Form factor calculations for mesonic and baryonic systems

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In collaboration with:

B. Brandt, S. Capitani, M. Della Morte, D. Djukanovic, E. Endreß, J. Gegelia, G. von Hippel, B. Jäger, A. Jüttner, B. Knippschild, H.B. Meyer

Future directions in lattice gauge theory - LGT10, CERN, 2 August 2010

Motivation & Outline

Form factors:

- provide information on hadron structure:

  $\longrightarrow$ distribution of electric charge and magnetisation; charge radii

- accurate experimental data available

- relatively simple to compute on the lattice:

  $\longrightarrow$ precise lattice estimates for $K_{\ell 3}$-decays

  [A. Vladikas @ Lat10]

- Large systematic uncertainties remain for baryonic form factors

$\Rightarrow$ "Next-generation benchmark" for lattice QCD

Outline:

1. Lattice Set-up
2. Pion electromagnetic form factor
3. Form factors and axial charge of the nucleon
4. Hadronic vacuum polarisation and $(g-2)_\mu$
5. Summary & Outlook

1. Lattice Set-up

- Perform a systematic study of mesonic and baryonic form factors with controlled uncertainties:
  - lattice artefacts
  - finite-volume effects
  - chiral extrapolations
  - excited state contamination

- Determine form factors with fine momentum resolution

- Eventually: include quark disconnected diagrams

1. Lattice Set-up

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- **C**oordinated **L**attice **S**imulations: *[https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]*

  Berlin – CERN – Madrid – Mainz – Milan – Rome – Valencia – Wuppertal – Zeuthen

- Share configurations and technology

CLS run tables

- $N_{\mathrm{f}} = 2$ flavours of non-perturbatively $\mathrm{O}(a)$ improved Wilson quarks
- Use deflation accelerated DD-HMC algorithm *[Lüscher 2003–07]*
- Generated ensembles without serious topology problems:

β	$a[\mathrm{fm}]$	lattice	$L[\mathrm{fm}]$	masses	$m_\pi L$	Labels
5.20	0.08	64×32^3	2.6	4 masses	4.8 – 9.0	A1 — A4
5.30	0.07	48×24^3	1.7	3 masses	4.6 – 7.9	D1 — D3
5.30	0.07	64×32^3	2.2	3 masses	4.7 – 7.9	E3 — E5
5.30	0.07	96×48^3	3.4	2 masses	5.0, 4.2	F6,F7
5.50	0.05	96×48^3	2.5	3 masses	5.3 – 7.7	N3 — N5
5.50	0.05	128×64^3	3.4	1 mass	4.7	O6

[Capitani, Della Morte, Endreß, Jüttner, Knippschild, H.W., Zambrana @ Lattice 2009, arXiv:0910.5578]

The “Wilson” cluster at Mainz

- 280 nodes: 2 AMD “Barcelona” processors @ 2.3 GHz: 2240 cores
- Infiniband network & switch
- Peak speed: ~ 20 TFlop/s
- Sustained speed: 3.7 TFlops/s $\Rightarrow$ 0.30 €/MFlops/s
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Scale setting

- Use mass of the Ω-baryon to set the scale

- Determine am_Ω from Jacobi smeared-local correlator

$\rightarrow \quad \beta = 5.5: \qquad a_\Omega = 0.053(1)\,\mathrm{fm}$ (preliminary)

- Determination of am_Ω still on-going at $\beta = 5.3$

$$\beta = 5.3: \quad \text{“}a_\Omega\text{”} = \left(\frac{a_{\mathsf{ref}}(\beta = 5.3)}{a_{\mathsf{ref}}(\beta = 5.5)}\right) 0.053(1)\,\mathrm{fm} = 0.069(2)\,\mathrm{fm}$$ (preliminary)

- To come: comparison with r_0, f_{K}, t_0

2. The pion form factor

- Provides information on pion structure:

$$\left\langle \pi^+(\vec{p}_f) \middle| \tfrac{2}{3}\overline{u}\gamma_\mu u - \tfrac{1}{3}\overline{d}\gamma_\mu d \middle| \pi^+(\vec{p}_i) \right\rangle = (p_f + p_i)_\mu \, f_\pi(q^2)$$

$$q^2 = (p_f - p_i)^2 : \qquad \text{momentum transfer}$$

- Pion charge radius derived from form factor at zero q^2:

$$f_\pi(q^2) = 1 - \frac{1}{6}\langle r^2 \rangle q^2 + \mathrm{O}(q^4) \quad \Rightarrow \quad \langle r^2 \rangle = 6 \, \frac{\mathrm{d}f_\pi(q^2)}{\mathrm{d}q^2}\bigg|_{q^2=0}$$

- Restriction on minimum accessible momentum transfer:

$$\vec{p}_{i,f} = \vec{n}\frac{2\pi}{L} \quad \Rightarrow \quad |q^2| \geq 2m_\pi\left(m_\pi - \sqrt{m_\pi^2 + (2\pi/L)^2}\right)$$

$$L = 2.5\,\mathrm{fm}, \quad m_\pi = 300\,\mathrm{MeV} \quad \Rightarrow \quad |q^2| \geq 0.17\,\mathrm{GeV}^2 = (0.41\,\mathrm{GeV})^2$$

$\rightarrow$ Lack of accurate data points near $q^2 = 0$

Twisted boundary conditions

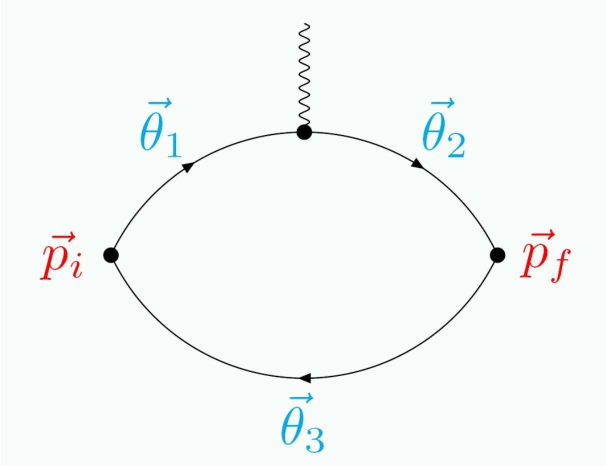
[Bedaque 2004; de Divitiis, Petronzio & Tantalo 2004; Flynn, Jüttner & Sachrajda 2005]

- Apply "twisted" spatial boundary conditions;

  Impose periodicity up to a phase $\vec{\theta}$:

$$\psi(x + L\hat{e}_k) = \mathrm{e}^{i\theta_k}\psi(x) \quad \Rightarrow \quad p_k = n_k\frac{2\pi}{L} + \frac{\theta_k}{L}, \quad k = 1, 2, 3$$

- Can tune $|q^2|$ to any desired value *[Boyle, Flynn, Jüttner, Sachrajda, Zanotti, hep-lat/0703005]*



$$\vec{\theta}_i = \vec{\theta}_1 - \vec{\theta}_3,$$
$$\vec{\theta}_f = \vec{\theta}_2 - \vec{\theta}_3$$

$$\Rightarrow \; q^2 = (p_i - p_f)^2 = \Big(E_\pi(\vec{p}_i) - E_\pi(\vec{p}_f)\Big)^2 - \left[\left(\vec{p}_i + \frac{\vec{\theta}_i}{L}\right) - \left(\vec{p}_f + \frac{\vec{\theta}_f}{L}\right)\right]^2$$

Preliminary results

[Brandt, Della Morte, Djukanovic, Endreß, Gegelia, Jüttner, H.W.]

β	$L^3 \cdot T$	a[fm]	L[fm]	m_π [MeV]	Lm_π
5.50	$48^3 \cdot 96$	0.053	2.5	600	7.7
5.50	$48^3 \cdot 96$	0.053	2.5	510	6.5
5.50	$48^3 \cdot 96$	0.053	2.5	410	5.3
5.30	$48^3 \cdot 96$	0.069	3.3	290	5.0

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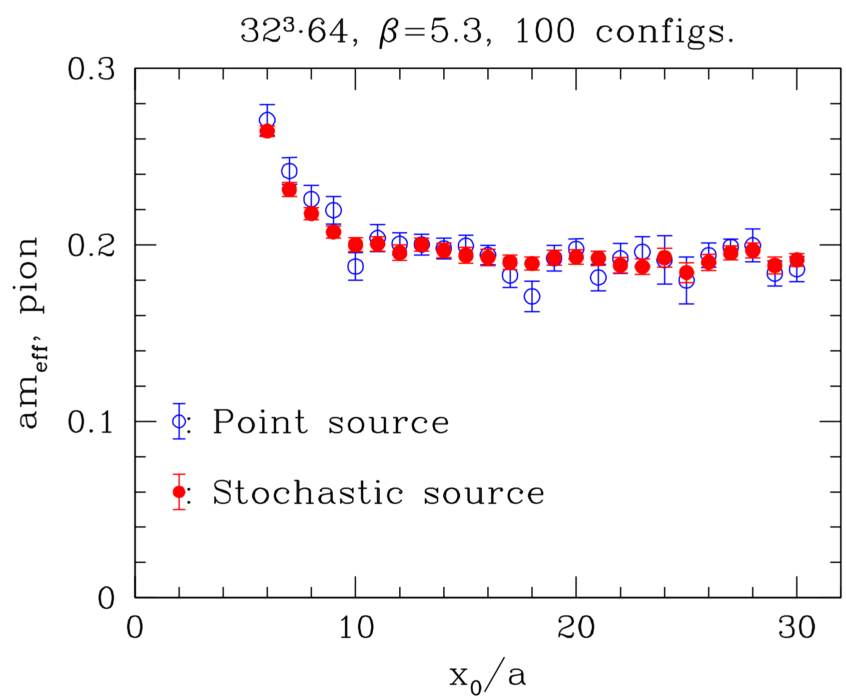
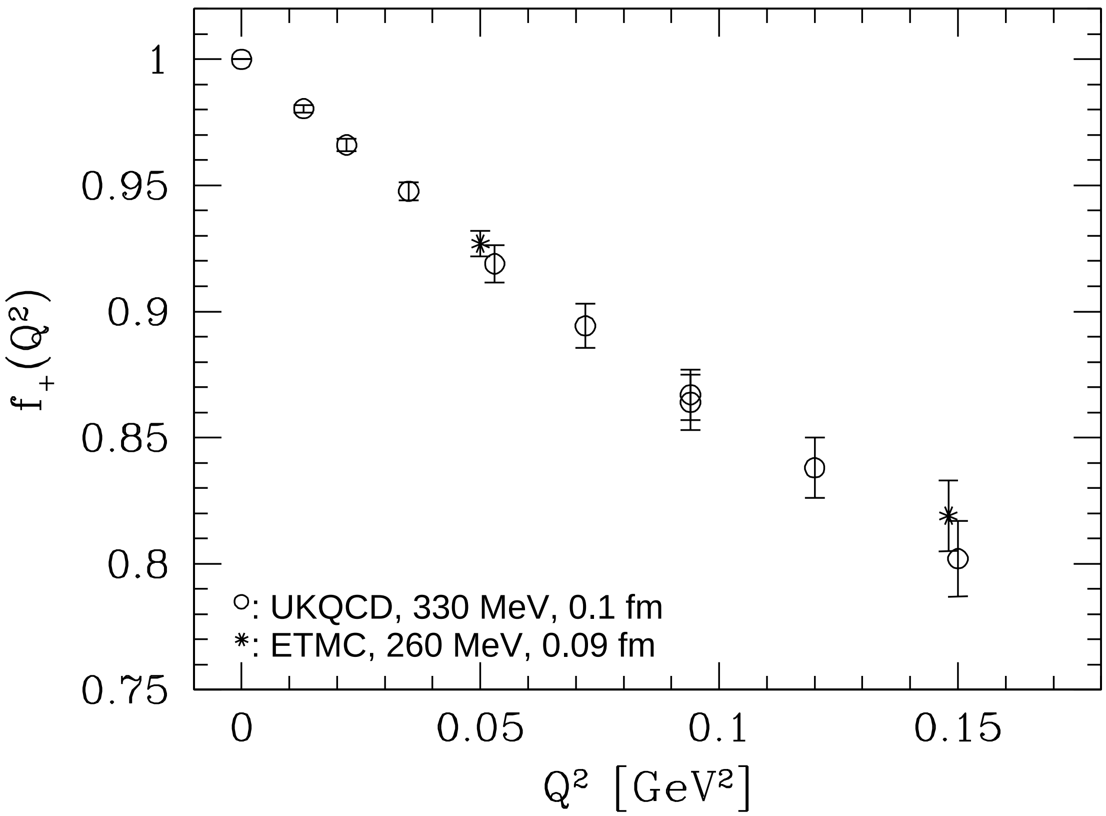
- Use stochastic noise source ("one-end trick") *[E. Endreß, Diploma thesis, 2009]*



Pion form factor

[Brandt, Della Morte, Djukanovic, Endreß, Gegelia, Jüttner, H.W.]

