma_0) = e^{i\theta} |\det(D + m + \mu\gamma_0)|$$

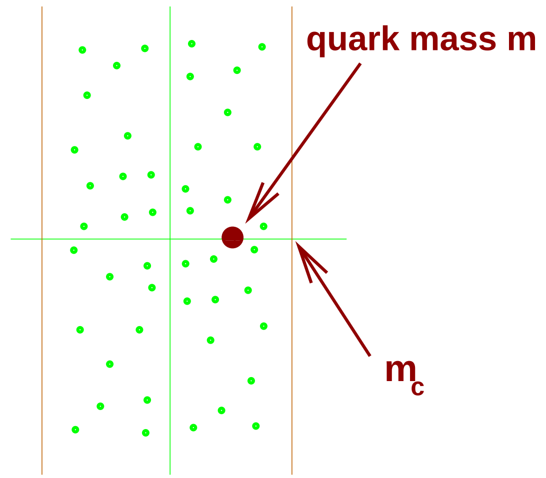
$\prod_k (\lambda_k + m)$ phase factor



Toussaint-1990



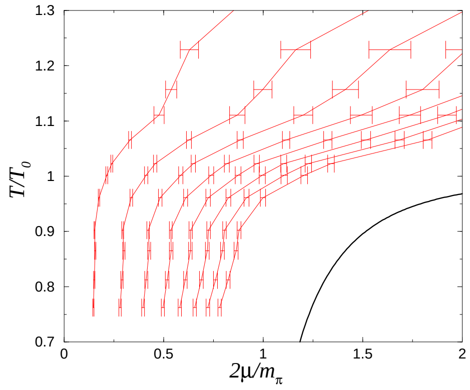
Barbour et al. 1986



Scatter plot of Dirac eigenvalues

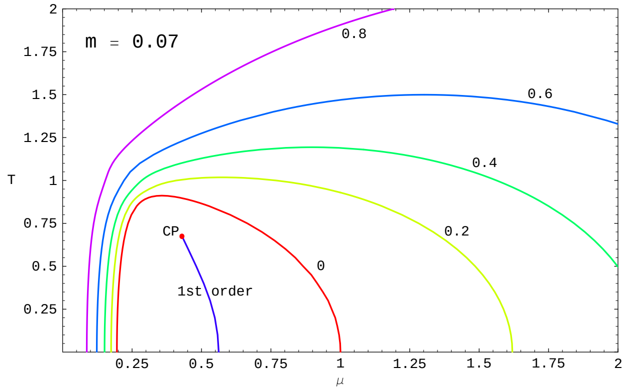
$$m < m_c \qquad \text{then} \qquad \langle e^{i\theta} \rangle \sim 0$$

Phase Diagram and Average Phase Factor



Lattice results showing contour lines with equal variance of the phase of the fermion determinant.

Allton-et al-2005, Splittorff-2006



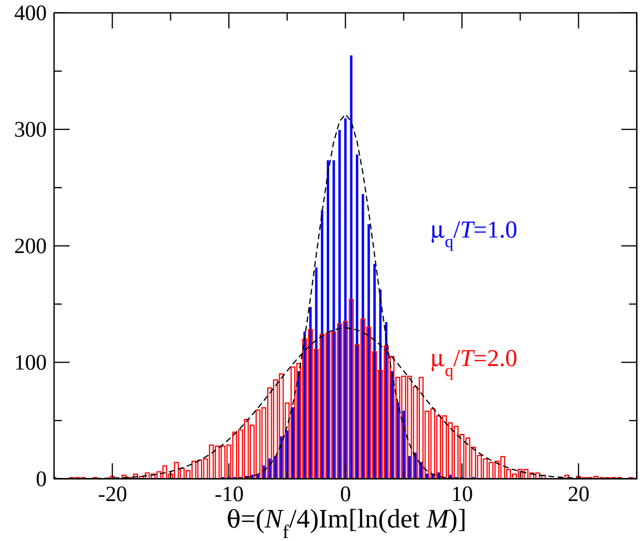
Analytical random matrix result for phase diagram of average phase factor. Curves show contours of equal average phase factor. Han-Stephanov-2008, Ravagli-JV-2008

