Momentum Space Proof of BPH Renormalization

to all orders in perturbation theory with applications to lattice perturbation theory

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Dedicated to the memory of Bill Caswell

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What is the BPH Theorem?

- The BPH theorem states that the divergences of local polynomial quantum field theories can be absorbed into local monomials (counterterms) in the action to all orders in perturbation theory.
- It does not say that there are only a finite number of such counterterms, or make any claims about their dimension.

$$\boxed{\text{Can be Renormalized} \neq \text{Renormalizable.}}$$

- If the assumptions of the theorem are not met it does not say that the theory cannot be renormalized.
- There needs to be some regulator to make the manipulations well-defined. It is possible to define a subtracted integrand such that all the loop integrals are absolutely convergent (Zimmermann forests, BPHZ), but without a regulator these cannot be directly related to the underlying Lagrangian (so properties like unitarity are not obvious).

Ancient History

- Dyson (1949) *(Power counting)*
- Stückelberg and Green (1951)
- Боголюбов and Парасюк (1957) *(R operation)*
- Hepp (1966) *(Proof of BPH theorem)*

> *"Unfortunately the papers of BOGOLIUBOV and PARASIUK come close to not satisfying SALAM's criterion: it is hard to find two theoreticians whose understanding of the essential steps of the proof is isomorphic. This is articularly regrettable, since the very ingenious and elaborate treatment of the authors is the most general discussion of renormalization in Lagrangian quantum field theory."*

> *"Unfortunately the argument relies on a splitting of the testing functions . . . which is in general impossible."*

Slightly More Recent History

- Hahn and Zimmermann (1968) *(Small momentum cutoff)*
- Epstein and Glaser (1973)
- Аникин, Поливанов, and Завьялов (1973) *(Equivalence to counterterms)*
- Lowenstein and Speer (1976) *(Euclidean ⇒ Minkowski convergence)*
- Тарасов and Владимиров; Четыркин, Катаев, and Ткачев (1980) *(Differentiation with respect to external momenta)*
- Symanzik (1981) *(Schrödinger functional)*
- Caswell and Kennedy (1982, 1983) *(Henges)*

Motivation

- Hepp's proof still has a fairly small "Salam number" — the number of theoreticians who understand the proof; indeed, it is not even obvious what sign the time derivative of this quantity has.
- It would be nice to have a method of proof which was simple enough that more people might understand it, and perhaps apply it to new problems. (This is probably wishful thinking).
- The momentum-space proof is directly applicable to lattice perturbation theory, where Feynman parametrization is not applicable. In particular, the proof works for staggered fermions.

What Else?

It can also be used to prove

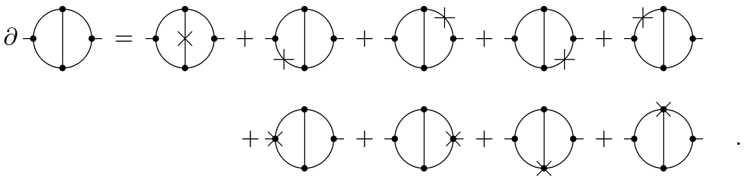
- Operator renormalization, and the operator product expansion.
- The cutoff dependence of an L loop lattice Feynman diagram is bounded by $a(\ln a)^L$, where a is the lattice spacing.
- The decoupling theorem, that all the effects of heavy particles can be absorbed into a renormalization of the interactions of light particles for external momenta at the light scale, up to powers of the mass ratio (subject to suitable power-counting conditions).
- That Zimmermann oversubtraction can reduce the cutoff dependence at the expense of introducing "non-renormalizable" interactions with explicit supression by powers of the cutoff. In particular, this justifies Symanzik improvement (removal of $O(a^\ell)$ effects in lattice perturbation theory).
- Renormalization of quantum field theories with boundaries (Schrödinger functional). (Work in progress with Stefan Sint).