- Recently published results



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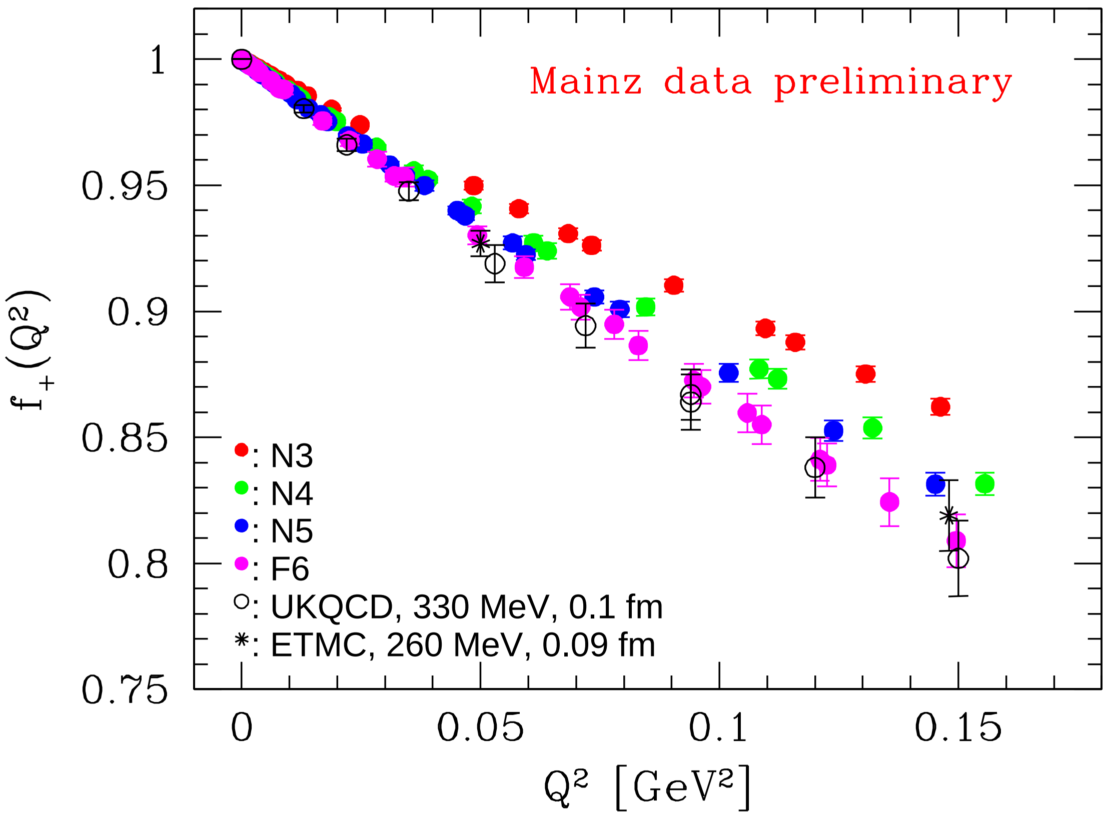

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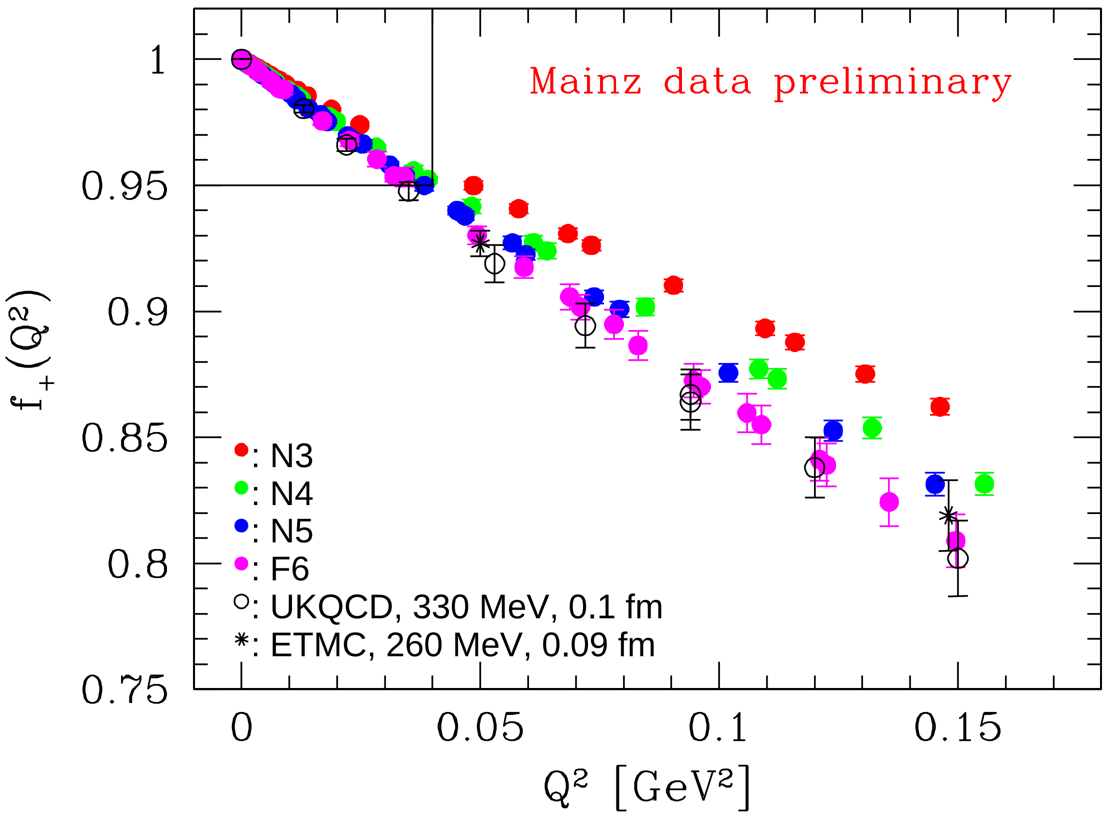

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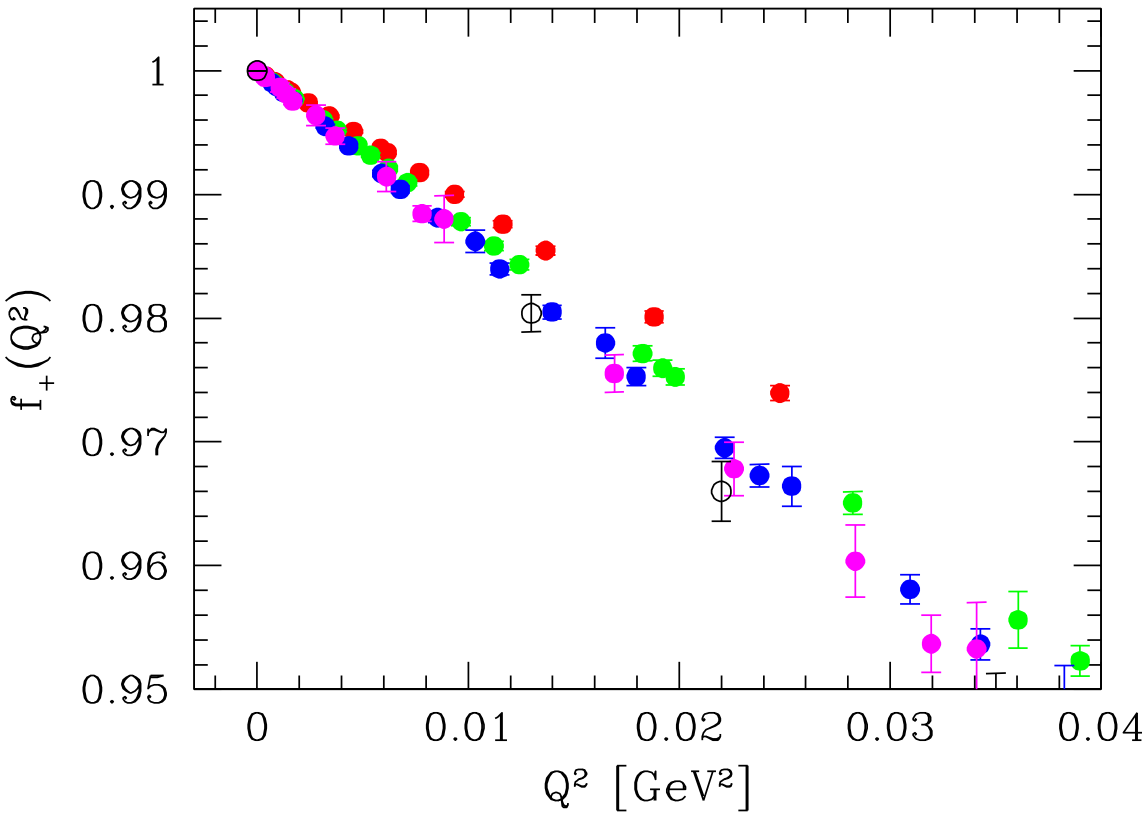

Pion charge radius

[Brandt, Della Morte, Djukanovic, Endreß, Gegelia, Jüttner, H.W.]

