Spectra of the Wilson Dirac operator at nonzero lattice spacing

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What Microscopic eigenvalue spectrum

New Lattice QCD effects included ($a \neq 0$)

Why Extract continium physics from the lattice

How Chiral Perturbation Theory

Warm up: Zero a

CPT in the ϵ-regime $m\Sigma V \sim 1$ $\quad a = 0$

The partition function in a sector of topological charge ν

$$Z^{\nu}_{N_f}(m; a = 0) \ = \ \int_{U(N_f)} dU \ \det{}^{\nu}(U) e^{\frac{m}{2}\Sigma V \mathrm{Tr}(U+U^\dagger)}$$

A group integral (*not a path integral*)

Σ is the chiral condensate

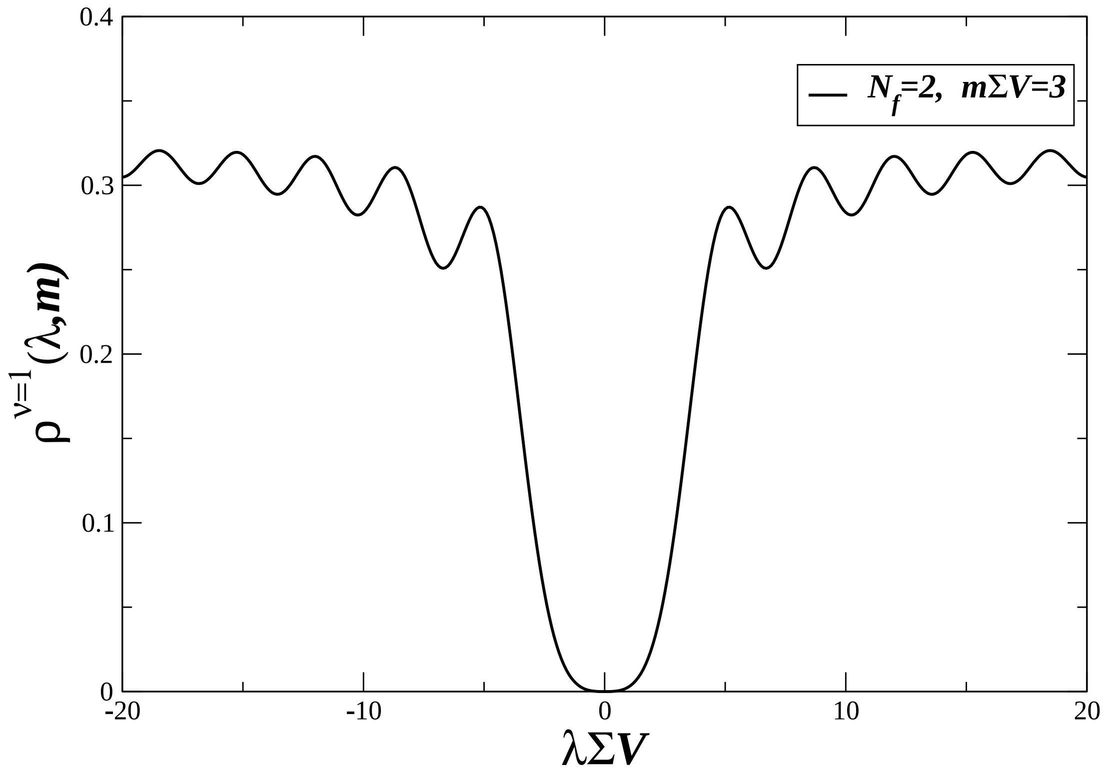
Gasser, Leutwyler, PLB **188**(1987) 477; NPB **307** (1988) 763

Leutwyler, Smilga, PRD 46 (1992) 5607

Eigenvalue density at $a = 0$:

$$\gamma_5 D = -D\gamma_5 \qquad \text{eigenvalues in pairs } (i\lambda, -i\lambda)$$

ν zero ev's



Verbaarschot Wettig Ann.Rev.Nucl.Part.Sci. 50 (2000) 343, hep-ph/0003017

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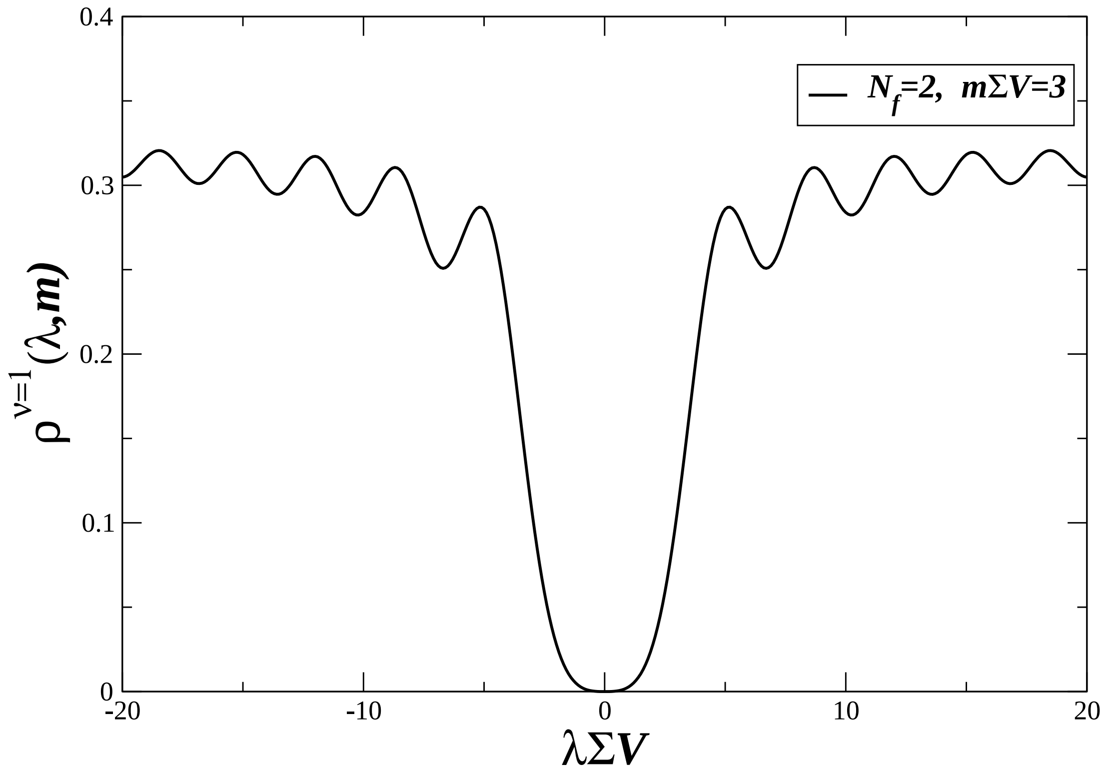