The Distribution of Phase

Both within one-loop chiral perturbation theory and in one-dimensional QCD we find for the distribution of the phase:

✓ $\mu < m_\pi/2$: $\rho(\theta)$ is a periodicized Gaussian

$$\langle \rho(\theta) \rangle_{1+1} = \frac{1}{\sqrt{2\pi \Delta G_0}} e^{-\frac{(\theta - i\Delta G)^2}{\underbrace{\Delta G}_{\text{one-loop chPT integral}}}}.$$



Quenched distribution, Ejiri-2009

✓ $\mu > m_\pi/2$: $\rho(\theta)$ is a periodicized Lorentzian
Lombardo-Splittorff-JV-2009

III. Overlap Problem

Overlap Problem

Distribution of the Baryon Number Density

Overlap Problem

It is possible to put the phase factor in the observable and use gauge field configurations generated by $Z_{|QCD|}$ (known as reweighting).

This might introduce the overlap problem, namely that observables for the ensemble that is generated seem to converge to the wrong value.

For example, the chiral condensate is close to the phase quenched value for each separate gauge field configuration

$$\frac{1}{V}\mathrm{Tr}\frac{1}{D+\mu\gamma_0+m} = \Sigma_{|\mathrm{QCD}|} + \delta\Sigma,$$

with $\delta\Sigma \ll \Sigma_{|\mathrm{QCD}|}$ for a $finite$ ensemble, whereas the true value of the condensate is reached from rare but very large fluctuations, many orders of magnitude bigger than a typical value.

The Baryon Number Density

$$n_B = \frac{1}{V}\mathrm{Tr}\frac{1}{\gamma_0(D+m)+\mu}.$$

It satisfies the charge conjugation relation

$$n_B^*(\mu) = -n_B(-\mu).$$

Therefore n_B generally has a nonzero real and imaginary part.

$$\begin{aligned}
\mathrm{Re}(n_B) &= \frac{1}{2}[n_B(\mu) - n_B(-\mu)] = \lim_{n\to 0}\frac{1}{2nV}\frac{d}{d\mu}\mathrm{det}^n(\gamma_0(D+m)+\mu)\mathrm{det}^n(\gamma_0(D+m)-\mu), \\
\mathrm{Im}(n_B) &= \frac{1}{2i}[n_B(\mu) + n_B(-\mu)] = \lim_{n\to 0}\frac{1}{2inV}\frac{d}{d\mu}\frac{\mathrm{det}^n(\gamma_0(D+m)+\mu)}{\mathrm{det}^n(\gamma_0(D+m)-\mu)}.
\end{aligned}$$

Therefore, $\langle \mathrm{Im}(n_B)\rangle = \langle\theta\rangle$ so that $\langle \mathrm{Im}(n_B)\rangle_{1+1^*} = 0$ and $\langle \mathrm{Im}(n_B)\rangle_{1+1^*} = i\nu_I$