- Twisted boundary conditions: accurate data near $Q^2 = 0$

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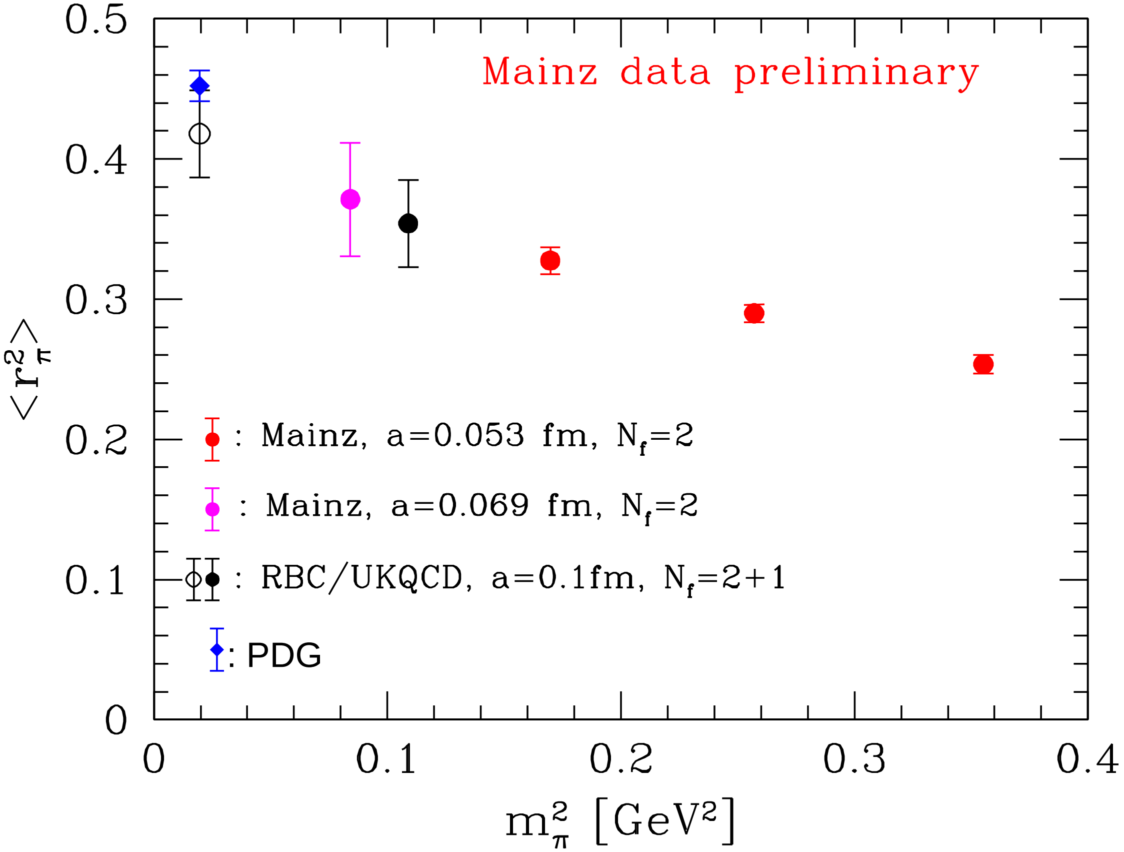

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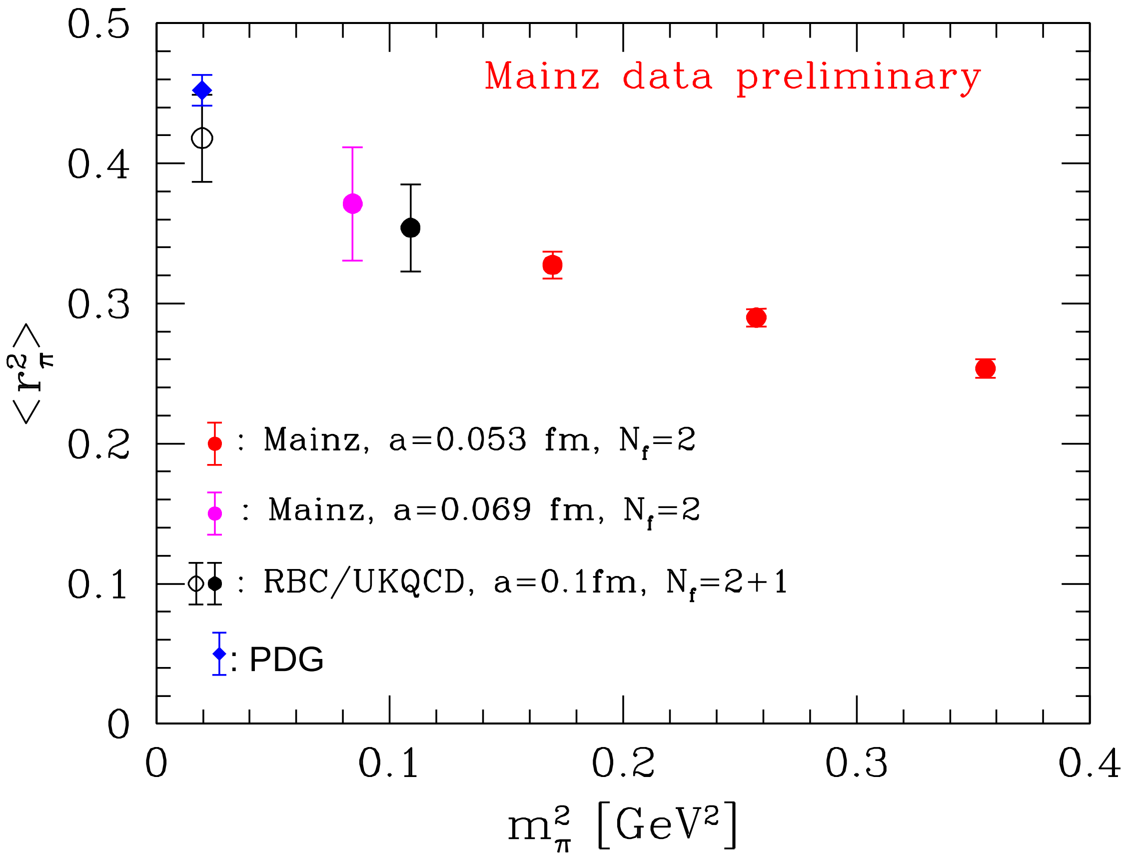


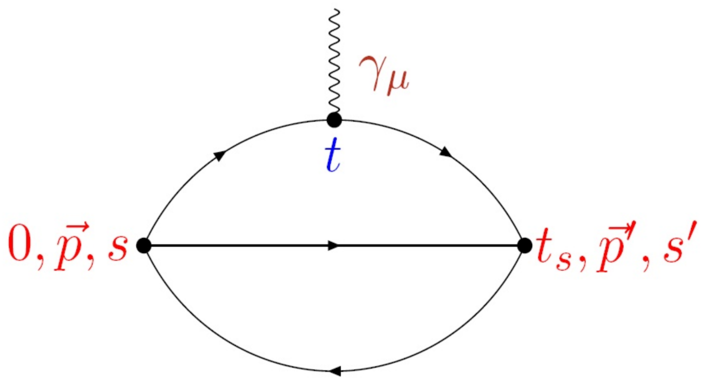
- Still to come: larger statistics, smaller pion masses, fits to ChPT

3. Form factors and axial charge of the nucleon

- Dirac and Pauli form factors

$$\langle N(p',s')\,|V_\mu(x)|\,N(p,s)\rangle = \overline{u}(p',s')\Big[\gamma_\mu F_1(q^2) + \sigma_{\mu\nu}\frac{q_\nu}{2m_{\rm N}}F_2(q^2)\Big]u(p,s)$$

$$G_{\rm E}(q^2) = F_1(q^2) - \frac{q^2}{(2m_{\rm N})^2}F_2(q^2), \qquad G_{\rm M}(q^2) = F_1(q^2) + F_2(q^2)$$



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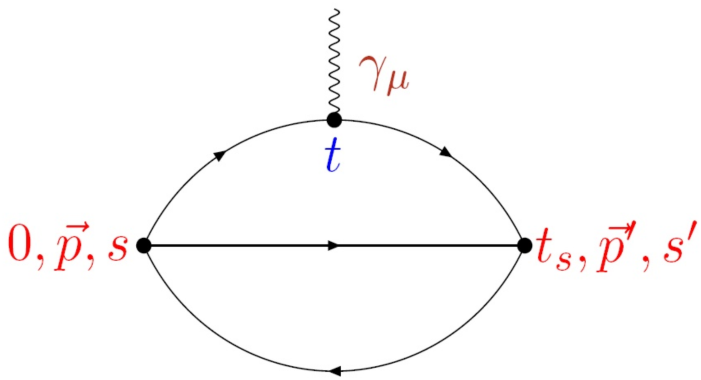


$$\sim \langle N(p',s')\,|V_\mu(x)|\,N(p,s)\rangle$$

- Disconnected diagrams usually neglected

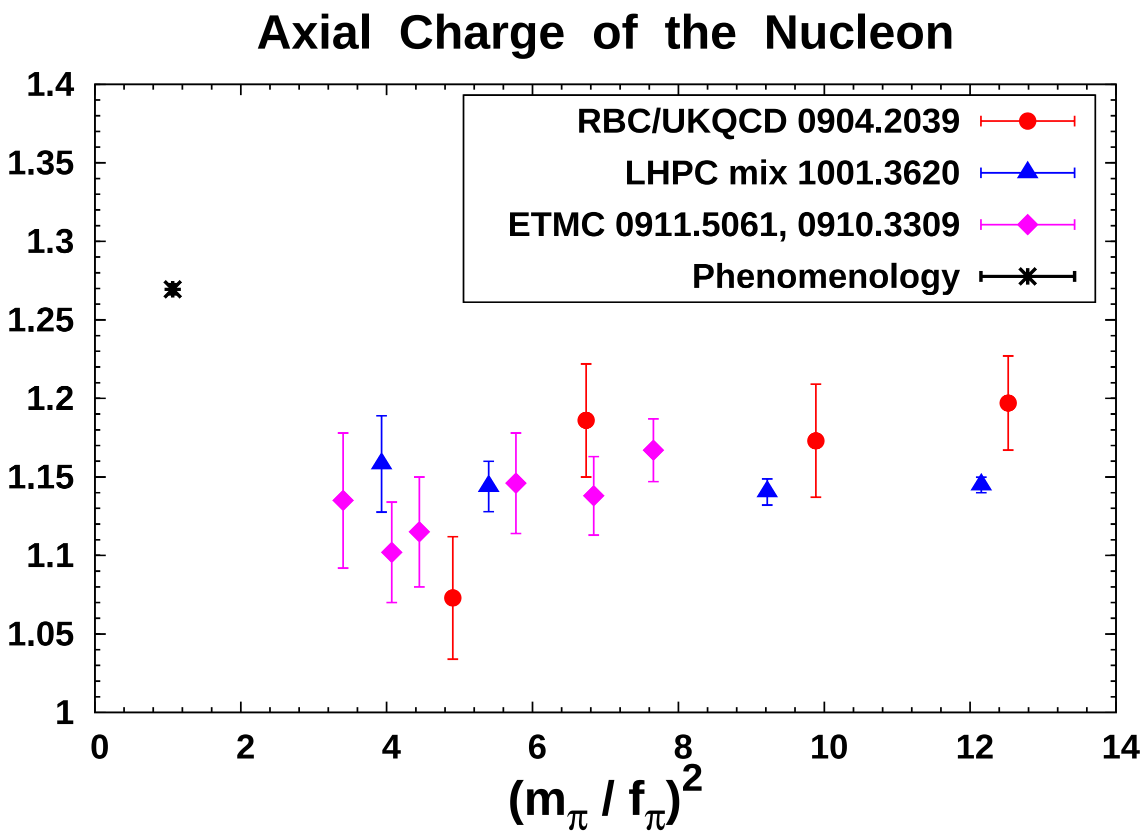
- Twisted boundary conditions: may incur large finite-volume effects

Current Status

[Dru Renner @ Lattice 2009, Dina Alexandrou @ Lattice 2010]

- Experimental Q^2-dependence & charge radii not reproduced

- Lattice simulations produce low values for axial charge g_A



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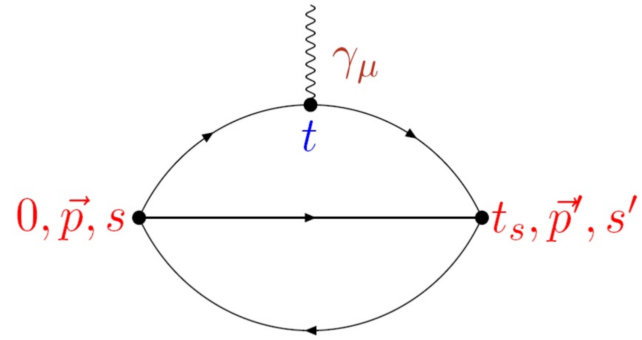
- Experimental Q^2-dependence & charge radii not reproduced
- Lattice simulations produce low values for axial charge g_{A}
- Possible origin:
  - Lattice artefacts
  - Chiral extrapolations (pion masses too large)
  - Finite-volume effects
  - Contamination from excited states