One fit parameter Σ

Verbaarschot Wettig Ann.Rev.Nucl.Part.Sci. 50 (2000) 343, hep-ph/0003017

New: non zero lattice spacing a

Goal: *analytic predictions for the Dirac spectrum with* $a \neq 0$

Discretization effects depend on the discretization

Here: Wilson fermions

$$\begin{aligned} \gamma_5 D_W &\neq -D_W \gamma_5 \\ D_W^\dagger &\neq -D_W \end{aligned}$$

γ_5-hermiticity

$$D_W^\dagger = \gamma_5 D_W \gamma_5$$

Itho, Iwasaki, Yoshie, PRD 36 (1987) 527

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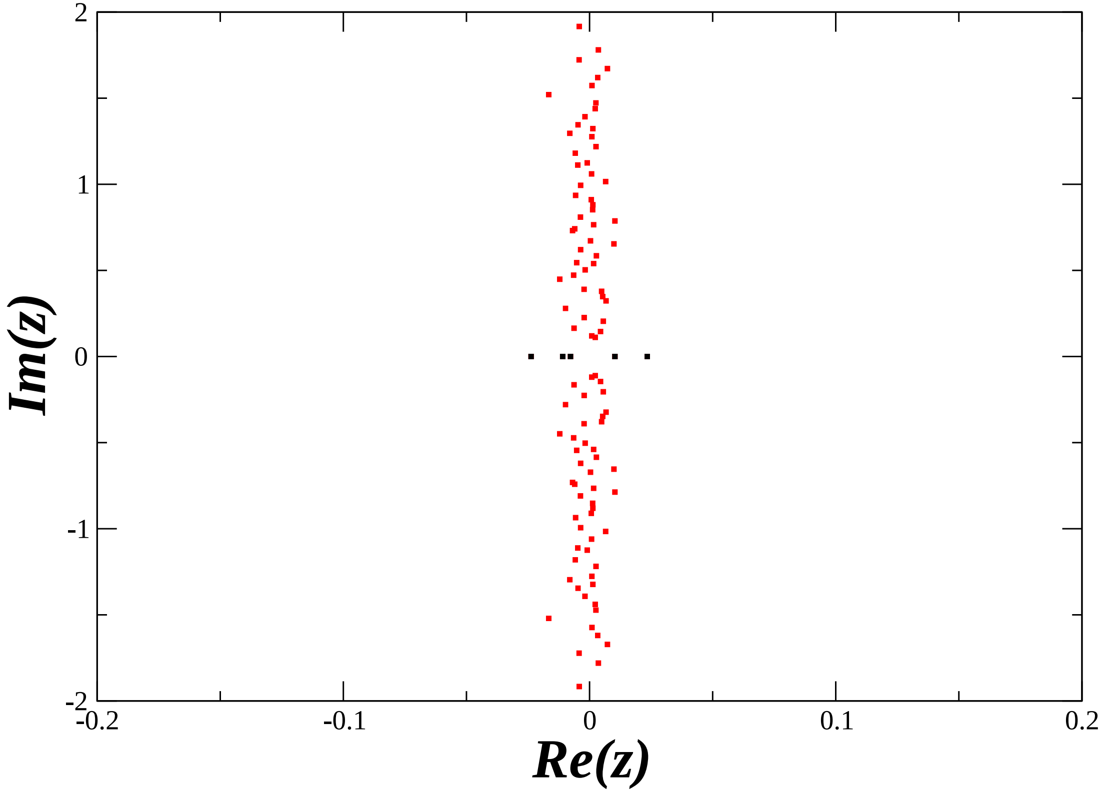
Eigenvalues, z, of D_W

- complex conjugate pairs (z, z^*)

- exact real eigenvalues

Itho, Iwasaki, Yoshie, PRD 36 (1987) 527

Eigenvalues, z, of D_W (*illustration*)



Goal: *analytic predictions for the Wilson Dirac spectrum with* $a \neq 0$

Lüscher JHEP0707:081,2007

Del Debbio Giusti Lüscher Petronzio Tantalo JHEP0702:082,2007

Method: *Wilson Chiral Perturbation Theory*

Sharpe PRD 74 (2006) 014512: p-regime

Wilson CPT

The chiral Lagrangian for Wilson fermions has new terms

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \frac{F_\pi^2}{4}\mathrm{Tr}\left(d_\mu U d_\mu U^\dagger\right) + \frac{m}{2}\Sigma\mathrm{Tr}(U+U^\dagger) \\
& -a^2 W_6[\mathrm{Tr}\left(U+U^\dagger\right)]^2 - a^2 W_7[\mathrm{Tr}\left(U-U^\dagger\right)]^2 \\
& -a^2 W_8 \mathrm{Tr}(U^2 + {U^\dagger}^2)
\end{aligned}
$$

with new constants W_6, W_7 and W_8

Sharpe Singleton PRD **58**, 074501 (1998)
Rupak Shoresh PRD **66**, 054503 (2002)
Bar Rupak Shoresh PRD **70**, 034508 (2004)
Sharpe Wu PRD **70**, 094029 (2004)
Golterman Sharpe Singleton PRD **71**, 094503 (2005)
Del Debbio Frandsen Panagopoulos Sannino JHEP0806:007 (2008)
Shindler PLB 672, 82 (2009)

Bar Necco Schaefer JHEP 0903, 006 (2009)

Wilson CPT in the ϵ-regime $(m\Sigma V \sim a^2 V W_i \sim 1)$

The partition function in a sector ν

$$Z_{N_f}^{\nu} = \int_{U(N_f)} dU \, \det{}^{\nu}(U) \, e^{S}$$

with

$$\begin{aligned} S = & +\frac{m}{2}\Sigma V \mathrm{Tr}(U + U^{\dagger}) \\ & -a^2 V W_6 [\mathrm{Tr}\left(U + U^{\dagger}\right)]^2 - a^2 V W_7 [\mathrm{Tr}\left(U - U^{\dagger}\right)]^2 \\ & -a^2 V W_8 \mathrm{Tr}(U^2 + {U^{\dagger}}^2) \end{aligned}$$

