Overture to lattice study of supersymmetric gauge theories

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30 July 2010 @ CERN

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Nonperturbative study of SUSY gauge theories from first principles?

- SUSY will play an important role in particle physics beyond SM
  - hierarchy (naturalness) problem
  - consistency of string theory (gauge/gravity correspondence)
- Nonperturbative phenomena
  - confinement, bound states, spontaneous chiral symmetry breaking, quantum tunneling, . . .
  - spontaneous SUSY breaking
- Quest for nonperturbative formulation. . . lattice!?

SUSY on the lattice? (cf. Dondi–Nicolai, Nuovo Cim. A41 (1977))

- Lattice SUSY would be *impossible*, because

$$\left\{Q^A_\alpha, (Q^B_\beta)^\dagger\right\} = 2\delta^{AB}\sigma^m_{\alpha\dot\beta}P_m$$

but infinitesimal translations P_m cannot be defined on lattice fields

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- However, at least a linear combination Q of Q_{α}^{A} and $(Q_{\beta}^{B})^{\dagger}$ such that

$$\{Q, Q\} = 2Q^2 = 0$$

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- However, at least a linear combination Q of Q_α^A and $(Q_\beta^B)^\dagger$ such that

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- Moreover, if the target continuum action S can be written as

$$S = QX$$

  Q-invariance could be promoted to a lattice symmetry!

2D $\mathcal{N} = (2,2)$ SYM

- action (dimensional reduction of 4D $\mathcal{N} = 1$ SYM to 2D)

$$S_{\text{2DSYM}} = \frac{1}{g^2}\int d^2x \,\text{tr}\left[\frac{1}{2}F_{MN}F_{MN} + \Psi^T C\Gamma_M D_M \Psi + \tilde{H}^2\right]$$

- SUSY

$$\delta A_M = i\epsilon^T C\Gamma_M\Psi, \qquad \delta\Psi = \frac{i}{2}F_{MN}\Gamma_M\Gamma_N\epsilon + i\tilde{H}\Gamma_5\epsilon$$

$$\delta\tilde{H} = -i\epsilon^T C\Gamma_5\Gamma_M D_M\Psi$$

- we set $\Gamma_0 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $\Gamma_3 = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\Gamma_{\uparrow,\downarrow} \equiv \frac{i}{2}\left(\Gamma_2 \mp i\Gamma_3\right)$$

$$\Psi^T \equiv (\psi_0, \psi_1, \chi, \eta/2), \qquad \epsilon^T \equiv -\left(\varepsilon^{(0)}, \varepsilon^{(1)}, \tilde{\varepsilon}, \varepsilon\right)$$

and decompose

$$\delta \equiv \varepsilon^{(0)}Q^{(0)} + \varepsilon^{(1)}Q^{(1)} + \tilde{\varepsilon}\tilde{Q} + \varepsilon Q$$

2D $\mathcal{N} = (2,2)$ SUSY algebra

- SUSY algebra in this spinor basis,

$$Q^2 = \tilde{Q}^2 = \delta_\phi, \qquad (Q^{(0)})^2 = (Q^{(1)})^2 = -\delta_{\bar{\phi}}$$
$$\{Q, Q^{(\mu)}\} = -2i\partial_\mu + 2\delta_{A_\mu}, \quad \{\tilde{Q}, Q^{(\mu)}\} = -\epsilon_{\mu\nu}(-2i\partial_\nu + 2\delta_{A_\nu})$$
$$\{Q, \tilde{Q}\} = \{Q^{(0)}, Q^{(1)}\} = 0$$

  where

$$\phi \equiv A_2 + iA_3, \qquad \bar{\phi} = A_2 - iA_3, \qquad \epsilon_{01} \equiv 1$$

  and δ_φ denotes the infinitesimal gauge transformation by φ: $\delta_\varphi = [\varphi, \cdot]$ for matter fields and $\delta_\varphi A_\mu = iD_\mu\varphi$