For QCD with, say with $N_f = 2$, we know that at low temperatures

$$\langle n_B\rangle_{1+1} = 0 \qquad \text{for} \quad \mu < m_N/3.$$

Expectation values of n_B for $\mu < m_\pi/2$

To one loop order in chiral perturbation theory we find

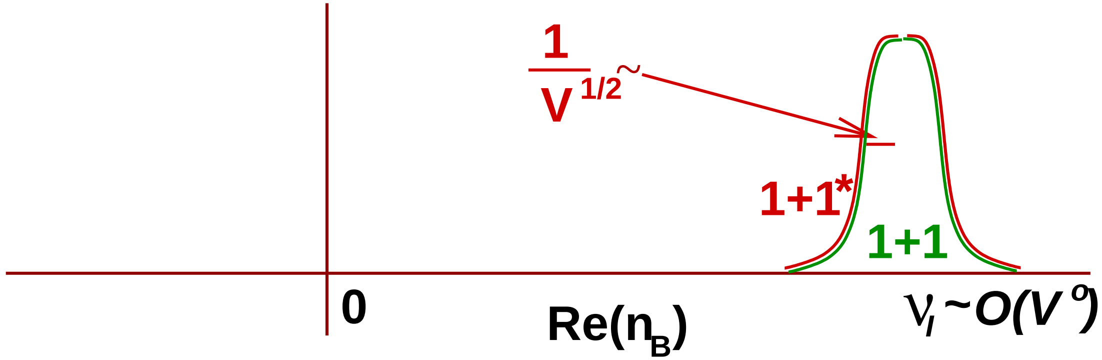
$$
\begin{aligned}
\langle \mathrm{Re}\, n_B \rangle_{1+1^*} &= \nu_I\,, \\
\langle \mathrm{Re}\, n_B \rangle_{1+1} &= \nu_I\,,
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathrm{Im}\, n_B \rangle_{1+1^*} &= 0, \\
\langle \mathrm{Im}\, n_B \rangle_{1+1} &= i\nu_I\,.
\end{aligned}
$$

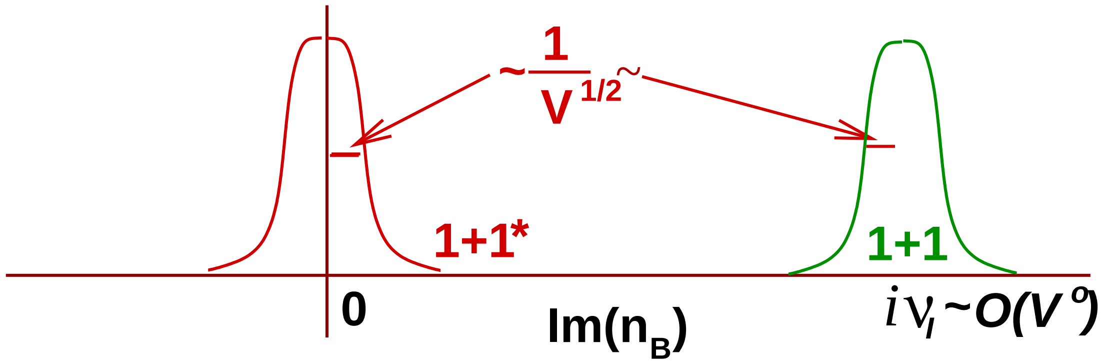
It it possible to evaluate all moments of both the real and the imaginary parts of the baryon density. Their distribution is a Gaussian with a width given by the sum and difference of the isospin number and the baryon number susceptibility, respectively.

Lombardo-Splittorff-JV-2009

Distribution of n_B for $\mu < m_\pi/2$



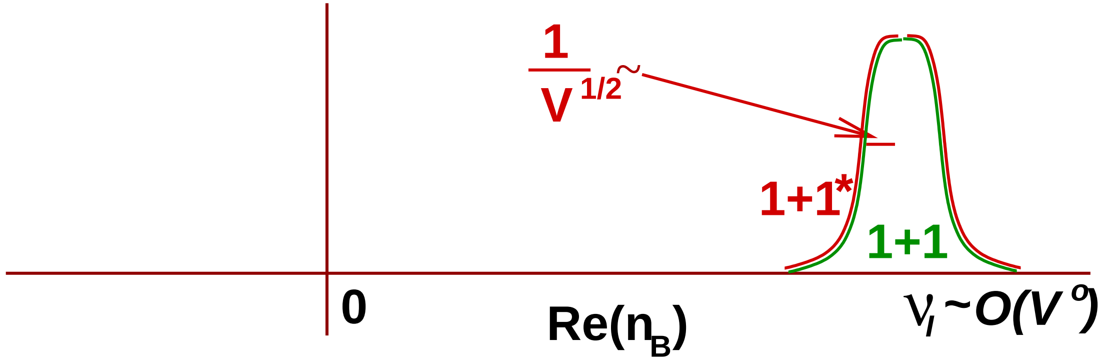
Distribution of the real part of the baryon number density for two dynamical fermions for full QCD (green) and phase quenched QCD (red).