Damgaard Splittorff Verbaarschot arXiv:1001.2937

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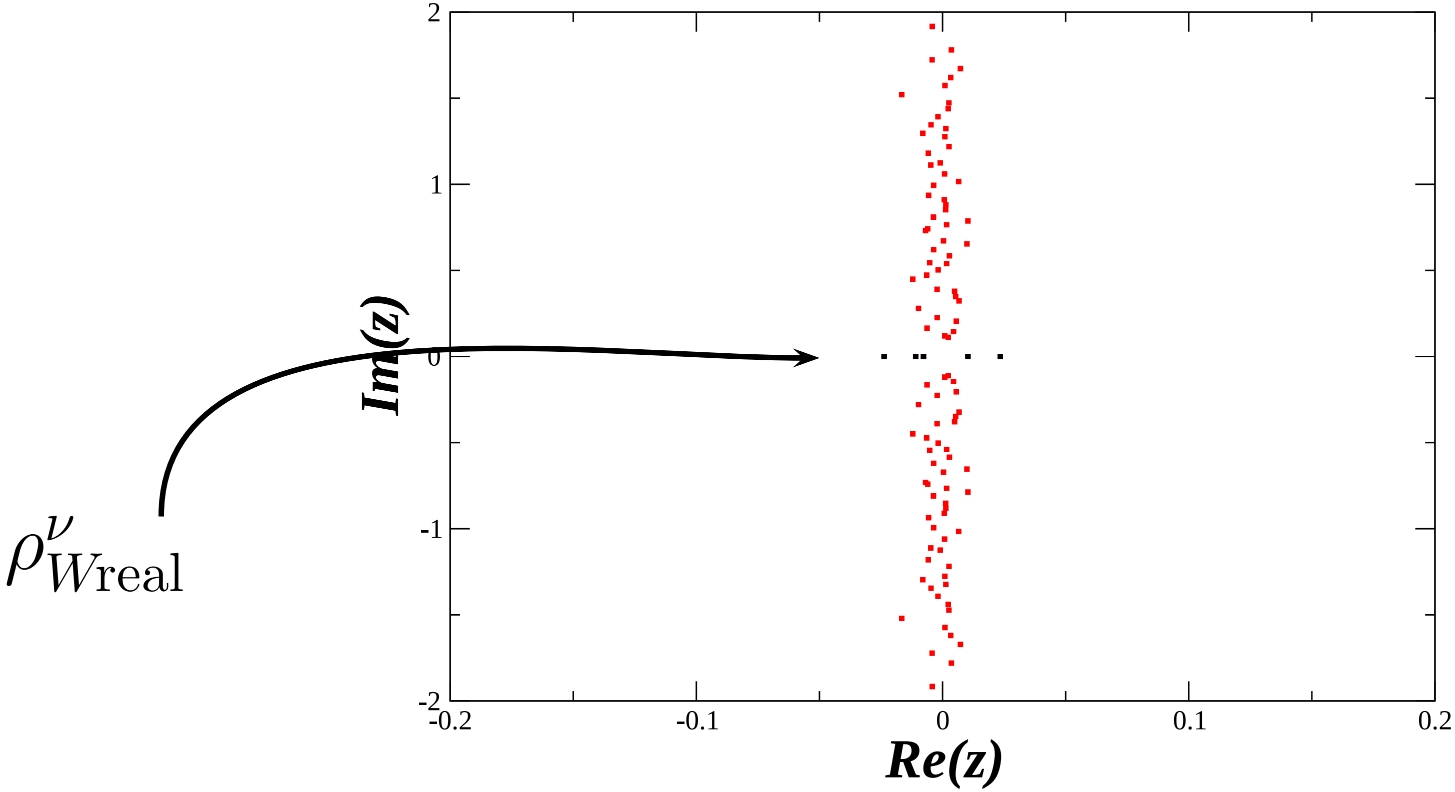
Non trivial fact: In sector ν the Wilson Dirac operator D_W has ν real eigenvalues

Damgaard Splittorff Verbaarschot arXiv:1001.2937

Eigenvalues, z, of D_W (*illustration*)



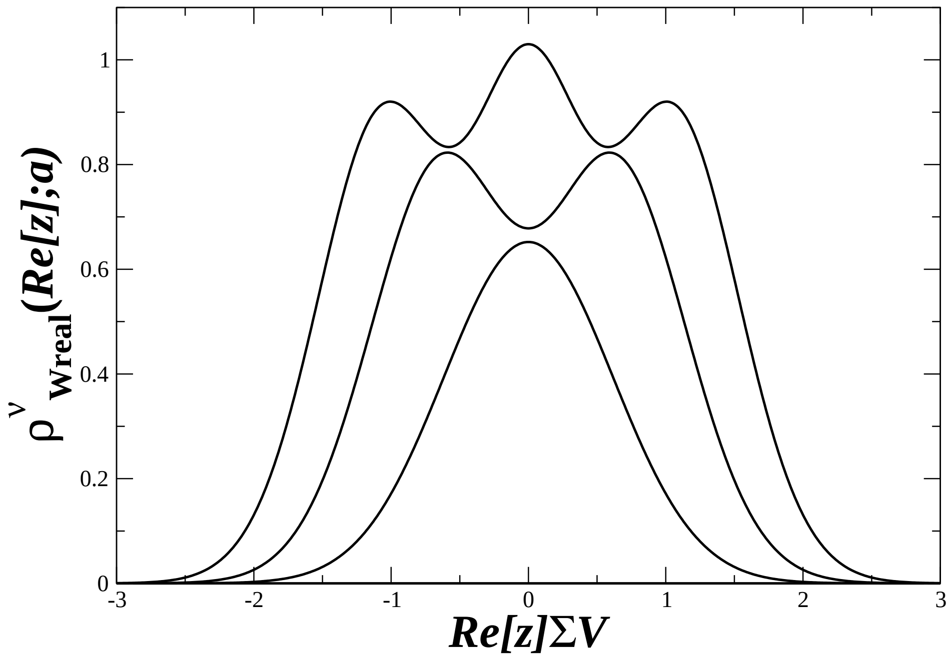
Microscopic density of D_W

The ν real eigenvalues of D_W in sector $\nu = 0, 1, 2, 3$

$N_f = 0$

$a\sqrt{W_8 V} = 0.2$

$W_6 = W_7 = 0$



Gattringer Hip NPB 536 (1998) 363

Hernandez NPB 536 (1998) 345

Damgaard Splittorff Verbaarschot arXiv:1001.2937

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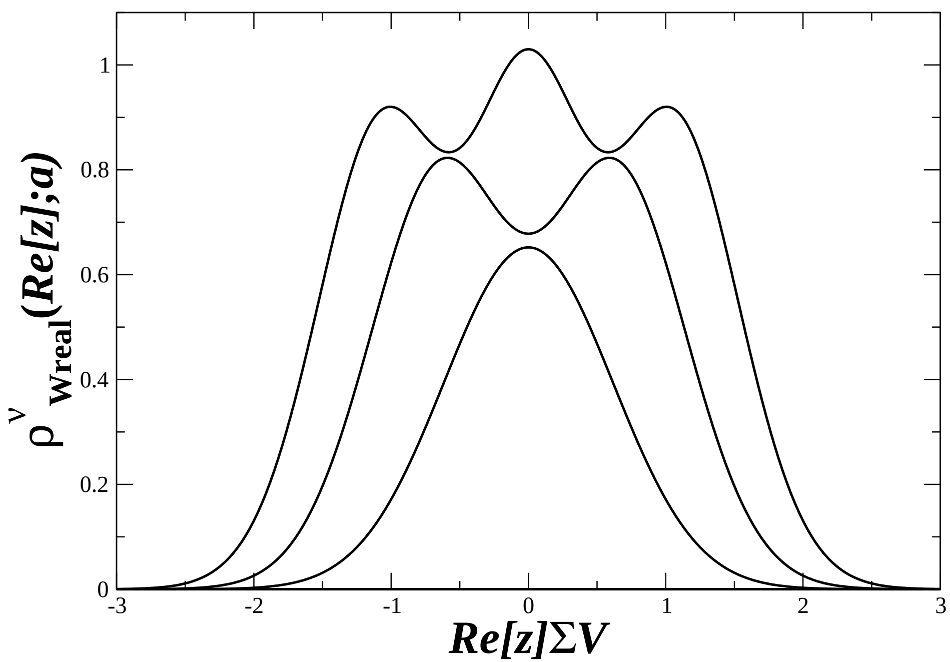

$$\langle Re[z]^2 \rangle_\nu = 8a_8^2 \nu(\nu + 4a_8^2) \qquad a_8 \equiv a\sqrt{W_8 V}$$