Graphs and Integrals

- A graph is connected if it cannot be partitioned into two sets of vertices which are not connected by an edge.
- A graph is one particle irreducible (1PI) if it remains connected after removing any edge. A single vertex is a 1PI graph.
- A Feynman integral $I(\mathcal{G})$ may be associated with any graph $\mathcal{G}$ by means of the Feynman rules for the theory. A propagator is associated with each line, a factor with each vertex, and a D-dimensional momentum integral with each independent closed loop.
- $I(\mathcal{G})$ is a function of the external momenta p, the lightest mass m (we assume $m > 0$ to avoid infrared divergences), some dimensionless couplings, and a cutoff Λ which is introduced to make the theory well defined.
- We extend the mapping $I : \mathcal{G} \mapsto I(\mathcal{G})$ to act linearly on sums of graphs.
- For simplicity we only consider Euclidean space.

Diagrammatic Differentiation

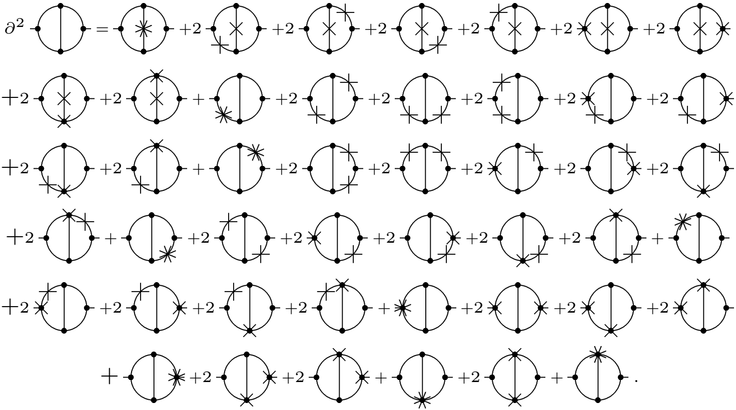
- It is useful to consider the derivative of a Feynman diagram with respect to its external momenta. This is drawn diagrammatically as



- Note that we view crossed and double crossed lines and vertices as associated with new Feynman rules: although one might view the cross as a new vertex inserted into a line this notation is not adequate in general when vertices (including such crosses themselves) have a non-trivial momentum dependence.

Further Diagrammatic Differentiation

The second derivative is



Taylor's Theorem

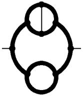
- Each of the graphs shown above is really a sum over all the components of all the independent external momenta, $I(\partial\mathcal{G}) = \frac{\partial I(\mathcal{G})}{\partial p_\mu}$, $I(\partial^2\mathcal{G}) = \frac{\partial^2 I(\mathcal{G})}{\partial p_\mu \partial p_\nu}$, etc.
- Viewing $I(\mathcal{G})$ as a function of its external momenta repeated application of the fundamental theorem of calculus gives us Taylor's theorem. In our notation

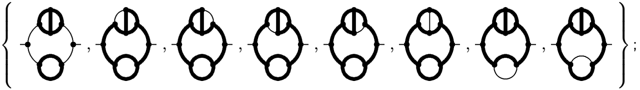
$$I(p) = \mathrm{T}^n I(p) + \int_{p_0}^{p} dp_1 \int_{p_0}^{p_1} dp_2 \ldots \int_{p_0}^{p_{n-1}} dp_n \, \partial^n I(p_n),$$

where

$$\begin{aligned} \mathrm{T}^n I(p) &\equiv \sum_{j=0}^{n} \frac{(p-p_0)^j}{j!} \partial^j I(p_0) \\ &= \sum_{j=0}^{n} \sum_{\mu_1,\ldots,\mu_j} \frac{(p-p_0)_{\mu_1} \ldots (p-p_0)_{\mu_j}}{j!} \frac{\partial^j I(p_0)}{\partial p_{\mu_1} \ldots \partial p_{\mu_j}}. \end{aligned}$$

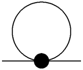
Diagrammatic Definition of Henges

- Any graph may be decomposed into a set of disjoint 1PI components and a set of edges which do not belong to any 1PI subgraph.
- Selecting any line from a graph defines a henge, which is just the set of 1PI components of the graph with the specified line removed. An example of a henge is , where the heavy lines indicate the set of 1PI subgraphs in the henge corresponding the light line.
- The set of all henges for a four-loop contribution to the two-point function of ϕ^3 theory is



the henges $\mathcal{H}(\mathcal{G}, \ell)$ shown as heavy lines correspond to ℓ being any of the light lines.