- *Q*-transformation is nilpotent, on gauge invariant combinations:

$$Q^2 = \delta_\phi \simeq 0$$

Q-transformation

- Q-transformation ($H \equiv \tilde{H} + iF_{01}$)

$$
\begin{aligned}
QA_\mu &= \psi_\mu, & Q\psi_\mu &= iD_\mu\phi \\
Q\phi &= 0 & & \\
Q\bar{\phi} &= \eta, & Q\eta &= [\phi, \bar{\phi}] \\
Q\chi &= H, & QH &= [\phi, \chi]
\end{aligned}
$$

is nilpotent on gauge invariant combinations

$$Q^2 = \delta_\phi \simeq 0$$

- moreover, the continuum action is Q-exact

$$S_{\text{2DSYM}} = Q\frac{1}{g^2}\int d^2x\, \text{tr}\left[-2i\chi F_{01} + \chi H + \frac{1}{4}\eta\,[\phi, \bar{\phi}] - i\psi_\mu D_\mu\bar{\phi}\right]$$

Lattice formulation (Sugino, JHEP 0401 (2004))

(cf. Kaplan et al., JHEP 0305 (2003))

- 2D lattice (a: lattice spacing)

$$\Lambda = \left\{ x \in a\mathbb{Z}^2 \mid 0 \leq x_0 < \beta,\ 0 \leq x_1 < L \right\}$$

- lattice Q-transformation ($U_\mu(x) \in SU(N)$: link variables)

$$QU_\mu(x) = i\psi_\mu(x)U_\mu(x)$$
$$Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+a\hat{\mu})U_\mu(x)^{-1}\right)$$
$$Q\phi(x) = 0$$
$$Q\bar{\phi}(x) = \eta(x), \qquad Q\eta(x) = \left[\phi(x), \bar{\phi}(x)\right]$$
$$Q\chi(x) = H(x), \qquad QH(x) = [\phi(x), \chi(x)]$$

is nilpotent on gauge invariant combinations on the lattice

$$Q^2 = \delta_\phi \simeq 0$$

Lattice formulation (cont'd)

- lattice action

$$
\begin{aligned}
&S_{\text{2DSYM}}^{\text{LAT}} \\
&= Q\frac{1}{a^2g^2}\sum_{x\in\Lambda}\text{tr}\Bigg[-i\chi(x)\hat{\Phi}(x)+\chi(x)H(x)+\frac{1}{4}\eta(x)\left[\phi(x),\bar{\phi}(x)\right] \\
&\qquad -i\sum_{\mu=0}^{1}\psi_\mu(x)\left(U_\mu(x)\bar{\phi}(x+a\hat{\mu})U_\mu(x)^{-1}-\bar{\phi}(x)\right)\Bigg]
\end{aligned}
$$

where $\hat{\Phi}(x)$ ($\simeq 2F_{01}$) is basically given by the plaquette

$$\hat{\Phi}(x) \simeq -iU_0(x)U_1(x+a\hat{0})U_0(x+a\hat{1})^{-1}U_1(x)^{-1} + \text{h.c.}$$

- Q is a manifest symmetry of $S_{\text{2DSYM}}^{\text{LAT}}$, $QS_{\text{2DSYM}}^{\text{LAT}} = 0$
- $U(1)_A$ is another manifest symmetry

$$\Psi(x) \to \exp\left(\alpha\Gamma_2\Gamma_3\right)\Psi(x),$$

$$\phi(x) \to \exp\left(2i\alpha\right)\phi(x), \qquad \bar{\phi}(x) \to \exp\left(-2i\alpha\right)\bar{\phi}(x)$$

Scalar mass term

- Only with $S_{\mathrm{2DSYM}}^{\mathrm{LAT}}$ (and $SU(2)$), no thermalization was archived, owing to the flat directions