Gattringer Hip NPB 536 (1998) 363

Hernandez NPB 536 (1998) 345

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The Hermitian Wilson Dirac operator D_5

Introduce

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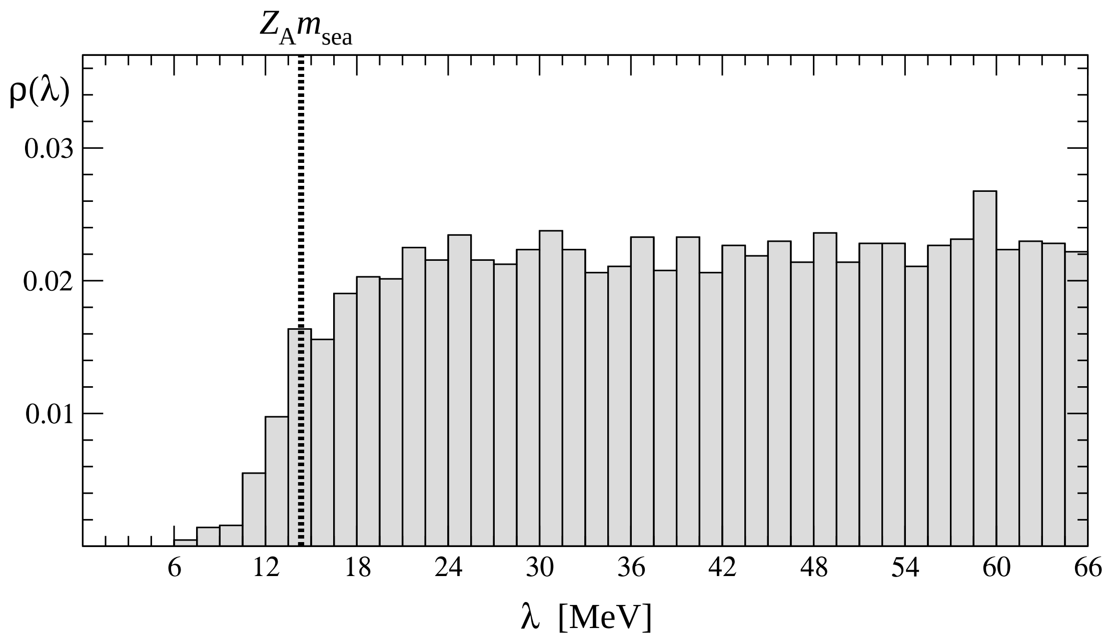
D_5 is hermitian but spectrum *not* symmetric: *not* $(x, -x)$

Lattice

Spectrum of D_5



- Aoki phase when gap closes

Lüscher JHEP0707:081,2007
Del Debbio Giusti Lüscher Petronzio Tantalo JHEP0702:082,2007
Aoki PRD 30 (1984) 2653

Bitar Heller Narayanan PLB **418** 167 (1998)

From Wilson CPT to the spectrum of D_5

The SUSY method

The SUSY way of writing the eigenvalue density

$$\rho_5^{N_f}(x,m;a) = \frac{1}{\pi}\mathrm{Im}[\lim_{x'\to x}\frac{d}{dx}Z^{\nu}_{N_f+1|1}(m,m,x,x';a)]$$

SUSY/graded *generating function* for the eigenvalue density

$$Z^{\nu}_{N_f+1|1}(m,m,x,x';a) = \int \mathrm{d}A\,\det(D_\eta\gamma_\mu+m)^{N_f}\,\frac{\det(D_\mu\gamma_\eta+m+x\gamma_5)}{\det(D_\mu\gamma_\eta+m+x'\gamma_5)}\,\mathrm{e}^{-S_{\mathrm{YM}}(A)}$$

Efetov *Supersymmetry in disorder and chaos* Cambridge Uni Press 1997

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integrate over the gauge fields

Efetov *Supersymmetry in disorder and chaos* Cambridge Uni Press 1997

The SUSY method in Wilson CPT

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SUSY/graded *generating function* for the eigenvalue density

$$\mathcal{Z}_{N_f+1|1}(m,m,x,x';a) = \int dU\ \mathrm{Sdet}(U)^{\nu} \times e^{i\frac{1}{2}\mathrm{Str}(\mathcal{M}[U-U^{-1}])+i\frac{1}{2}\mathrm{Str}(\mathcal{X}[U+U^{-1}])+a^2W_8V\mathrm{Str}(U^2+U^{-2})}$$

Damgaard Osborn Toublan Verbaarschot NPB **547** 305 (1999): $a = 0$

Splittorff, Verbaarschot, NPB 683 (2004) 467: $\mu \neq 0$

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SUSY/graded *generating function* for the eigenvalue density

$$\mathcal{Z}_{N_f+1|1}(m,m,x,x';a) =$$

$$\int dU\ \mathrm{Sdet}(U)^{\nu}$$

$$\times e^{i\frac{1}{2}\mathrm{Str}(\mathcal{M}[U-U^{-1}])+i\frac{1}{2}\mathrm{Str}(\mathcal{X}[U+U^{-1}])+a^2W_8V\mathrm{Str}(U^2+U^{-2})}$$

integrate over graded Goldstone manifold $Gl(N_f+1|1)$

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Quenched microscopic density of $D_5 = \gamma_5(D_W + m)$