Henges and Feynman Integrals

- We shall write $\mathcal{G}/\mathcal{H}$ to indicate the graph obtained by shrinking each 1PI subgraph Θ in $\mathcal{H}$ to a point.
- If $\mathcal{G}$ is a 1PI graph and $\ell \in \mathcal{G}$ some edge, then $\mathcal{G}$ may be considered as a single loop $\mathcal{G}/\mathcal{H}(\mathcal{G},\ell)$ with the 1PI subgraphs in the henge $\mathcal{H}(\mathcal{G},\ell)$ acting as "effective vertices." For the example above the graph $\mathcal{G}/\mathcal{H}$ is  .
- We define $I_\lambda(\mathcal{G})$ to be the Feynman integral corresponding to $\mathcal{G}$ where all the lines carry momentum greater than λ; that is $|k_\ell| > \lambda \quad (\forall \ell \in \mathcal{G})$ where we use the usual Euclidean norm. This corresponds to Feynman rules in which an extra step function $\theta(k_\ell^2 - \lambda^2)$ is associated with each line.
- $i_\lambda(\mathcal{G})$ is the integrand of the graph $\mathcal{G}$.
- $i_\lambda(\mathcal{G}/\mathcal{H})$ is the integrand of the graph $\mathcal{G}$ with all the 1PI subgraphs in $\mathcal{H}$ removed (i.e., set to unity).

Definition of the R operation

- We now apply the simple momentum space decomposition which says that at every point in the space of loop momenta k some line has to be carrying the smallest momentum:

$$I_\lambda(\mathcal{G}) = \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk\, i_k\big(\mathcal{G}/\mathcal{H}(\mathcal{G},\ell)\big) \prod_{\Theta \in \mathcal{H}(\mathcal{G},\ell)} I_k(\Theta).$$

- For each henge all possible subdivergences of $I(\mathcal{G})$ must live within one of the "effective vertices," so it is most natural to define the $\bar{\mathrm{R}}$ operation, which subtracts all subdivergences, as

$$\boxed{\bar{\mathrm{R}} I_\lambda(\mathcal{G}) \equiv \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk\, i_k\big(\mathcal{G}/\mathcal{H}(\mathcal{G},\ell)\big) \prod_{\Theta \in \mathcal{H}(\mathcal{G},\ell)} \mathrm{R} I_k(\Theta),}$$

where R is the operation which subtracts all divergences

$$\boxed{\mathrm{R} I_\lambda(\mathcal{G}) \equiv \bar{\mathrm{R}} I_\lambda(\mathcal{G}) - \mathrm{K}\bar{\mathrm{R}} I_0(\mathcal{G}).}$$

The Subtraction Operation $-\mathrm{K}$

The subtraction operator $-\mathrm{K}$ removes the divergent part of $I(\mathcal{G})$. Various choices are possible

- For minimal subtraction $-\mathrm{K}I(\mathcal{G})$ subtracts the pole terms in the Laurent expansion of $I(\mathcal{G})$ in the dimension D. In this case the BPH theorem states that these subtractions are local; i.e., polynomial in the external momenta p.
- K can be chosen to be the Taylor series subtraction operator $T^{\deg \mathcal{G}} I(\mathcal{G})$ with respect to the external momenta p, where $\deg \mathcal{G}$ is the overall (power counting) degree of divergence of $\mathcal{G}$. In this case the BPH theorem states that the subtracted Feynman integrals are convergent, i.e., they have a finite limit as the cutoff $\Lambda \to \infty$.