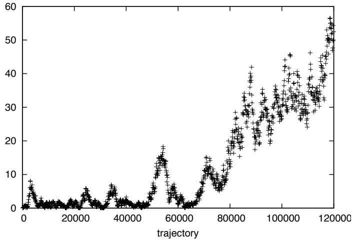

Figure: Monte Carlo evolution of $a^2\,\mathrm{tr}[\bar{\phi}(x)\phi(x)]$. 18×12, $ag = 0.1179$, antiperiodic BC

- We add a (soft) SUSY breaking scalar mass term

$$S_{\mathrm{mass}}^{\mathrm{LAT}} = \frac{\mu^2}{g^2} \sum_{x \in \Lambda} \mathrm{tr}\left[\bar{\phi}(x)\phi(x)\right]$$

SUSY Ward–Takahashi identity in the continuum

- In the target SUSY theory, one would expect (for gauge invariant $\mathcal{O}$)

$$\partial_\mu \langle \hat{s}_\mu(x)\, \mathcal{O}(y_1, \ldots, y_n) \rangle = \frac{\mu^2}{g^2} \left\langle \hat{f}(x)\, \mathcal{O}(y_1, \ldots, y_n) \right\rangle - i \frac{\delta}{\delta\epsilon(x)} \langle \mathcal{O}(y_1, \ldots, y_n) \rangle ,$$

where

$\hat{s}_\mu(x)$: (renormalized) supercurrent

$\hat{f}(x)$: (renormalized) variation of the scalar mass term

- This must hold, irrespective of
  - boundary conditions ($\because$ used localized SUSY transformations)
  - whether SUSY is spontaneously broken or not ($\Rightarrow$ Nambu-Goldstone fermion)

Identity on the lattice

- On the lattice, we have

$$\partial_\mu^* \langle s_\mu(x)\,\mathcal{O}(y_1,\dots,y_n)\rangle$$
$$= \frac{\mu^2}{g^2}\langle f(x)\,\mathcal{O}(y_1,\dots,y_n)\rangle - i\frac{\delta}{\delta\epsilon(x)}\langle\mathcal{O}(y_1,\dots,y_n)\rangle$$
$$+\frac{1}{g^2}\langle X(x)\,\mathcal{O}(y_1,\dots,y_n)\rangle\,,$$

where

$s_\mu(x)$: (bare) lattice supercurrent $\propto 1/g^2$

$$f(x)\equiv\frac{1}{a^{5/2}}2iC\left(\Gamma_\uparrow\,\mathrm{tr}[\phi(x)\Psi(x)]+\Gamma_\downarrow\,\mathrm{tr}[\bar{\phi}(x)\Psi(x)]\right)$$

and

$$X(x)\equiv a\,\mathcal{O}_{11/2}(x)=\begin{pmatrix}*\\ *\\ *\\ 0\end{pmatrix}$$

Identity on the lattice

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and

$$X(x) \equiv a\,\mathcal{O}_{11/2}(x) = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} \Leftarrow Q \text{ symmetry!}$$

Restoration of full SUSY

(cf. Bochicchio et al., NPB 262 (1985), Donini et al., NPB 523 (1998))

- For simplicity, assume x is away from y_i (no contact term)
- dimensional counting, gauge symmetry, $U(1)_A$ symmetry (and assuming no SUSY anomaly),

$$\mathcal{O}^R_{11/2}(x) = \mathcal{O}_{11/2}(x) + a^{-1} Z_f g^2 f(x) + g^2 \sum_j Z^{(j)}_{7/2} \mathcal{O}^{(j)R}_{7/2}(x),$$

which means

$$\partial^*_\mu \langle s_\mu(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle = \frac{\mu^2 - Z_f g^2}{g^2} \langle f(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle + O(a)$$

- However, since $\mathcal{O}_{11/2}(x)^T = (*,*,*,0)$, we conclude

$$Z_f = 0$$

and

$$\lim_{a\to 0} \partial^*_\mu \langle s_\mu(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle = \frac{\mu^2}{g^2} \lim_{a\to 0} \langle f(x)\, \mathcal{O}(y_1,\ldots,y_n)\rangle$$