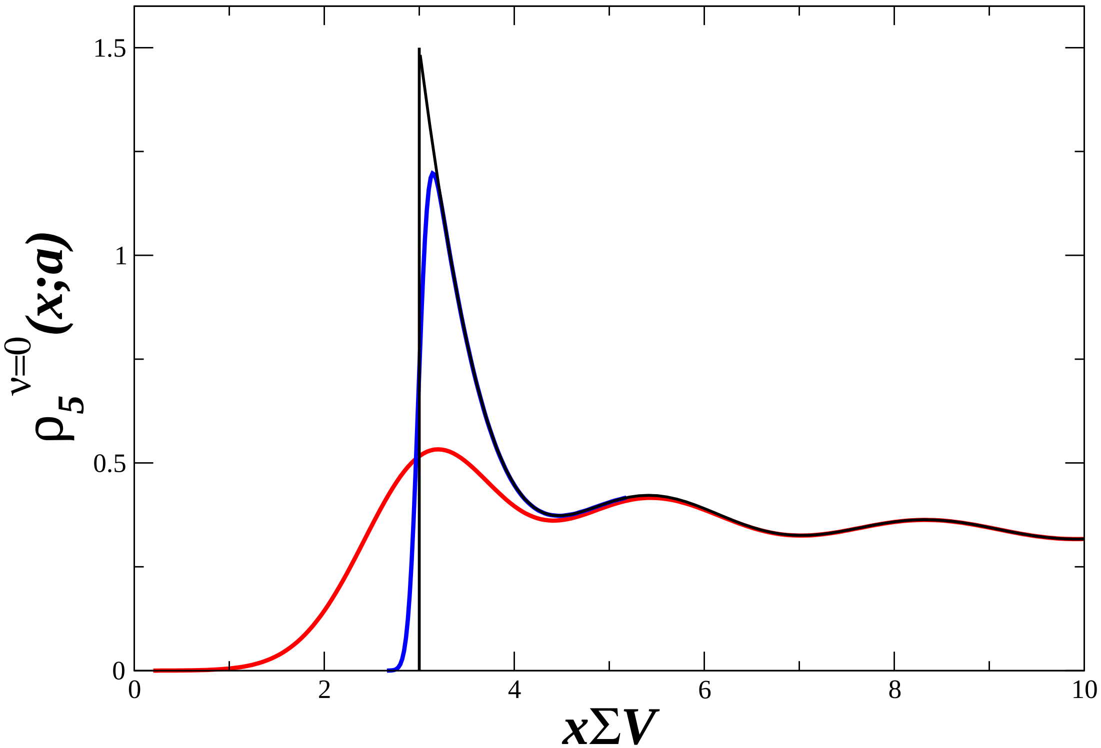
Sector $\nu = 0$

$m\Sigma V = 3$

$a\sqrt{W_8 V} = 0$

$a\sqrt{W_8 V} = 0.03$

$a\sqrt{W_8 V} = 0.250$



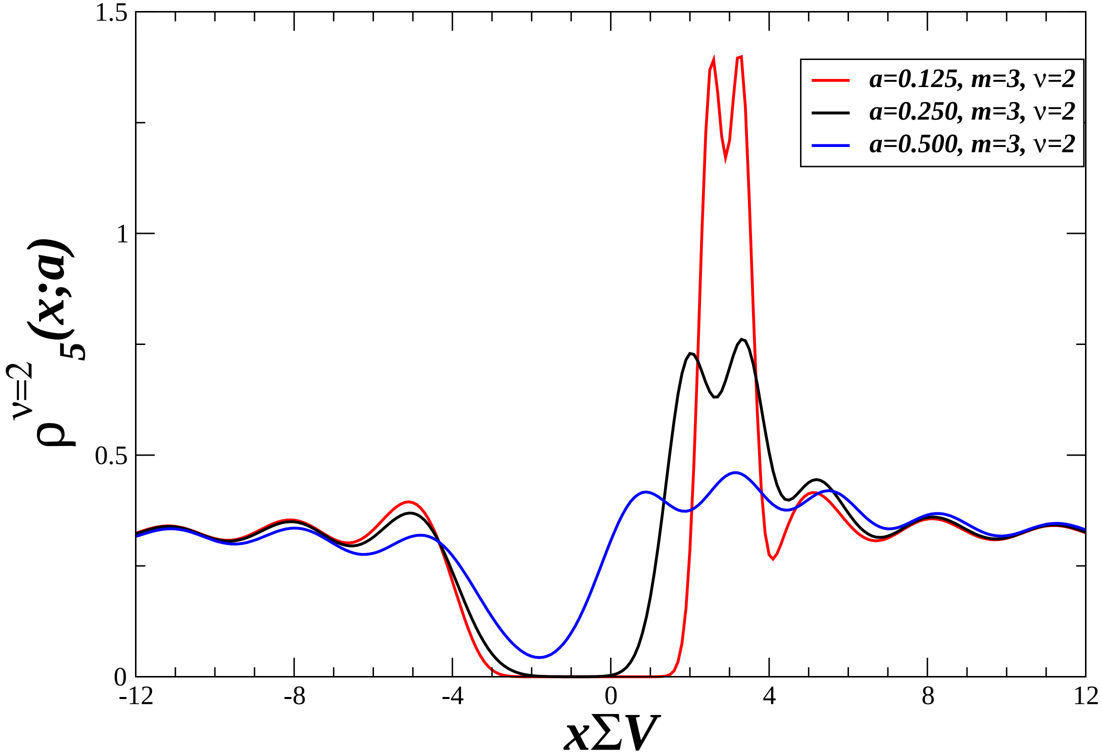
For $\nu = 0$ the density is symmetric: $\rho_5^{\nu=0}(x;a) = \rho_5^{\nu=0}(-x;a)$

Damgaard Splittorff Verbaarschot arXiv:1001.2937

Quenched microscopic density of $D_5 = \gamma_5(D_W + m)$

Sector $\nu = 2$ increasing $a\sqrt{W_8 V}$

$m\Sigma V = 3$

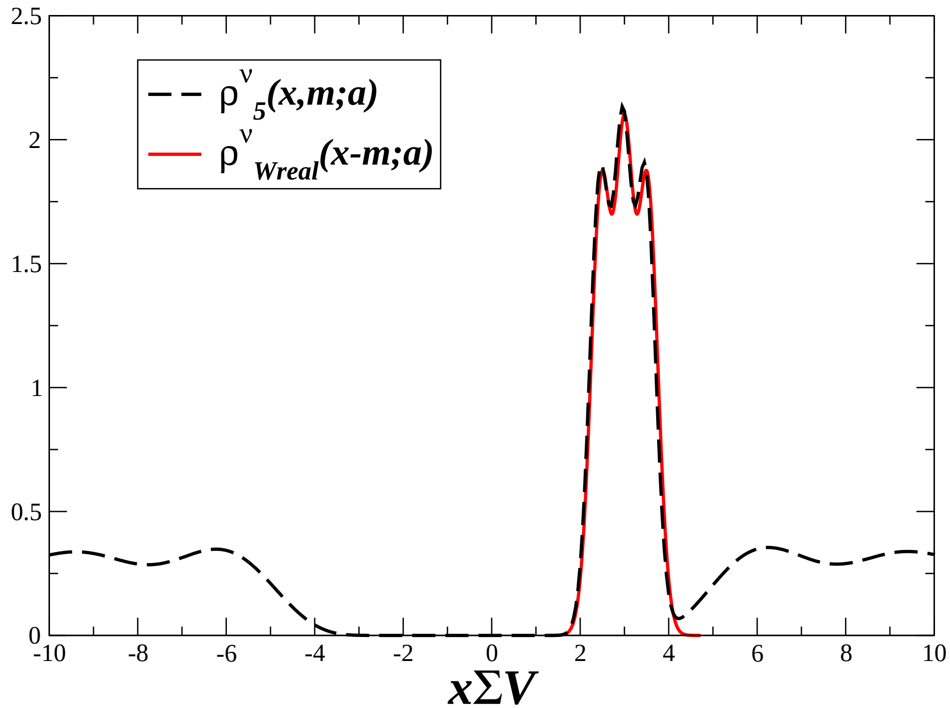


Quenched microscopic density of D_5 and D_W

Sector $\nu = 3$:

$a\sqrt{W_8 V} = 0.1$

$m\Sigma V = 3$



The ν real modes, ϕ, of D_W are almost chiral: $\phi^\dagger \gamma_5 \phi \simeq 1$

Itho, Iwasaki, Yoshie, PRD 36 (1987) 527

Unquenched

- $N_f = 1$ solved
- General N_f solved in the limit $a\sqrt{VW_8} \ll 1$

Unquenched

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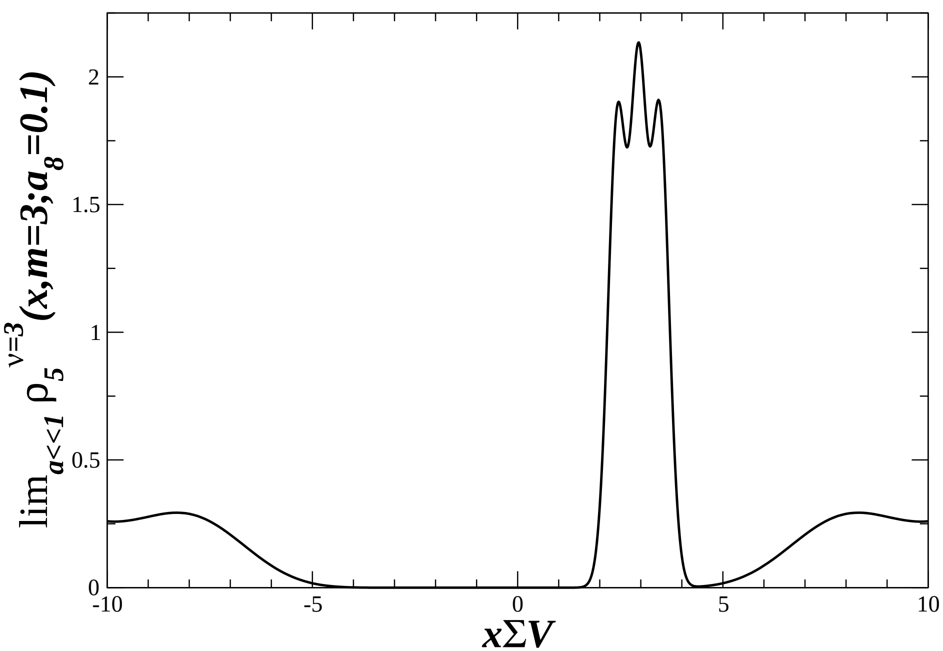
$N_f = 2$

$m\Sigma V = 3$

$a\sqrt{VW_8} = 0.1$



small $a\sqrt{VW_8}$ limit

The sign of W_8

γ_5-Hermiticity $\Rightarrow \det^2(D_W + m) \geq 0$

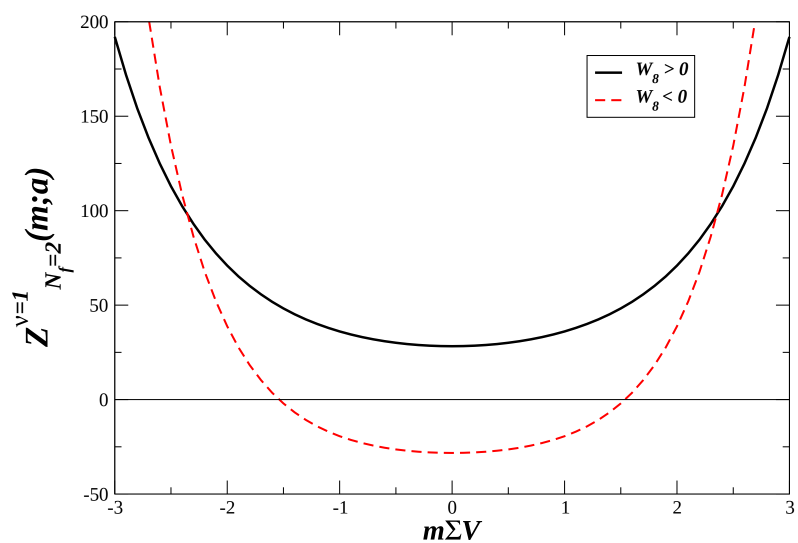
QCD inequality $\qquad Z^{\nu}_{N_f=2}(m; a) \geq 0$

The sign of W_8

γ_5-Hermiticity $\Rightarrow \det^2(D_W + m) \geq 0$

$$\text{QCD inequality} \qquad Z^{\nu}_{N_f=2}(m; a) \geq 0$$

Only satisfied if $W_8 > 0$ (the sign which gives an Aoki phase)



$a^2VW_8 = 1$ (full) $\qquad a^2VW_8 = -1$ (dashed)

The sign of W_8

Wilson CPT with $W_8 < 0$ corresponds to an *anti hermitian* D_W !

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Natural explanation in Wilson Random Matrix Theory:

γ_5-Hermitian

$$D_{WRMT} = \begin{pmatrix} aA & iW \\ iW^\dagger & aB \end{pmatrix} \qquad \rightleftharpoons \qquad W_8 > 0$$

Anti-Hermitian (not γ_5-Hermitian)

$$D_{WRMT} = \begin{pmatrix} iaA & iW \\ iW^\dagger & iaB \end{pmatrix} \qquad \rightleftharpoons \qquad W_8 < 0$$

Conclusions

Derived the microscopic eigenvalue density from WCPT

- for the real eigenvalues of D_W

- for $D_5 = \gamma_5(D_W + m)$

in sectors with fixed number of real eigenvalues of D_W

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$N_f = 0$ *and* $N_f = 1$ *solved (SUSY method incl* W_6 *and* W_7*)*

For general N_f *we have solved the small* $a^2 V W_8$ *limit*

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For general N_f *we have solved the small* a^2VW_8 *limit*

Next:

Unquenched $N_f = 2$, summation over ν, twisted mass, individual eigenvalues, dist of chirality ... *suggestions are welcome!*