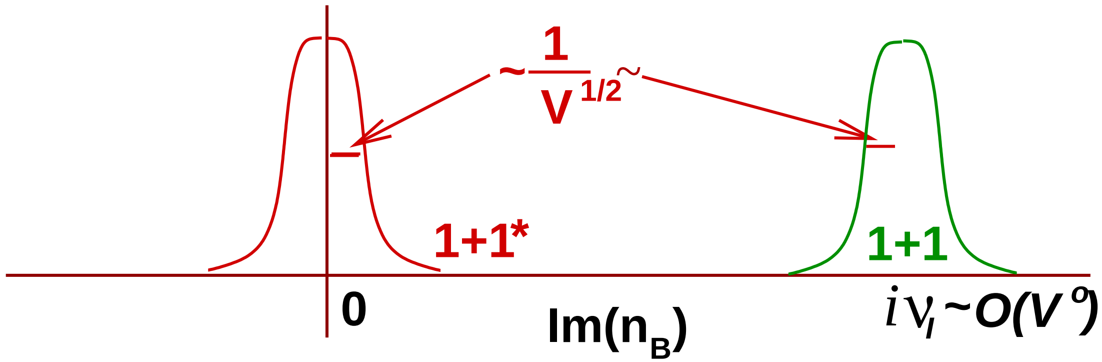
Distribution of the imaginary part of the baryon density.

$$\nu_I = \frac{m_\pi^2 T}{\pi^2} \sum_{n=1}^{\infty} \frac{K_2(\frac{m_\pi n}{T})}{n} \sinh \frac{2\mu n}{T}.$$

Distribution of n_B for $\mu < m_\pi/2$



Distribution of the real part of the baryon number density for two dynamical fermions for full QCD (green) and phase quenched QCD (red).

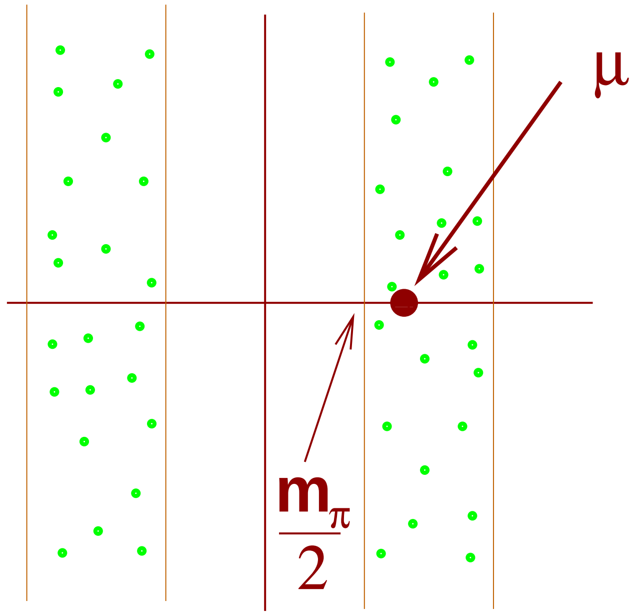


Distribution of the imaginary part of the baryon density.

$$\nu_I = \frac{m_\pi^2 T}{\pi^2} \sum_{n=1}^{\infty} \frac{K_2(\frac{m_\pi n}{T})}{n} \sinh \frac{2\mu n}{T}.$$

$$\langle \mathbf{n_B} \rangle_{\mathbf{1+1}} = \langle \mathrm{Re}(\mathbf{n_B}) \rangle_{\mathbf{1+1}} + \mathbf{i} \langle \mathrm{Im}(\mathbf{n_B}) \rangle_{\mathbf{1+1}} = \nu_{\mathbf{I}} + \mathbf{ii}\nu_{\mathbf{I}} = \mathbf{0}.$$