Monte Carlo verification (Kanamori-H.S., NPB 811 (2009))

- Took a lowest-dimensional gauge invariant operator

$$\mathcal{O}(y) \equiv -\frac{i}{a^{5/2}g^2}\Gamma_0\left(\Gamma_\uparrow \operatorname{tr}[\phi\Psi(y)] + \Gamma_\downarrow \operatorname{tr}\left[\bar{\phi}\Psi(y)\right]\right)$$

and an appropriate supercurrent $s'_\mu(x) = s_\mu(x) + O(a)$, examined

$$\lim_{a\to 0}\partial_\mu^{(s)}\left\langle (s'_\mu)_i(x)\,(\mathcal{O})_i(0)\right\rangle = \frac{\mu^2}{g^2}\lim_{a\to 0}\langle (f)_i(x)\,(\mathcal{O})_i(0)\rangle ? \qquad \text{for } x \neq 0,$$

or equivalently

$$\frac{\partial_\mu^{(s)}\left\langle (s'_\mu)_i(x)\,(\mathcal{O})_i(0)\right\rangle}{\langle (f)_i(x)\,(\mathcal{O})_i(0)\rangle} \xrightarrow{a\to 0} \frac{\mu^2}{g^2} ? \qquad \text{for } x \neq 0$$

$N_f = 1/2$ RHMC simulation ($\sim 20,000\,\text{CPU} \cdot \text{hour}$)

μ^2/g^2	lattice size	ag	number of configurations	set label
0.04	12×6	0.2357	800	I (a)
0.04	16×8	0.1768	800	I (b)
0.04	20×10	0.1414	800	I (c)
0.25	12×6	0.2357	800	II (a)
0.25	16×8	0.1768	800	II (b)
0.25	20×10	0.1414	800	II (c)
0.49	12×6	0.2357	800	III (a)
0.49	16×8	0.1768	1800	III (b)
0.49	20×10	0.1414	1800	III (c)
1.0	12×6	0.2357	800	IV (a)
1.0	16×8	0.1768	1800	IV (b)
1.0	20×10	0.1414	1800	IV (c)
1.69	12×6	0.2357	800	V (a)
1.69	16×8	0.1768	1800	V (b)
1.69	20×10	0.1414	1800	V (c)

Monte Carlo verification (Kanamori-H.S., NPB 811 (2009))

- For $\mu^2/g^2 = 1.0$,

$$\frac{\partial_\mu^{(s)} \langle (s'_\mu)_i(x)\,(\mathcal{O})_i(0)\rangle}{\langle (f)_i(x)\,(\mathcal{O})_i(0)\rangle}$$ with $i = 1$, along the line $x_1 = L/2$

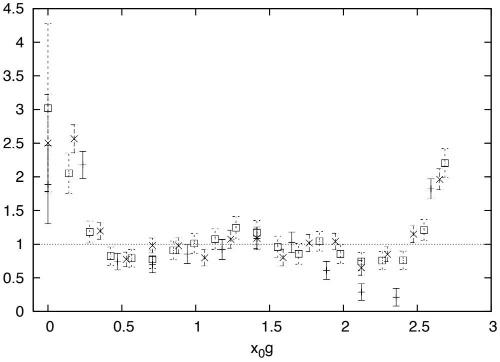

Figure: set IV(a) ($+$), IV(b) ($\times$), IV(c) ($\square$), antiperiodic BC

Monte Carlo verification (Kanamori-H.S., NPB 811 (2009))

- Continuum limit of the ratio

$$\frac{\partial_\mu^{(s)} \left\langle (s'_\mu)_i(x)\,(\mathcal{O})_i(0)\right\rangle}{\left\langle (f)_i(x)\,(\mathcal{O})_i(0)\right\rangle} \xrightarrow{a\to 0} \frac{\mu^2}{g^2}? \qquad \text{for } x \neq 0$$