Additional slides

W_6 and W_7

The double-trace terms re-expressed as gaussian integrals

$$Z^{\nu}_{N_f}(m,x;a_6,a_8) = \frac{1}{4\sqrt{\pi}a_6}\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{16|a_6^2|}}\, Z^{\nu}_{N_f}(m+y,x;a_6=0,a_8)$$

where $a_6 = a\sqrt{W_6 V}$ and $a_8 = a\sqrt{W_8 V}$

W_6 and W_7

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Also works for the density

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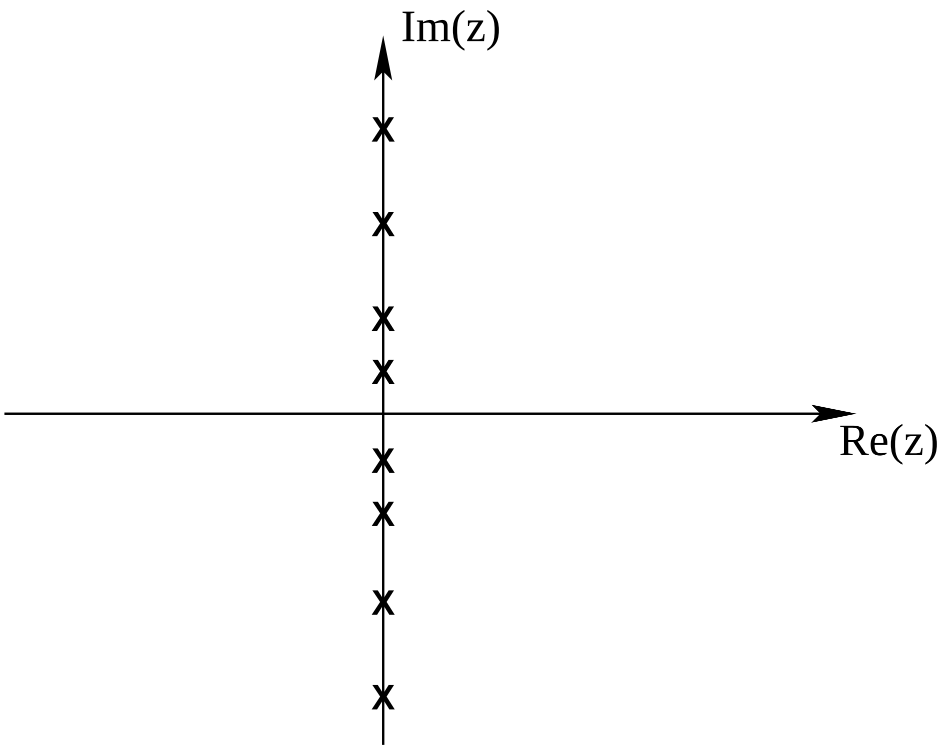
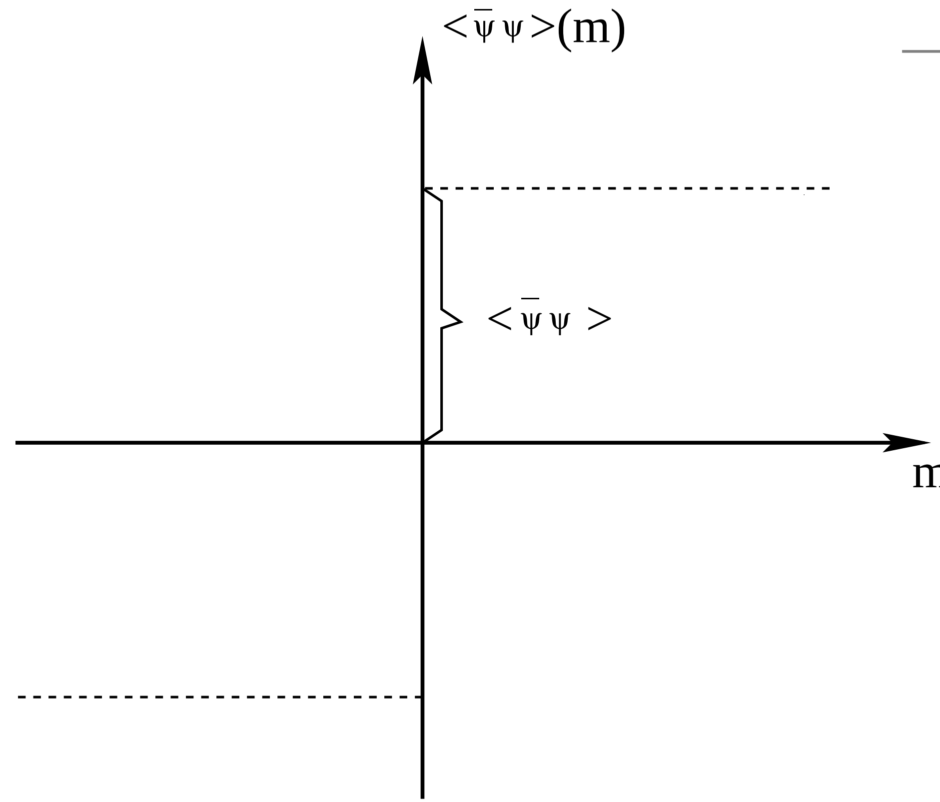
W_7 averaged x instead of m