Properties of the Subtraction Operation

- The subtraction operation commutes with differentiation:
  - For minimal subtraction $[\partial, \mathrm{K}] = 0$ trivially.
  - For Taylor series subtraction $\partial T^n = T^{n-1}\partial$ but, as we shall see, $\deg \partial\mathcal{G} = \deg \mathcal{G} - 1$, so $[\partial, T^{\deg}] = 0$.
- Strictly speaking we define $-\mathrm{K}$ to replace the divergent part with a finite polynomial of degree $\deg \mathcal{G}$ in the external momenta.
- The finite part of a subtracted graph is specified unambiguously by some set of renormalization conditions, which fix the values of $I(p_0), \partial I(p_0), \ldots, \partial^{\deg \mathcal{G}} I(p_0)$ at the subtraction point p_0.
- If $\mathcal{V}$ is a single vertex then it is convenient to define $\mathrm{K}I(\mathcal{V}) = -I(\mathcal{V})$, $\bar{\mathrm{R}}I(\mathcal{V}) = I(\mathcal{V})$, and $\mathrm{R}I(\mathcal{V}) = -\mathrm{K}\bar{\mathrm{R}}I(\mathcal{V}) = I(\mathcal{V})$.

Bounding Inequalities

- We require tree level bounds with the following properties:
  - All vertices and propagators Γ satisfy

$$|I_\lambda(\Gamma)| \leq c \cdot \chi(\lambda)^{\deg \Gamma},$$

  where c is a constant, and the overall degree of divergence $\deg \Gamma$ is a number which will be used for power counting.
  - The monotonically increasing bounding function χ must satisfy

$$\begin{aligned} \int_\lambda^\infty dk\, \chi(k)^\nu &\leq c \cdot \chi(\lambda)^{\nu+1} & (\nu+1<0) \\ \int_0^\lambda dk\, \chi(k)^\nu &\leq c \cdot \chi(\lambda)^{\nu+1+0} & (\nu+1\geq 0) \end{aligned}$$

  - Differentiation with respect to external momenta must lower the degree of divergence, $\deg(\partial\mathcal{G}) = \deg \mathcal{G} - 1$. This means that we also require that all derivatives of vertices and propagators must satisfy the bounds

$$|\partial^n I_\lambda(\Gamma)| \leq c \cdot \chi(\lambda)^{\deg \Gamma - n}.$$

- All external momenta and all masses are proportional to m.

Bounding Functions

- All these conditions are met by

$$\chi(k) \equiv \max(m,k) = \begin{cases} m & 0 \le k < m \\ k & m \le k < \infty \end{cases}$$

- This is trivially established by splitting up the integration region

$$\int_\lambda^\infty dk\ \max(m,k)^\nu = \begin{cases} \int_\lambda^m dk\ m^\nu + \int_m^\infty dk\ k^\nu & \lambda < m \\ \int_\lambda^\infty dk\ k^\nu & \lambda \ge m \end{cases}$$
$$\le c \cdot \max(m,\lambda)^{\nu+1} \qquad \nu < -1$$

$$\int_0^\lambda dk\ \max(m,k)^\nu = \begin{cases} \int_0^\lambda dk\ m^\nu & \lambda < m \\ \int_0^m dk\ m^\nu + \int_m^\lambda dk\ k^\nu & \lambda \ge m \end{cases}$$
$$\le c \cdot \max(m,\lambda)^{\nu+1+0} \qquad \nu \ge -1$$

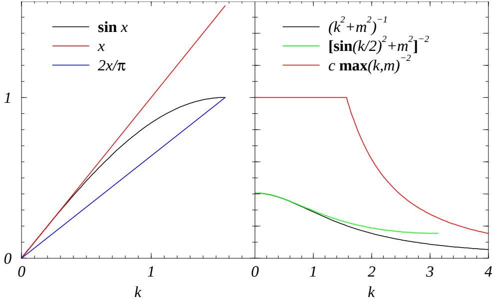
Lattice Bounds

- As an simple example, consider the one dimensional propagator $\Delta = (\tilde{k}^2 + m^2)^{-1}$ where $\tilde{k} = \left|\frac{2}{a} \sin \frac{ak}{2}\right|$.
- Using the inequalities $\frac{2}{\pi}|q| \leq |\sin q| \leq |q|$ within the Brillouin zone $-\frac{\pi}{2} \leq q \leq \frac{\pi}{2}$ we find that

$$|\Delta| \leq \left(\frac{\pi}{2}\right)^2 \max(k, m)^{-2}.$$

- Likewise, since $|\cos q| \leq 1$ we have

$$\left|\frac{\partial \Delta}{\partial k}\right| \leq \frac{\pi^4}{8} \max(k, m)^{-3}.$$