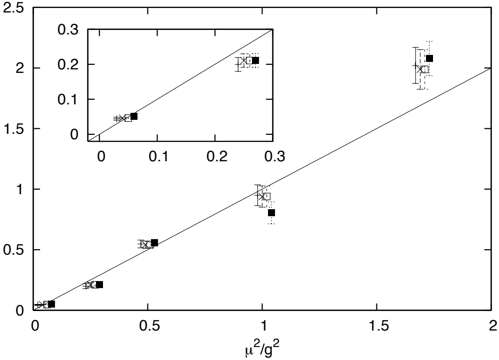

Figure: $i = 1$ ($+$), $i = 2$ ($\times$), $i = 3$ ($\square$), $i = 4$ ($\blacksquare$), antiperiodic BC

Application: a proper choice of the zero-point energy

- In the lattice identity in the massless limit $\mu^2 \to 0$,

$$\partial_\mu^* \langle (s_\mu)_i(x)\,\mathcal{O}(y)\rangle = -i\frac{\delta}{\delta\epsilon_i(x)}\langle\mathcal{O}(y)\rangle + \frac{1}{g^2}\langle (X)_i(x)\,\mathcal{O}(y)\rangle$$

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  set

$$i = 4 \quad \Leftrightarrow \quad Q\text{-transformation and } (X)_4 = 0$$

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and

$$\mathcal{O}(y) = (s_0')_1(y) \quad \Leftrightarrow \quad Q^{(0)}\text{-transformation,}$$

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we have (*for periodic BC*)

$$0 \equiv -ia^2 \sum_{x\in\Lambda} \partial_\mu^* \big\langle (s_\mu)_4(x)\,(s_0')_1(y)\big\rangle = \big\langle Q\,(s_0')_1(y)\big\rangle$$

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- The RHS can be identified with the hamiltonian density

$$\left\langle Q\, (s_0')_1(y) \right\rangle \equiv 2 \langle \mathcal{H}(y) \rangle \quad \Leftrightarrow \quad \{Q, Q^{(0)}\} = -2i\partial_0$$

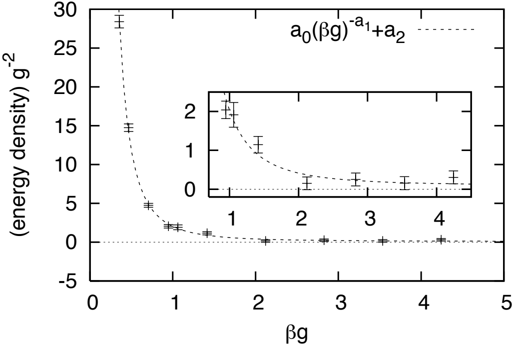
(Kanamori-Sugino-H.S., PRD 77 (2008))

Vacuum energy density: order parameter of SSUSYB

- can be obtained by a zero temperature limit of the thermal (i.e., with *antiperiodic BC*) average

$$\mathcal{E}_0 = \lim_{\beta\to\infty} \langle \mathcal{H}(x) \rangle$$

- Kanamori, PRD 79 (2009):



$$\mathcal{E}_0/g^2 = 0.09 \pm 0.09(\text{sys})\,{}^{+0.10}_{-0.08}(\text{stat})$$

Summary

- Compared with state-of-the-art lattice QCD calculation, small-scale and “primitive” but (I believe) conceptually interesting
- Illustrated an example in which numerical lattice calculation gives a clue to a question which is difficult to answer in other ways
- Naturally, we want to enlarge the range of target theories, for example, to
  - 2D $\mathcal{N} = (2, 2)$ SQCD
  - 2D $\mathcal{N} = (4, 4)$ SYM
  - 2D $\mathcal{N} = (8, 8)$ SYM
  - 4D $\mathcal{N} = 1$ SYM (Kaplan ’84, Curci–Veneziano ’87) (Montvay et al., Giedt et al., Endres)