Distribution of n_B for $\mu > m_\pi/2$



Spectrum of $\gamma_0(D+m)$

For $\mu > m_\pi/2$ moments of the baryon number diverge due to eigenvalues close to μ .
For the p -th moment we obtain after excluding a disc around μ with radius ϵ ,

$$\langle |n|^{2p} \rangle_{1+1^*} \sim \epsilon^{2p-4}.$$

Therefore the distribution of $|n|$ has a power tail ($1/|n|^5$ in this case).

It becomes virtually impossible to sample the baryon number.

Lombardo-Splittorff-JV-2009

VI. Teflon Plated Observables

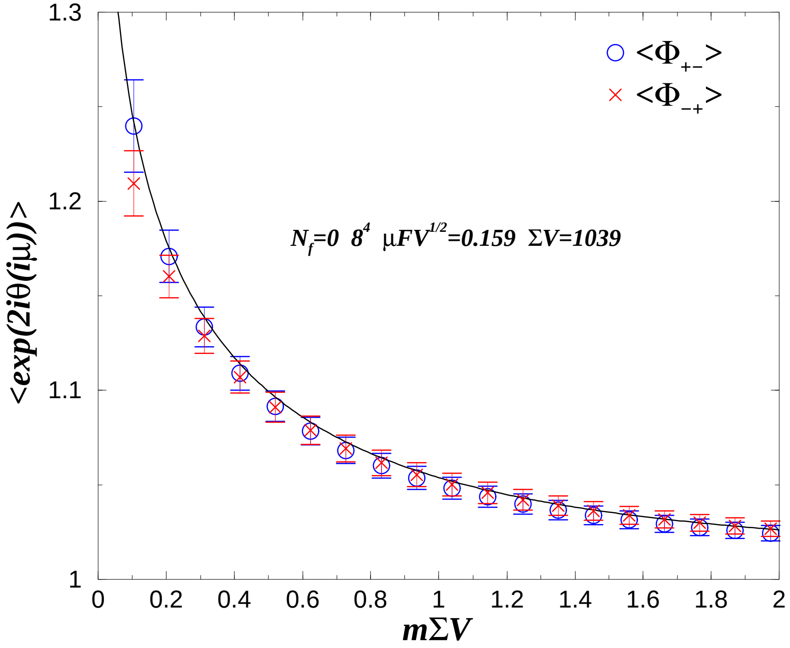
Infrared Dominance of the Phase Factor

Correlations with Phase Factor

Infrared Dominance of the Phase Factor

Both in the ϵ and p domain the mass and chemical potential dependence of QCD and QCD like partition functions can be obtained from chiral perturbation theory.

Therefore the average phase factor in this domain is determined by chPT, or in QCD, by the infrared part of the Dirac spectrum



"Phase" of the fermion determinant for imaginary chemical potential.

Splittorff-Svetitsky-2007

Analytical continuation of average phase factor:

$$\left\langle \frac{\det(D+i\mu)}{\det(D-i\mu)} \right\rangle = 1 - 4\hat{\mu}^2 I_0(\hat{m}) K_0(\hat{m}).$$

Here, $\hat{m} = mV\Sigma$ and $\hat{\mu}^2 = \mu^2 F_\pi^2 V$. The analytical result has been obtained in the microscopic domain

Damgaard-Splittorff-2006, Splittorff-JV 2007.

Quest for Teflon Plated Observables

√ Observables that are not sensitive to the infrared part of the Dirac spectrum can be measured in QCD at nonzero chemical potential.