- Note that lattice propagators and vertices vanish outside the Brillouin zone, $|k| > \pi/a$.
- On the lattice we subtract polynomials in $\tilde{p}$ rather than polynomials in p.

Induction Hypothesis

Induction hypothesis:

$$\boxed{|\mathrm{R}I_\lambda(\mathcal{G})| \leq c \cdot \chi(\lambda)^{\deg \mathcal{G}+0}}$$

for all graphs with less than L loops.

Proof for Overall Convergent Diagrams

Overall convergent diagram with L loops:

- Use the definition of $\bar{\mathrm{R}}$

$$|\bar{\mathrm{R}} I_\lambda(\mathcal{G})| \leq \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk \, |i_k(\mathcal{G}/\mathcal{H})| \prod_{\Theta \in \mathcal{H}(\mathcal{G},\ell)} |\mathrm{R} I_k(\Theta)|.$$

- Use the induction hypothesis for the subgraphs Θ and the tree level bounds

$$\begin{aligned} |\bar{\mathrm{R}} I_\lambda(\mathcal{G})| &\leq c \cdot \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk \, \chi(k)^{\deg(\mathcal{G}/\mathcal{H})-1} \prod_{\Theta \in \mathcal{H}(\mathcal{G},\ell)} \chi(k)^{\deg \Theta + 0} \\ &= c \cdot \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty dk \, \chi(k)^{\deg \mathcal{G} - 1 + 0}. \end{aligned}$$

- Integrate the bounding function

$$|\bar{\mathrm{R}} I_\lambda(\mathcal{G})| \leq c \cdot \chi(\lambda)^{\deg \mathcal{G} + 0} \qquad (\deg \mathcal{G} < 0).$$

This establishes the induction hypothesis, since $\mathrm{R} I_\lambda(\mathcal{G}) = \bar{\mathrm{R}} I_\lambda(\mathcal{G})$ in this case.

Proof for Overall Divergent Diagrams

Overall divergent diagrams with L loops:

- Taylor's theorem for the function $\bar{\mathrm{R}} I_0(p)$ gives

$$\bar{\mathrm{R}} I_0(p) = T^{\deg \mathcal{G}} \bar{\mathrm{R}} I_0(p) + \int_{p_0}^{p} dp_1 \ldots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1}\, \partial^{\deg \mathcal{G}+1} \bar{\mathrm{R}} I_0 \left(p_{\deg \mathcal{G}+1}\right).$$

- Since $\bar{\mathrm{R}}$ and ∂ commute (this follows from the equivalence of our definition of R with Bogoliubov's, which we will establish later)

$$\bar{\mathrm{R}} I_0(p) = T^{\deg \mathcal{G}} \bar{\mathrm{R}} I_0(p) + \int_{p_0}^{p} dp_1 \ldots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1}\, \bar{\mathrm{R}} \partial^{\deg \mathcal{G}+1} I_0 \left(p_{\deg \mathcal{G}+1}\right).$$

- The (sum of) graphs $\partial^{\deg \mathcal{G}+1} I_0(\mathcal{G})$ are overall convergent which we will show are absolutely convergent.
- The integral is over a compact region, so any divergences must be in the polynomial part.