Standard method

- Extract nucleon form factors from ratios of three- and two-point functions:



$$R(\vec{q}; t, t_s) = \frac{C_3(\vec{q}, t, t_s)}{C_2(\vec{0}, t_s)} \cdot \left\{ \frac{C_2(\vec{q}, t_s - t)\, C_2(\vec{0}, t)\, C_2(\vec{0}, t_s)}{C_2(\vec{0}, t_s - t)\, C_2(\vec{q}, t)\, C_2(\vec{q}, t_s)} \right\}^{1/2} \propto G_{\mathrm{E}}(q^2),\ G_{\mathrm{M}}(q^2)$$

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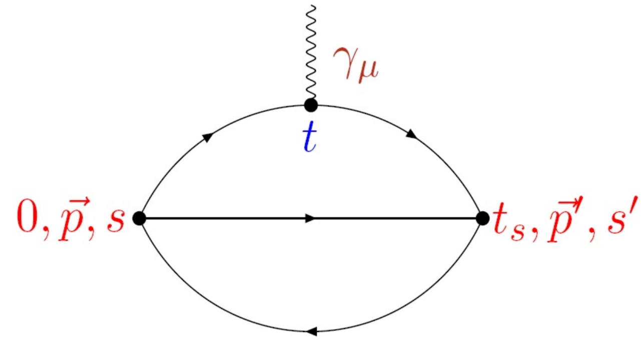

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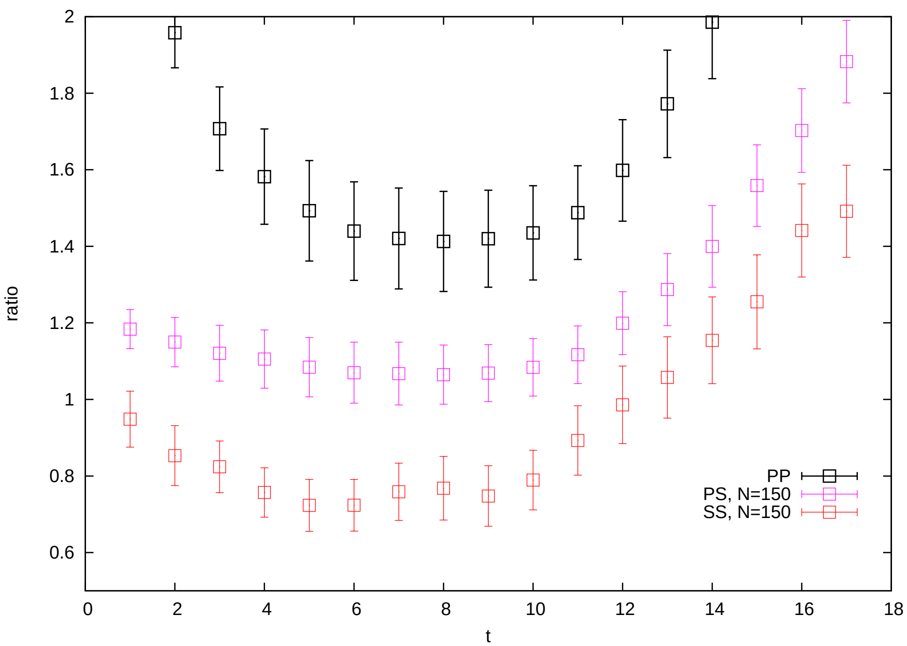
- $\beta = 5.3, \quad 32^3 \cdot 64, \quad a = 0.069\,\mathrm{fm}$
- $R(\vec{q}; t, t_s)$, iso-scalar V_0 (connected) at $Q^2 = 0.87\,\mathrm{GeV}$, $t_s = 18$
- Several source/sink combinations



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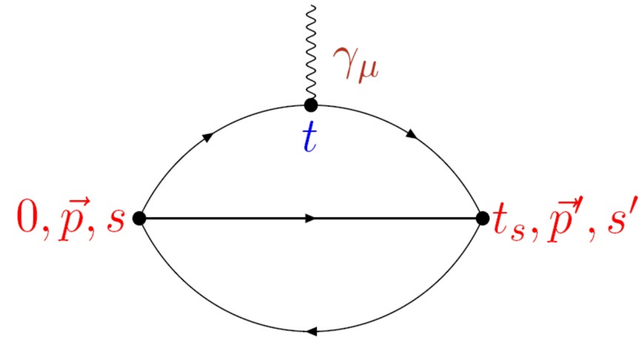

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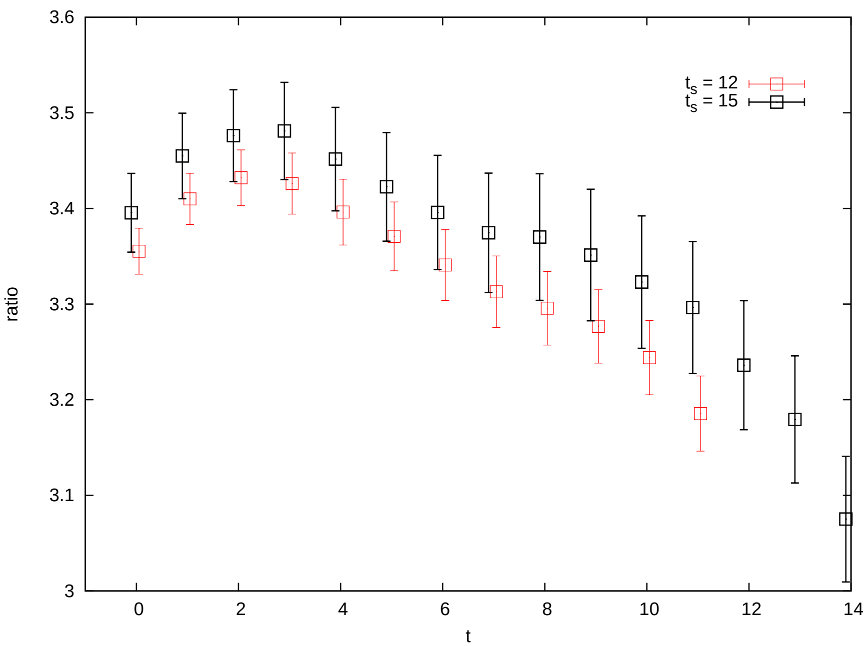
- $\beta = 5.3, \quad 32^3 \cdot 64, \quad a = 0.069\,\mathrm{fm}$
- Iso-vector magnetic form factor at $Q^2 = 0.30\,\mathrm{GeV}$, $t_s = 12, 15$
- Smeared-local correlator



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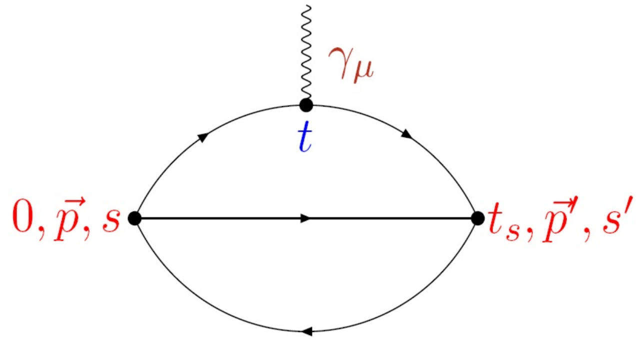

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- $\beta = 5.3, \quad 32^3 \cdot 64, \quad a = 0.069\,\mathrm{fm}$
- Iso-vector axial charge at $m_\pi = 550\,\mathrm{MeV}$, $t_s = 12, 15$
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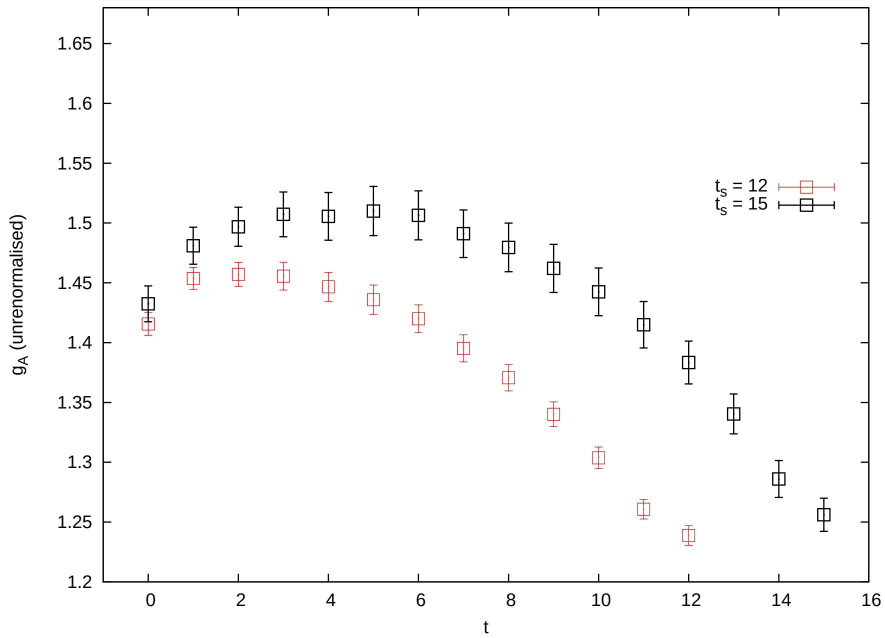

Summed insertions

[Maiani et al. 1987, B. Knippschild @ Lattice2010, J. Bulava @ LGT10]

- Standard method:

$$R(\vec{q}, t, t_s) = R_{\rm G}(\vec{q}) + {\rm O}({\rm e}^{-\Delta t}) + {\rm O}({\rm e}^{-\Delta'(t_s - t)})$$

- Summed insertion:

$$\sum_{t=1}^{t_s} R(\vec{q}, t, t_s) = R_{\rm G}(\vec{q}) \cdot t_s + K(\Delta, \Delta') + {\rm O}({\rm e}^{-\Delta t_s}) + {\rm O}({\rm e}^{-\Delta' t_s})$$

- Excited state contributions more strongly suppressed

- Determine $R_{\rm G}(\vec{q})$ from linear slope of summed ratio

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- $\beta = 5.3, \quad 32^3 \cdot 64,$
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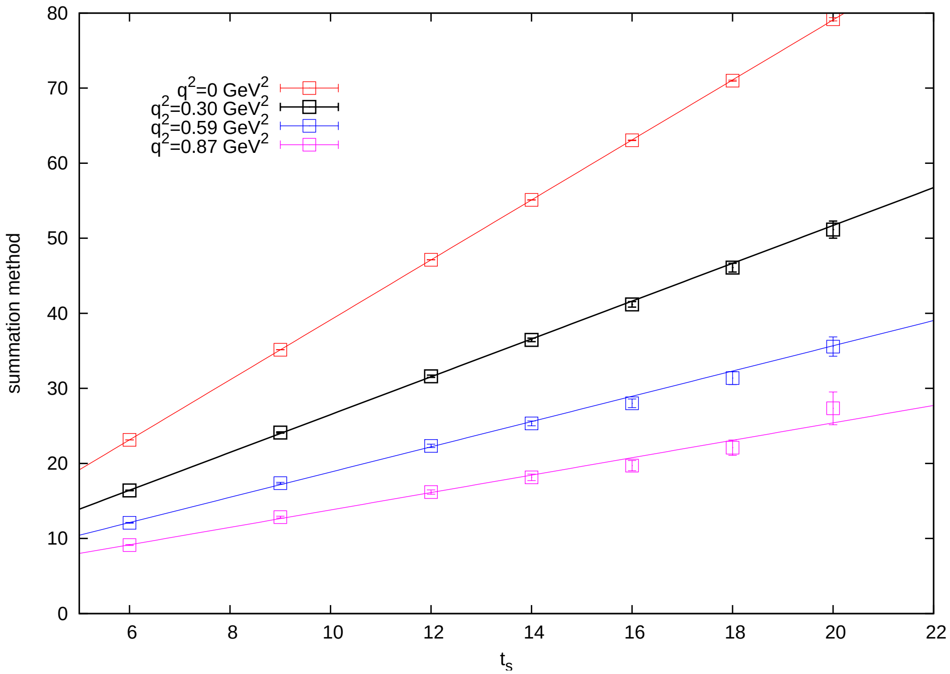
- Connected Iso-scalar form factor