$a = 0$

Banks Casher

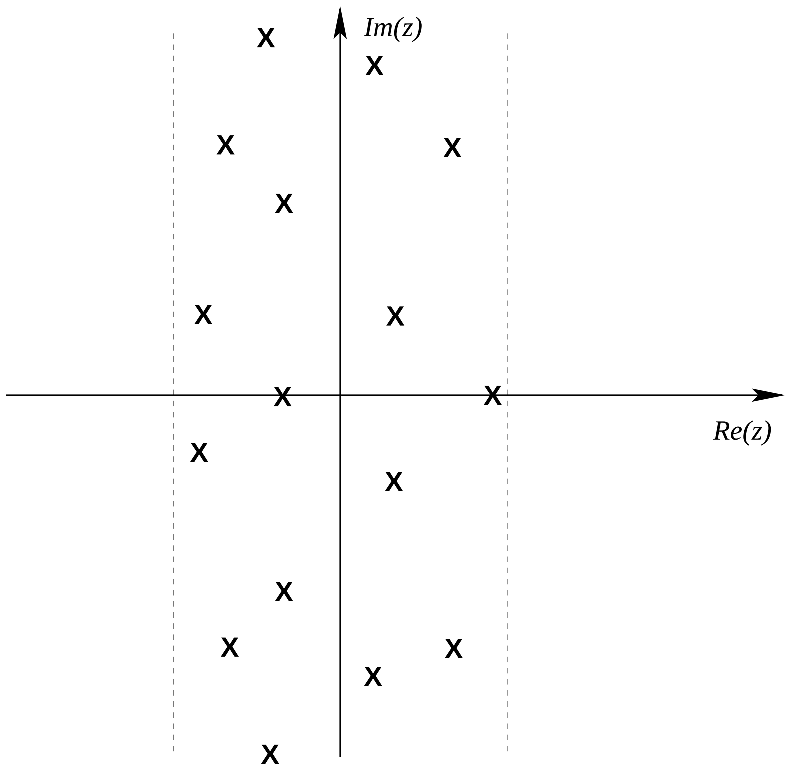


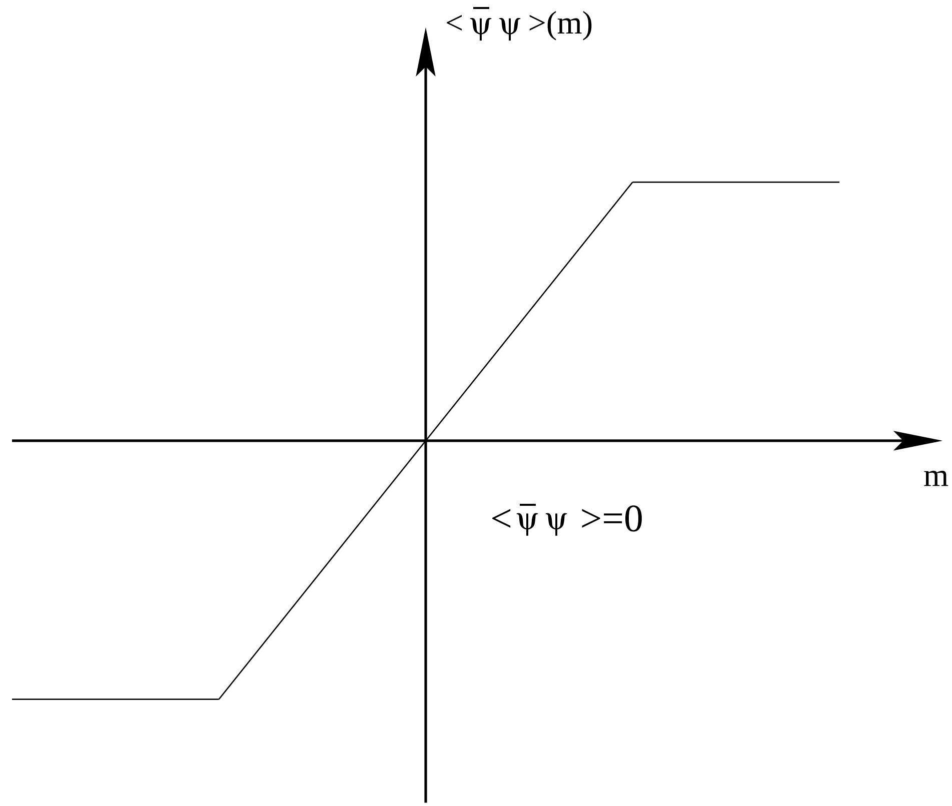


$$\langle \bar{\psi}\psi \rangle = \frac{\pi}{V}\rho(0)$$

Banks Casher NPB 169 (1980) 103

$a \neq 0$ Aoki phase (parity broken phase)





Electrostatic analogy:

Eigenvalues = charges, quark mass = test charge

Aoki PRD **30** 2653 (1984)

Barbour et al. NPB 275 (1986) 296 (nonzero μ)

RMT for Wilson Lattice QCD

Properties of the Wilson Dirac operator

γ_5-Hermiticity

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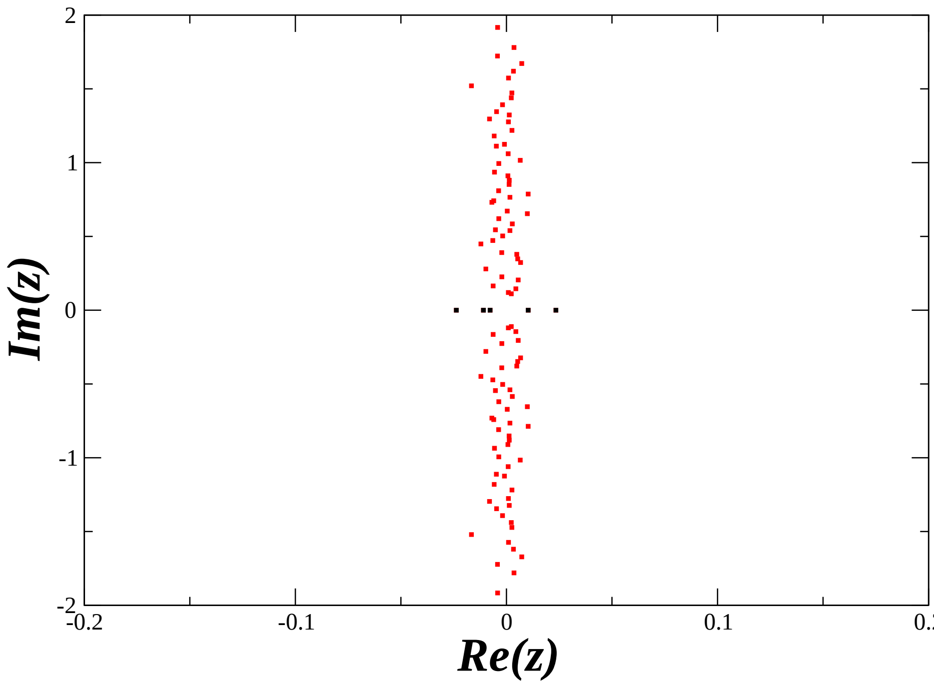
$$D^\dagger = \gamma_5 D \gamma_5$$

$$D = \begin{pmatrix} aA & iW \\ iW^\dagger & aB \end{pmatrix}$$

A ($N \times N$) and B ($(N+\nu) \times (N+\nu)$) are hermitian
W is a general complex matrix

The spectrum of one Random Matrix



Damgaard Splittorff Verbaarschot arXiv:1001.2937

The Wilson RMT partition function

$$\mathcal{Z}_{N_f}^{\nu} \equiv \int dW \det(D+m)^{N_f} e^{-\frac{N}{2}\mathrm{Tr}(A^2+B^2) - N\mathrm{Tr}W^\dagger W}$$

where

$$D+m = \begin{pmatrix} aA+m & iW \\ iW^\dagger & aB+m \end{pmatrix}$$

Shuryak, Verbaarschot, NPA **560**, 306 (1993), Verbaarschot, PRL **72**, 2531 (1994)

Damgaard Splittorff Verbaarschot arXiv:1001.2937

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Same low energy theory in the ϵ-regime

Shuryak, Verbaarschot, NPA **560**, 306 (1993), Verbaarschot, PRL **72**, 2531 (1994)

Damgaard Splittorff Verbaarschot arXiv:1001.2937

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$$\mathcal{Z}_{N_f}^{\nu} \equiv \int dW\, dA dB\, \det(D+m)^{N_f}\, e^{-\frac{N}{2}\mathrm{Tr}(A^2+B^2)-N\mathrm{Tr}W^\dagger W}$$

where

$$D+m = \begin{pmatrix} aA+m & iW \\ iW^\dagger & aB+m \end{pmatrix}$$

same flavor symmetries as QCD and same breaking by m and a

$$\mathcal{Z}_{N_f}^{\nu} = \int_{U(N_f)} dU\ \det{}^{\nu}(U)\ e^{Nm\mathrm{Tr}(U+U^\dagger)-\frac{Na^2}{2}\mathrm{Tr}(U^2+U^{\dagger\,2})}$$

for $N \to \infty$ with mN and a^2N fixed

Shuryak Verbaarschot NPA **560**, 306 (1993), Verbaarschot PRL **72**, 2531 (1994)

Damgaard Splittorff Verbaarschot arXiv:1001.2937

Why Wilson RMT

Usually: *easier to compute spectral correlation functions with RMT than the SUSY method*

- any N_f
- higher order correlation functions
- individual eigenvalue distributions

Akemann Damgaard Splittorff Verbaarschot arXiv:to.appear

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Work in progress .....

Akemann Damgaard Splittorff Verbaarschot arXiv:to.appear