More generally, these are observables that have no correlations with the phase factor,

$$\underbrace{\langle \mathcal{O} e^{2i\theta} \rangle_{|QCD|}} = \langle \mathcal{O} \rangle_{|QCD|} \underbrace{\langle e^{2i\theta} \rangle_{|QCD|}}.$$

Then

$$\langle \mathcal{O} \rangle_{QCD} = \langle \mathcal{O} \rangle_{|QCD|}.$$

Examples:
- √ Chiral condensate for $\mu < m_\pi/2$ and very low temperatures.
- √ Baryon density for $T \gtrsim T_{\rm co}$.
- √ $T_{\rm co}(\mu)$ Allton-et al-2002

There is no sign problem or overlap problem but do we learn anything about QCD at nonzero chemical potential?

Correlators in Chiral Perturbation Theory

Correlators between operator such as n_q , n_I , $\langle\bar{\psi}\psi\rangle$ and the phase factor can be calculated in chiral perturbation theory.

For example in a small chemical potential and small temperature expansion we obtain

$$\langle \mathrm{Re}(n_B)\, e^{2i\theta}\rangle_{1+1^*} - \langle \mathrm{Re}(n_B)\rangle_{1+1^*}\langle e^{2i\theta}\rangle_{1+1^*} = 0,$$

$$\frac{\langle \mathrm{Im}(n_B)\, e^{2i\theta}\rangle_{1+1^*} - \langle \mathrm{Im}(n_B)\rangle_{1+1^*}\langle e^{2i\theta}\rangle_{1+1^*}}{\langle \mathrm{Im}(n_B) e^{2i\theta}\rangle_{1+1^*}} = 1.$$

Lombardo-Splittorff-JV-2010

V. Ergodicity

Ergodicity (Self-Averaging)

- Ergodicity: Space-Time average of an observable is equal to the ensemble average.
- QCD at nonzero chemical potential is maximally nonergodic: space-time averaging gives the phase quenched result.
- Master configurations do not exist for QCD at nonzero chemical potential.
- One way out might be to complexify the fields so that cancellations can be achieved by spatial averaging.
- One method that may achieve this is the complex Langevin algorithm which has its own issues. de Forcrand-2009

Not being self-averaging is a fundamental problem for QCD at $\mu \neq 0$.

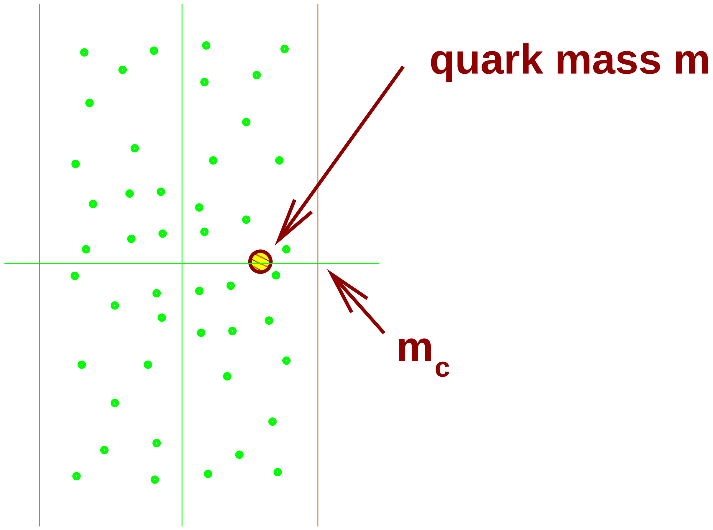
VI. Spectral Representations

Dirac Spectra

Alternative to Banks-Casher Relations

Spectral Representations

Spectral representations of the Dirac operator have been extremely useful for nonhermitean theories with a real determinant (see talk by Steve Sharpe).