- Smeared-local correlators



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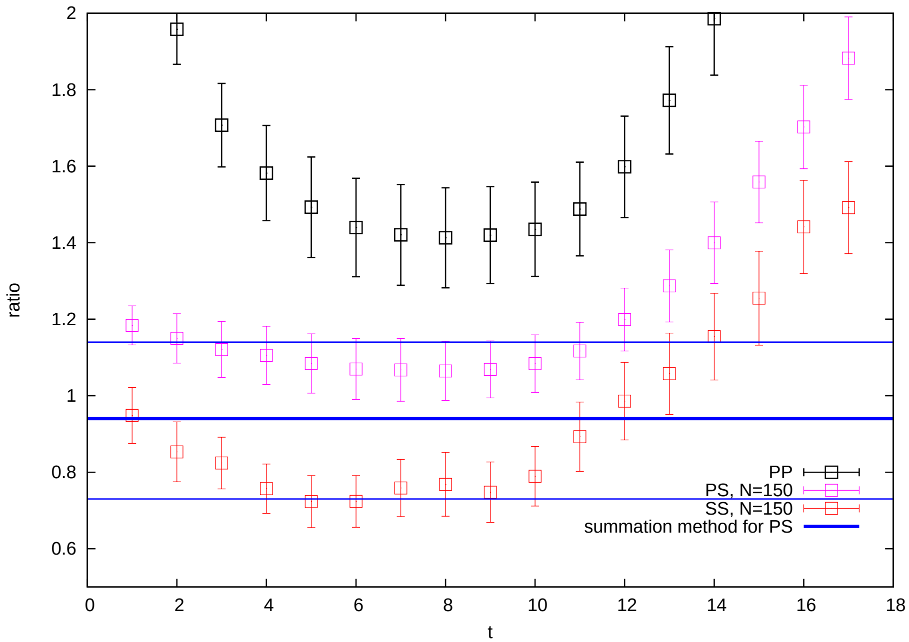

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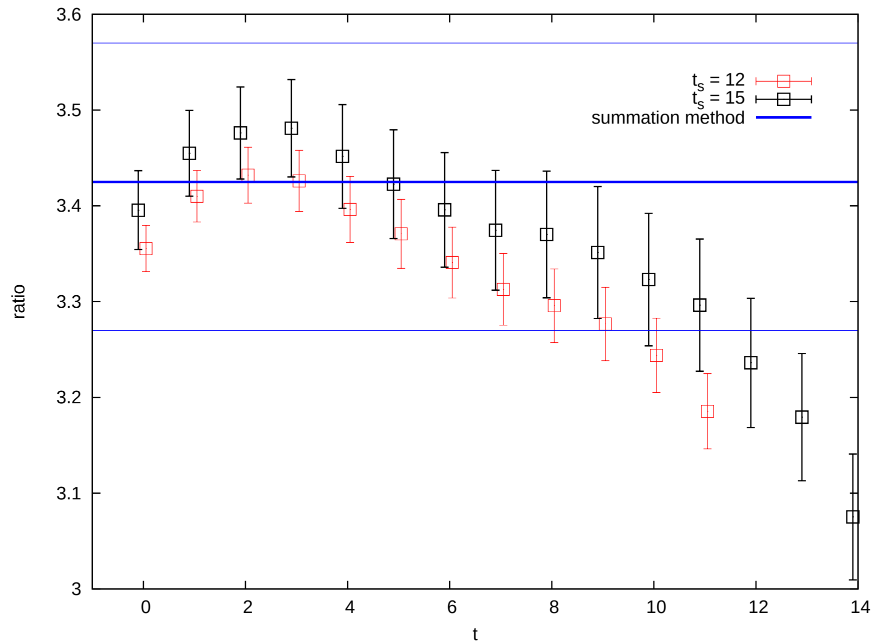

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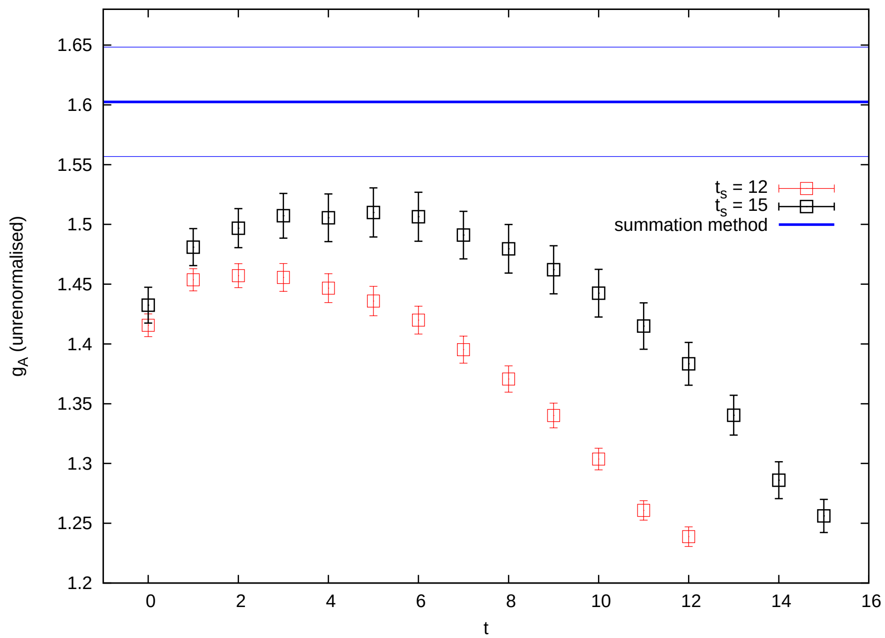

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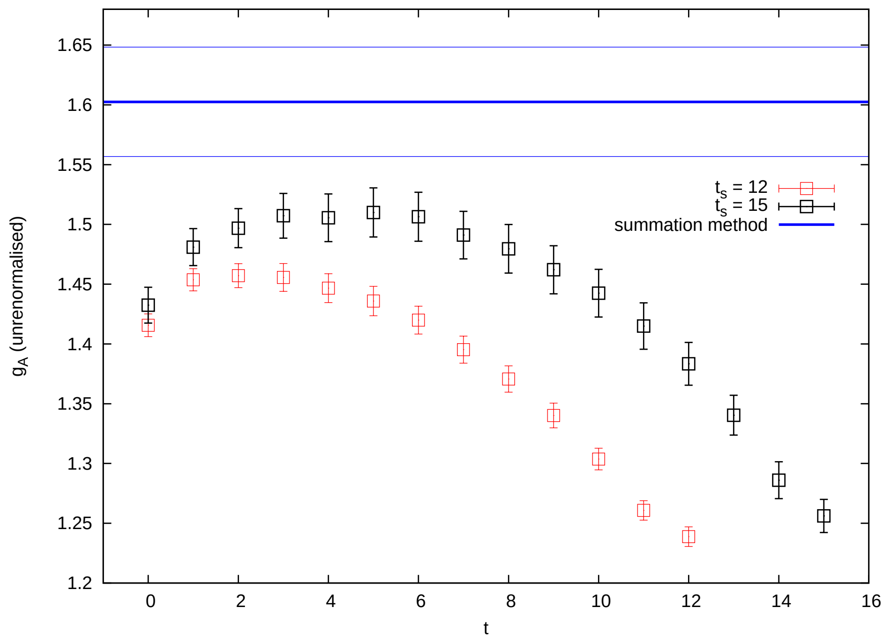

- Better control over excited state contamination

- Larger statistical errors

- Requires more values of t_s

Preliminary results

[Capitani, Della Morte, Knippschild, Meyer, H.W.]

- Ensembles at $\beta = 5.3, \quad a = 0.069\,\mathrm{fm}, \quad 32^3{\cdot}64$ and $48^3{\cdot}96$

  $m_\pi = 290 - 590\,\mathrm{MeV}, \quad L = 2.2$ and $3.3\,\mathrm{fm}, \quad Lm_\pi \gtrsim 5$

  Employ summed insertions to suppress excited state contributions

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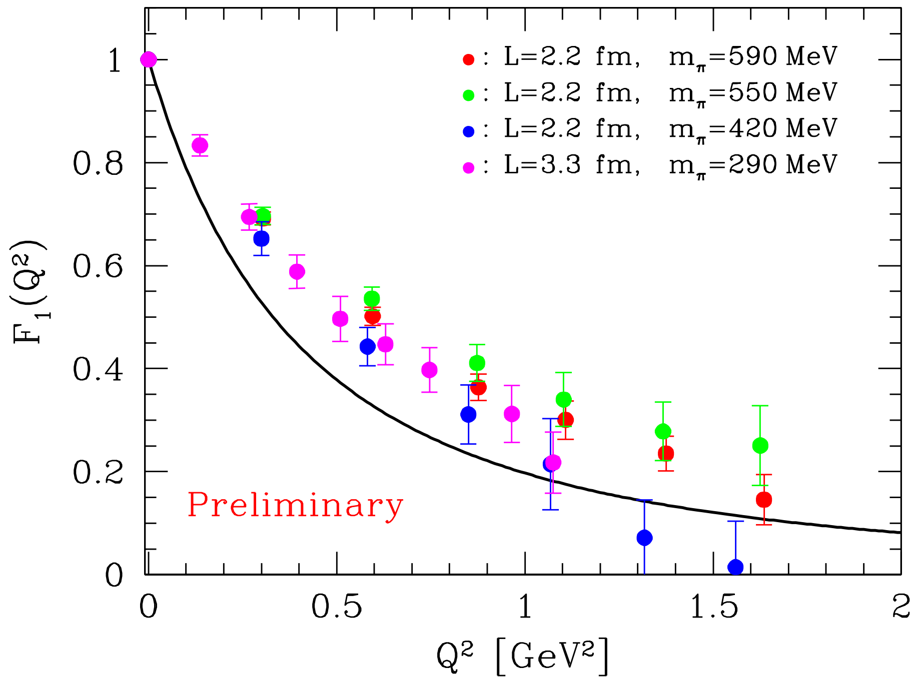
- Dirac form factor:



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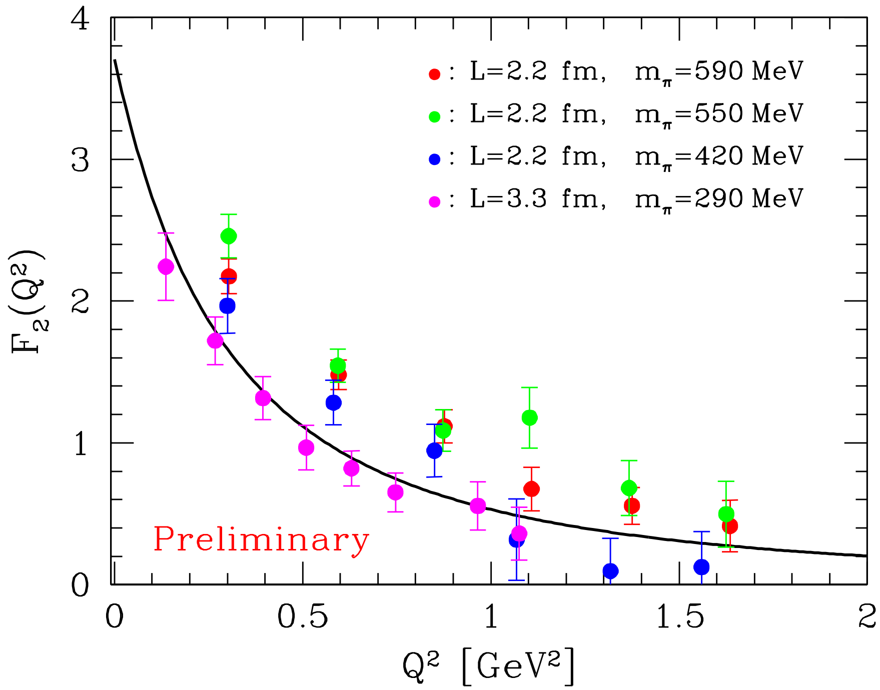
- Ensembles at $\beta = 5.3, \quad a = 0.069\,\mathrm{fm}, \quad 32^3{\cdot}64$ and $48^3{\cdot}96$

  $m_\pi = 290 - 590\,\mathrm{MeV}, \quad L = 2.2$ and $3.3\,\mathrm{fm}, \quad Lm_\pi \gtrsim 5$

  Employ summed insertions to suppress excited state contributions

- Pauli form factor:



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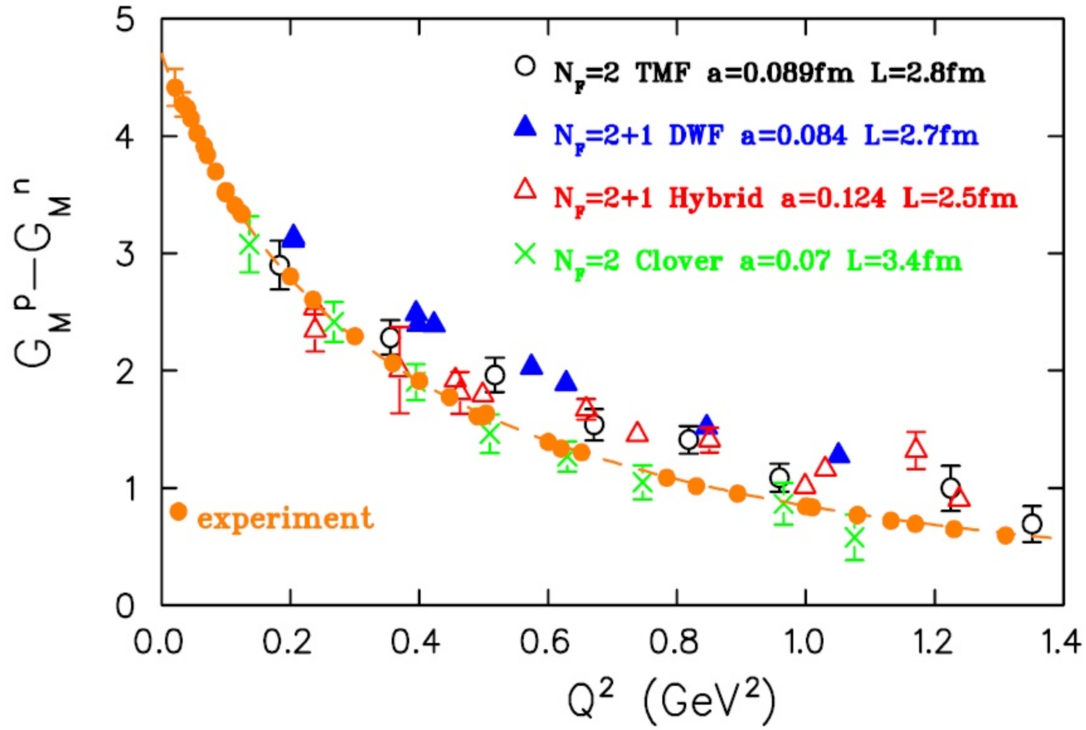
- Comparison: magnetic form factor @ $m_\pi \simeq 300\,\text{MeV}$ *[D. Alexandrou @ Lattice 2010]*



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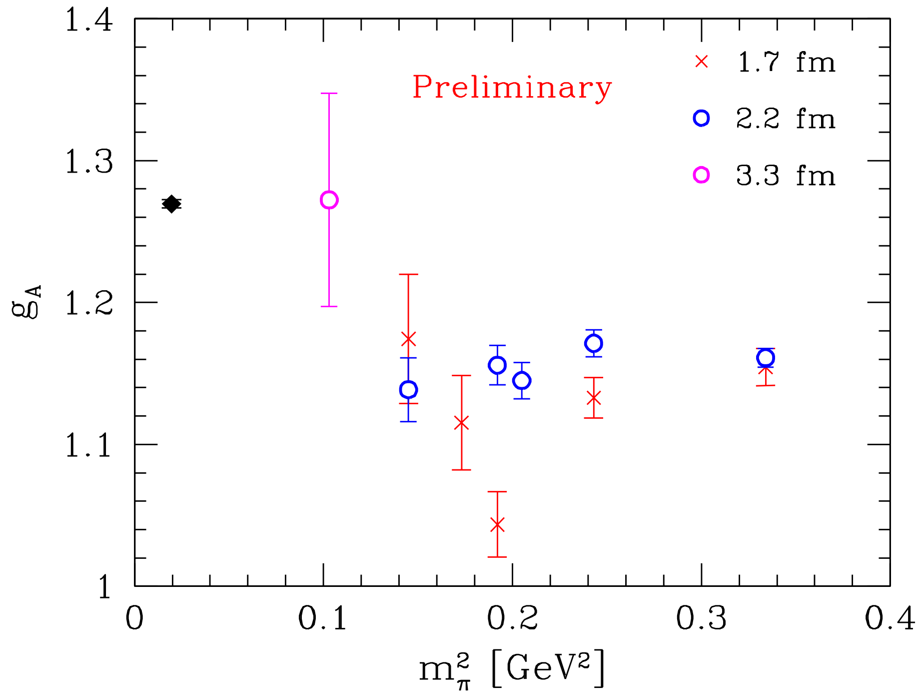
- Axial charge:



4. Hadronic vacuum polarisation and $(g-2)_\mu$

- Muon anomalous magnetic moment: $a_\mu = \frac{1}{2}(g-2)_\mu$

$$a_\mu = \begin{cases} 11\,659\,208(6.3)\cdot 10^{-10} & \text{Experiment} \\ 11\,659\,179(6.5)\cdot 10^{-10} & \text{SM prediction,} \end{cases} \qquad (3.2\sigma \text{ tension})$$

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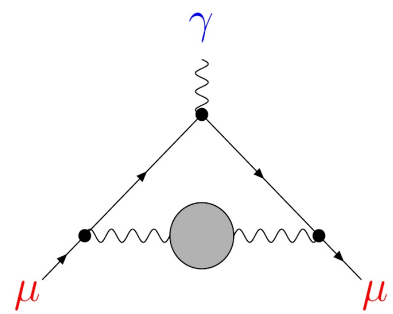
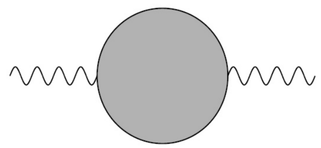
- Hadronic vacuum polarisation; leading contribution:



- Vacuum polarisation tensor:



$$\Pi_{\mu\nu}(q^2) = \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot(x-y)}\, \langle J_\mu(x) J_\nu(y)\rangle \equiv (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

- a_μ^{had} determined from convolution integral: *[Blum; Blum & Aubin]*

$$a_\mu^{\mathsf{had}} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\{\Pi(Q^2) - \Pi(0)\}$$

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Problems for lattice calculations:

- Convolution integral dominated by momenta near m_μ :

  maximum of $f(Q^2)$ located at: $(\sqrt{5}-2)m_\mu^2 \approx 0.003\,\mathrm{GeV}^2$

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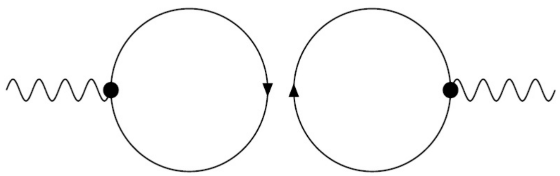
- Contributions from quark disconnected diagrams

  – Large noise-to-signal ratio

  – Twisted boundary conditions useless: effect of twist angle cancels



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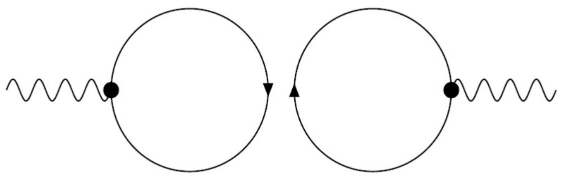

- Resonance effects: $\rho \to \pi\pi$

Strategy for two-flavour QCD

[Della Morte & Jüttner, arXiv:0910.3755]

- QCD with $N_\mathrm{f}=2$ flavours: $J_\mu(x)=\left(\frac{2}{3}j_\mu^{uu}-\frac{1}{3}j_\mu^{dd}\right)(x)$

$$\langle J_\mu(x)J_\nu(y)\rangle=\frac{4}{9}\left\langle j_\mu^{uu}j_\nu^{uu}\right\rangle-\frac{2}{9}\left\langle j_\mu^{uu}j_\nu^{dd}\right\rangle-\frac{2}{9}\left\langle j_\mu^{dd}j_\nu^{uu}\right\rangle+\frac{1}{9}\left\langle j_\mu^{dd}j_\nu^{dd}\right\rangle$$

- Impose isospin symmetry, $m_u=m_d$, set $y\equiv 0$; Correlation function:

$$C_{\mu\nu}(q)=\frac{5}{9}C_{\mu\nu}^{(\mathrm{con})}(q)+\frac{1}{9}C_{\mu\nu}^{(\mathrm{disc})}(q)$$

$$C_{\mu\nu}^{(\mathrm{con})}(q)=\sum_x \mathrm{e}^{iq\cdot x}\left\langle \mathrm{Tr}\left[\overline{\psi}(x)\gamma_\mu\psi(x)\,\overline{\psi}(0)\gamma_\mu\psi(0)\right]\right\rangle$$

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- $C_{\mu\nu}^{(\mathrm{con})}(q)$ and $C_{\mu\nu}^{(\mathrm{disc})}(q)$ have individual continuum and finite volume limits

- $C_{\mu\nu}^{(\mathrm{con})}(q)$ can be evaluated using twisted boundary conditions

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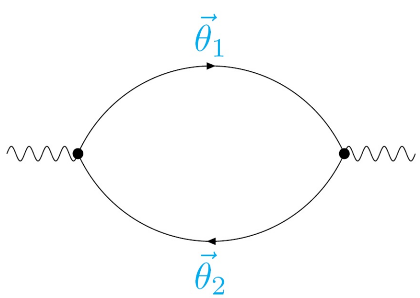

Relative size of the disconnected contribution

[Della Morte & Jüttner, arXiv:0910.3755]

- Compute polarisation tensor in SU(2) ChPT @ NLO
- Construct flavour-singlet and non-singlet contributions:

$$\Pi_{\mu\nu}^{(a,a)}(q^2) \quad \leftrightarrow \quad \overline{\psi}\tfrac{1}{2}\tau^a\gamma_\mu\psi$$

$$\Pi^{(\mathrm{con})}(q^2) = \tfrac{10}{9}\Pi^{(3,3)}(q^2), \quad \Pi^{(\mathrm{disc})}(q^2) = \tfrac{1}{9}\Big(\Pi^{(0,0)}(q^2) - \Pi^{(3,3)}(q^2)\Big)$$

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- Combination $\Pi(q^2) - \Pi(0)$ enters convolution integral
- $\Pi^{(0,0)}(q^2)$ is momentum-independent in NLO ChPT!

$$\Rightarrow \quad \frac{\Pi^{(\text{disc})}(q^2) - \Pi^{(\text{disc})}(0)}{\Pi^{(\text{con})}(q^2) - \Pi^{(\text{con})}(0)} = -\frac{1}{10}$$

$\longrightarrow$ Effect of disconnected contribution estimated to be a 10% downward shift

Calculation of a_μ^{had} in two-flavour QCD

- Compute connected contribution using twisted boundary conditions

  $\longrightarrow$ can extend q^2-range to smaller values

  $\longrightarrow$ probe region where convolution integral receives dominant contribution

- Compute disconnected contribution for Fourier modes only:

$$|q^2| = \left(\vec{n}\frac{2\pi}{T}\right)^2$$

  $\longrightarrow$ validate its relative suppression predicted by ChPT

- Include contributions of partially quenched strange quark

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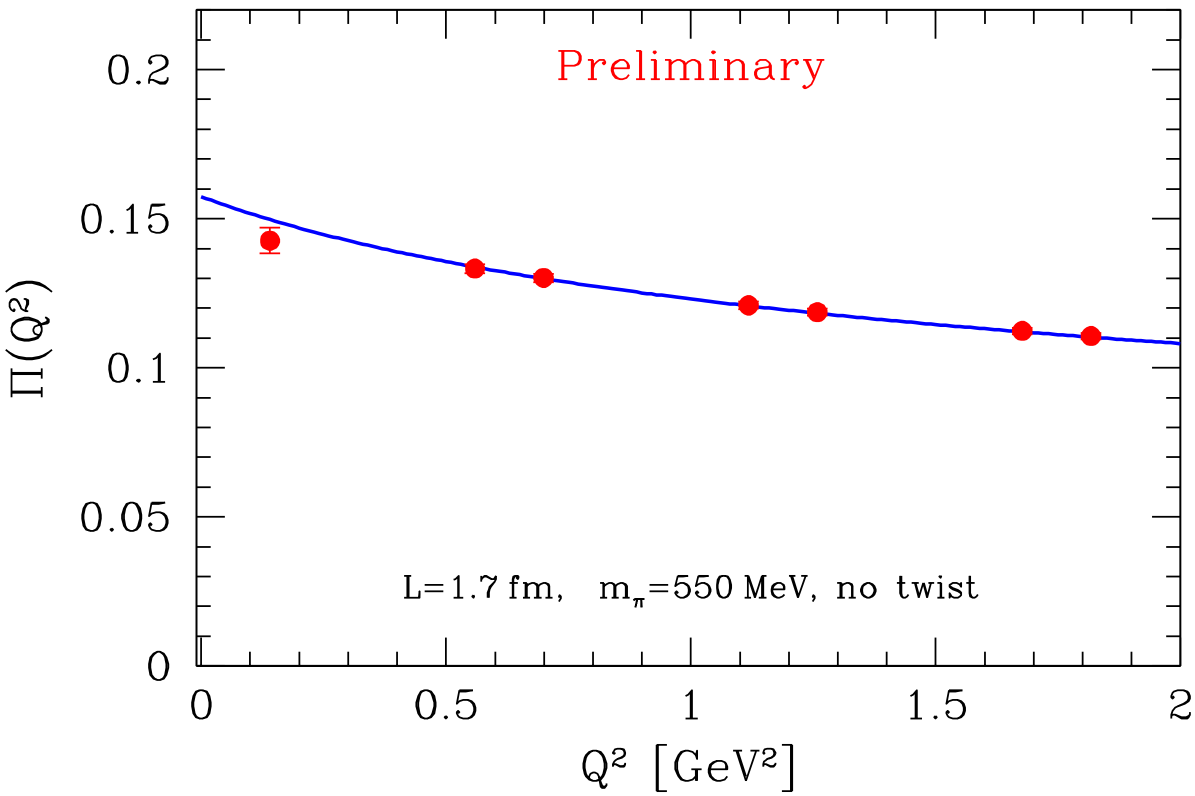

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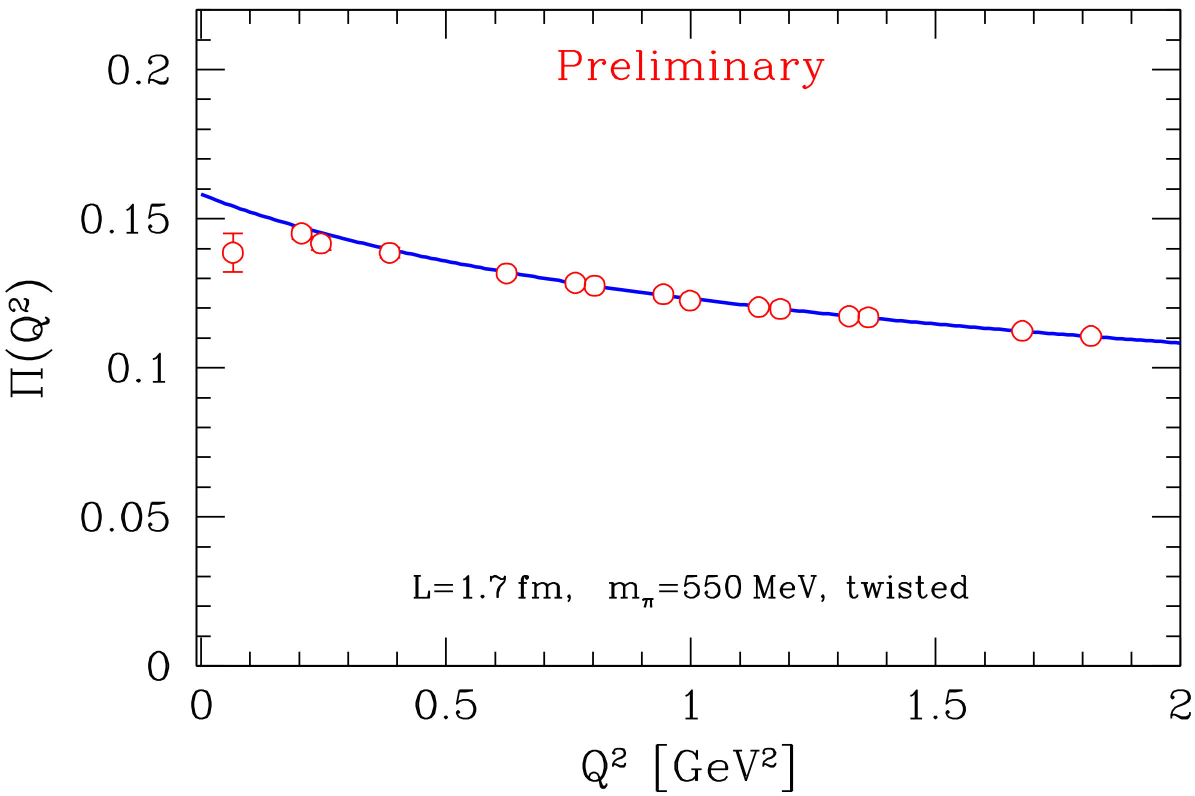

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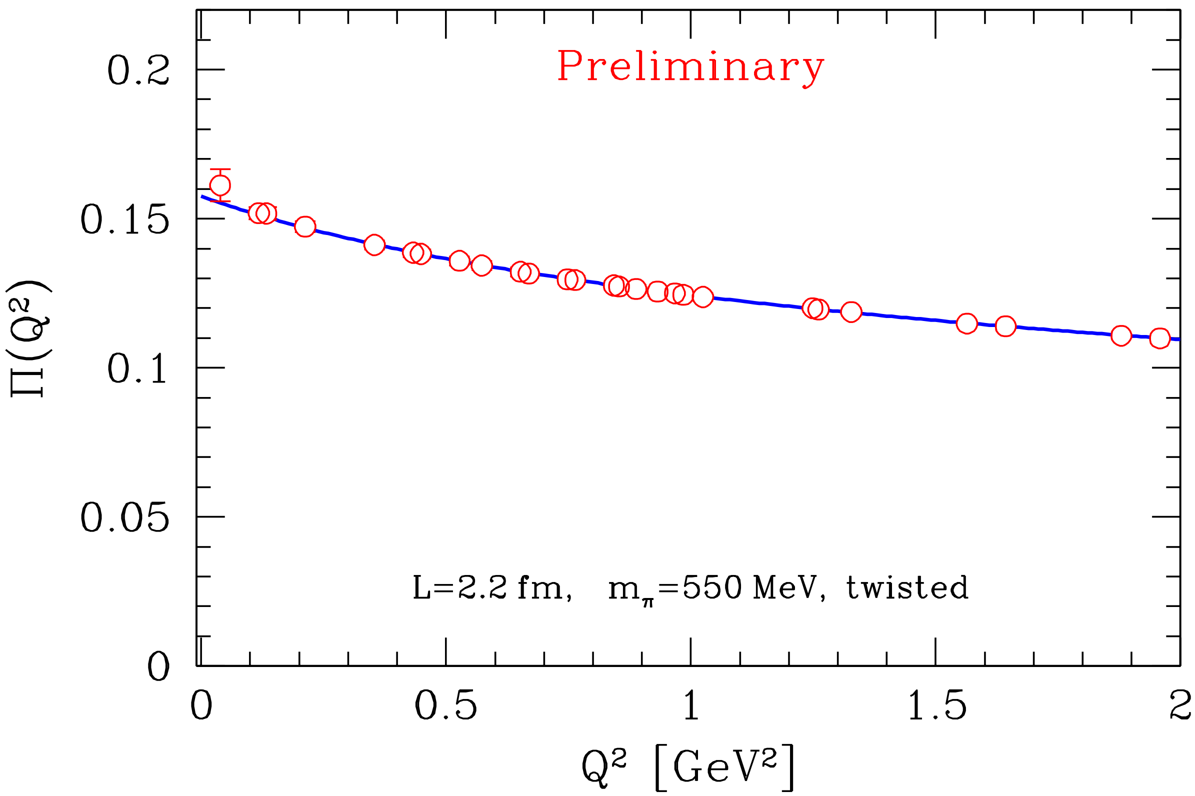