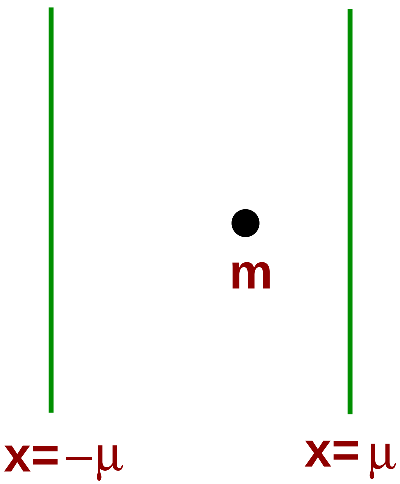


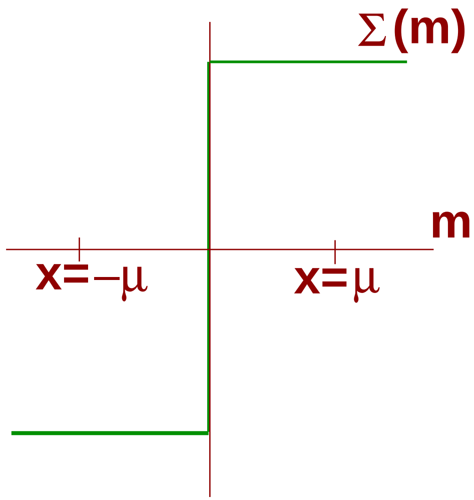
Scatter plot of Dirac eigenvalues

- √ The critical point is when the quark mass hits the cloud of eigenvalues.
- √ For phase quenched QCD this is the point when $\mu = m_\pi/2$.
- √ For Wilson fermions this is the onset of the Aoki phase.

- √ For nonhermitean theories theories with a complex determinant, the support of the Dirac spectrum does not depend on the complex phase of the determinant.
- √ Exponential cancellations can wipe out the critical point and reveal a completely different physical system. This is the case of QCD at nonzero baryon density.

Alternative to the Banks-Casher Relation





Based on the situation for QCD in 1d we consider the chiral condensate of a complex eigenvalue density with support only on $x \pm \mu$.

$$\Sigma(m) = \int \frac{dxdy}{\pi} \frac{1}{x+iy-m} \underbrace{\frac{e^{V(x+iy)}\delta(x-\mu)+e^{-V(x+iy)}\delta(x+\mu)}{e^{Vm}+e^{-Vm}}}_{\rho(x,y)} = \tanh(Vm).$$

In the thermodynamic limit $(V \to \infty)$ this results in a discontinuity across $m = 0$, but only after exponentially large cancellations. Osborn-Splittorff-JV-2005, Ravagli-JV-2008

Conclusions and Outlook

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.
- √ The physics of the phase quenched partition function has to be nullified before the physics of full QCD will be revealed.

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.
- √ The physics of the phase quenched partition function has to be nullified before the physics of full QCD will be revealed.
- √ QCD at $\mu \neq 0$ is not selfaveraging.

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.
- √ The physics of the phase quenched partition function has to be nullified before the physics of full QCD will be revealed.
- √ QCD at $\mu \neq 0$ is not selfaveraging.
- √ The sign problem is a fundamental problem and substantial progress requires a complete reformulation of QCD at nonzero chemical potential.

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.
- √ The physics of the phase quenched partition function has to be nullified before the physics of full QCD will be revealed.
- √ QCD at $\mu \neq 0$ is not selfaveraging.
- √ The sign problem is a fundamental problem and substantial progress requires a complete reformulation of QCD at nonzero chemical potential.
- √ Lattice QCD simulations are not feasible in the region of phase space where interesting baryonic effects occur.

VII. Conclusions and Outlook

- √ The physics of QCD at finite baryon density is obscured by both the sign problem and the overlap problem.
- √ The physics of the phase quenched partition function has to be nullified before the physics of full QCD will be revealed.
- √ QCD at $\mu \neq 0$ is not selfaveraging.
- √ The sign problem is a fundamental problem and substantial progress requires a complete reformulation of QCD at nonzero chemical potential.
- √ Lattice QCD simulations are not feasible in the region of phase space where interesting baryonic effects occur.
- √ As was emphasized in the talk by David Kaplan, rather than studying QCD at $\mu \neq 0$, to make substantial progress we first have to rethink the problem for much simpler model systems.