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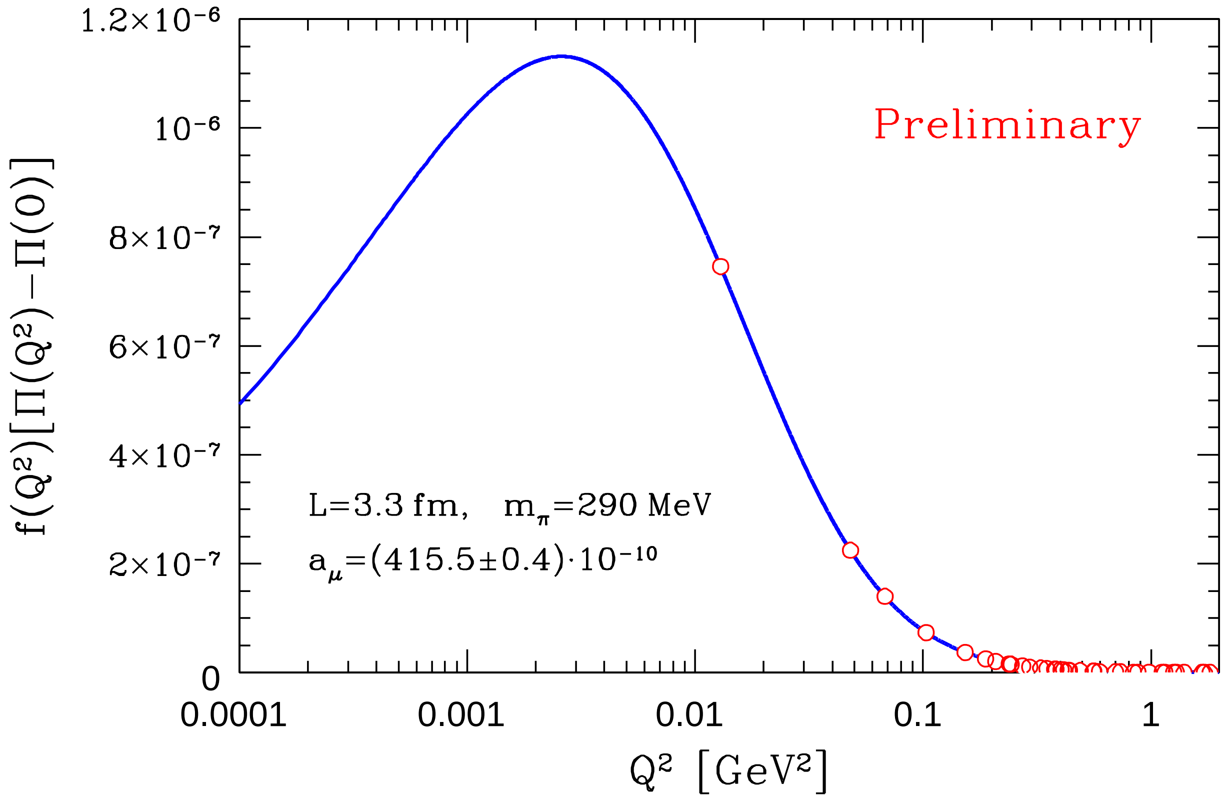

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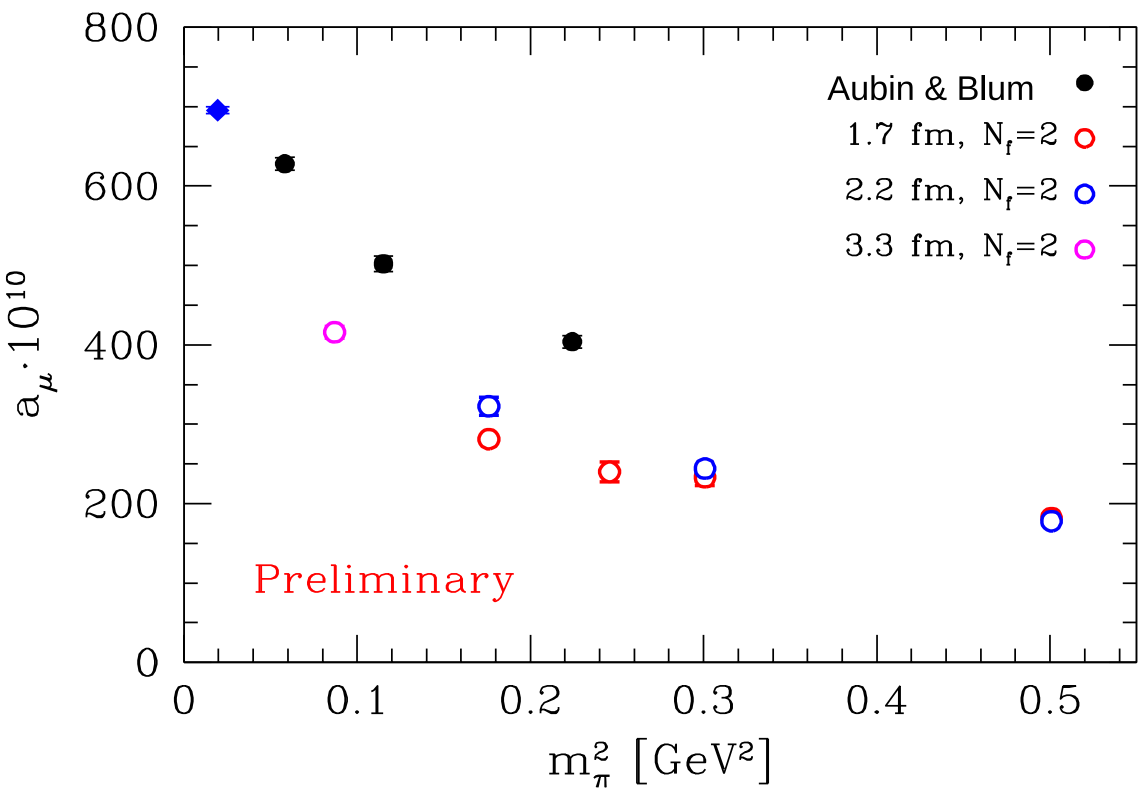
- Twisted boundary conditions stabilise fits to Q^2-dependence and extrapolation to $\Pi(0)$
- Pion mass & volume dependence (u, d-contributions):



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[Della Morte, Jäger, Jüttner, H.W.]

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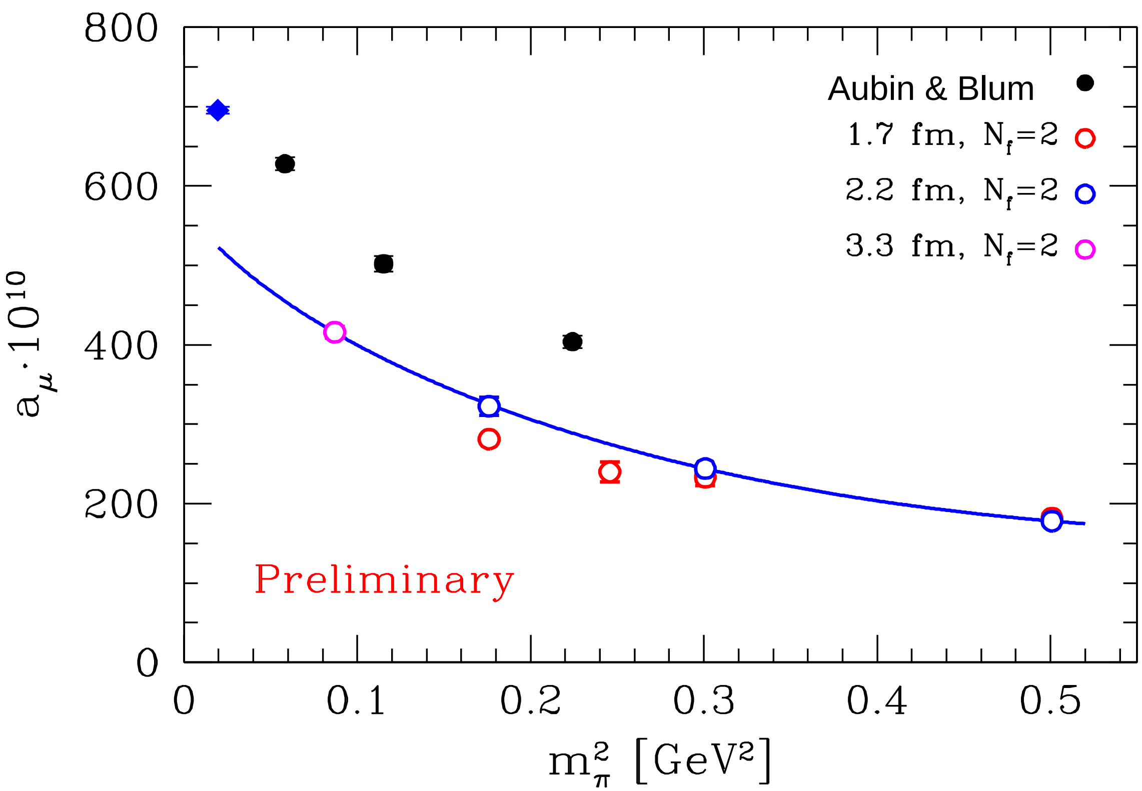
- Chiral fit: $A + Bm_\pi^2 + Cm_\pi^2 \ln m_\pi^2, \qquad (L = 2.2\,\text{fm})$



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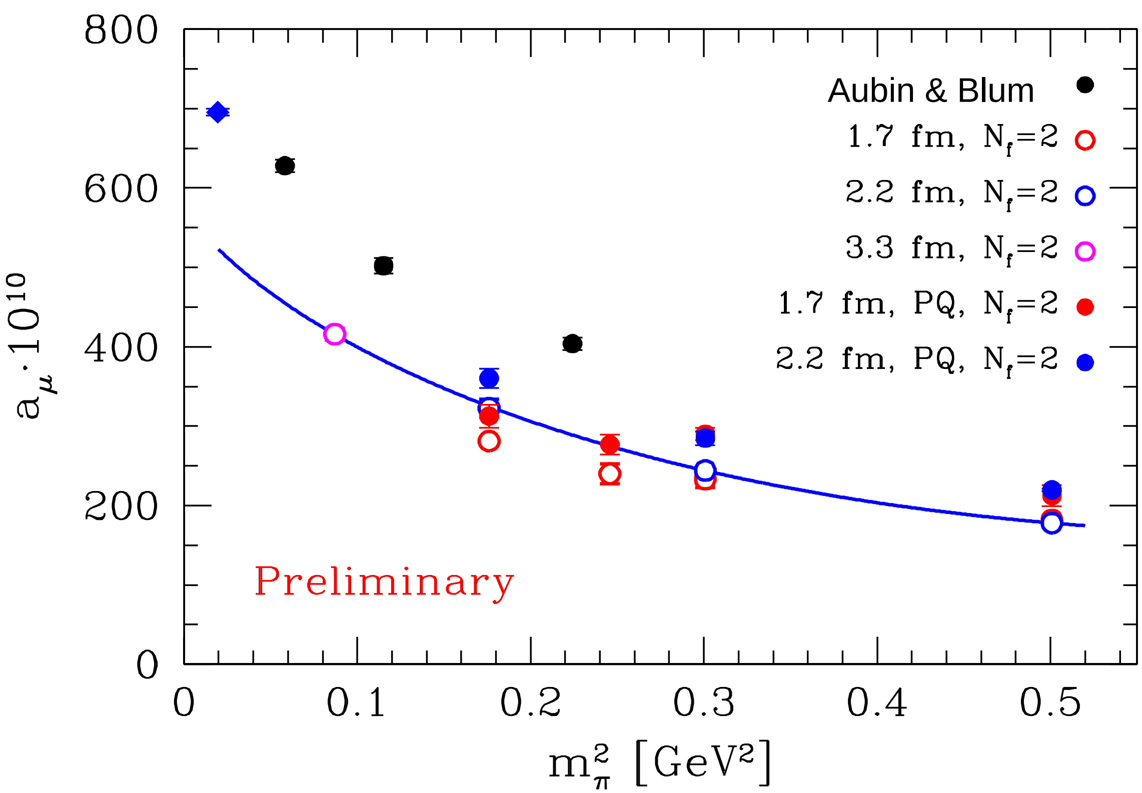

Summary

Progress in controlling systematic uncertainties in form factor calculations using CLS ensembles

- Pion form factor:
  - precise, model-independent estimates of $\langle r_\pi^2 \rangle$ via twisted boundary conditions
- Nucleon form factors and g_A :
  - summed insertions help control excited state contamination
  - situation far from settled (pion masses, volumes, discretisation effects)
- Lattice calculations of a_μ^{had} :
  - twisted boundary conditions help stabilise fits
  - ChPT prediction for contribution from quark-disconnected diagrams

Outlook

- Include smaller pion masses; increase statistics
- Study lattice artefacts
- Small lattice spacings: study form factors at large Q^2 (c.f. $12\,\mathrm{GeV}$ upgrade at JLab)
- Include quark-disconnected diagrams