Proof for Overall Divergent Diagrams — UV Part

- Using the definition of R, the polynomial $T^{\deg \mathcal{G}}\bar{\mathrm{R}}I_0(p)$ is replaced by a finite polynomial in the external momenta specified by the renormalization conditions. This polynomial satisfies the tree level bounds, so

$$|\mathrm{R}I_0(p)| \le c\cdot\chi(0)^{\deg \mathcal{G}} + \int_{p_0}^{p} dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1} \left| R\partial^{\deg \mathcal{G}+1} I_0\left(p_{\deg \mathcal{G}+1}\right)\right|.$$

- Use the inductive bound on the overall convergent integrand

$$\begin{aligned} |\mathrm{R}I_0(p)| &\le c\cdot\chi(0)^{\deg \mathcal{G}} + \int_{p_0}^{p} dp_1 \dots \int_{p_0}^{p_{\deg \mathcal{G}}} dp_{\deg \mathcal{G}+1}\, c\cdot\chi(0)^{-1+0} \\ &\le c\cdot\chi(0)^{\deg \mathcal{G}+0}. \end{aligned}$$

We have thus proved that $\mathrm{R}I_0(\mathcal{G})$ is made finite by local subtractions, but we still need to establish the induction hypothesis.

Proof for Overall Divergent Diagrams — IR Part

- In the definition of $\bar{\mathrm{R}}I_0(\mathcal{G})$ we may split the integration region $\int_0^\infty dk = \int_0^\lambda dk + \int_\lambda^\infty dk$, hence

$$\bar{\mathrm{R}}I_0(\mathcal{G}) = \bar{\mathrm{R}}I_\lambda(\mathcal{G}) + \sum_{\ell\in\mathcal{G}} \int_0^\lambda dk\; i_k(\mathcal{G}/\mathcal{H}) \prod_{\Theta\in\mathcal{H}(\mathcal{G},\ell)} \mathrm{R}I_k(\Theta).$$

- Subtract $K\bar{\mathrm{R}}I_0(\mathcal{G})$ from both sides,

$$\mathrm{R}I_0(\mathcal{G}) = \mathrm{R}I_\lambda(\mathcal{G}) + \sum_{\ell\in\mathcal{G}} \int_0^\lambda dk\; i_k(\mathcal{G}/\mathcal{H}) \prod_{\Theta\in\mathcal{H}(\mathcal{G},\ell)} \mathrm{R}I_k(\Theta).$$

Establishing the Induction Hypothesis for L Loops

- Finally, all we need to do is to bound the integral over the "infrared region"

$$
\begin{aligned}
|\mathrm{R}I_\lambda(\mathcal{G})| \leq |\mathrm{R}I_0(\mathcal{G})| \quad &+ \quad \sum_{\ell\in\mathcal{G}} \int_0^\lambda dk\, |i_k(\mathcal{G}/\mathcal{H})| \prod_{\Theta\in\mathcal{H}(\mathcal{G},\ell)} |\mathrm{R}I_k(\Theta)| \\
\leq c\cdot\chi(0)^{\deg\mathcal{G}+0} \quad &+ \quad c\cdot\sum_{\ell\in\mathcal{G}} \int_0^\lambda dk\, \chi(k)^{\deg(\mathcal{G}/\mathcal{H})-1} \prod_{\Theta\in\mathcal{H}(\mathcal{G},\ell)} \chi(k)^{\deg\Theta+0} \\
\leq c\cdot\chi(0)^{\deg\mathcal{G}+0} \quad &+ \quad c\cdot\sum_{\ell\in\mathcal{G}} \int_0^\lambda dk\, \chi(k)^{\deg\mathcal{G}-1+0} \\
\leq c\cdot\chi(0)^{\deg\mathcal{G}+0} \quad &+ \quad c\cdot\chi(\lambda)^{\deg\mathcal{G}+0} \leq c\cdot\chi(\lambda)^{\deg\mathcal{G}+0} \qquad (\deg\mathcal{G}\geq 0).
\end{aligned}
$$

Connection with Hepp's Proof

- Hepp's proof divides the space of Feynman parameters $x_1, \ldots, x_N$ into sectors in which the parameters have a definite ordering, e.g., $x_1 > x_2 > \cdots > x_N$.
- Feynman parameters are introduced using the identity

$$\int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{[x_1 A_1 + x_2 A_2 + x_3 A_3]^3} = \frac{1}{A_1 A_2 A_3}.$$

- If we consider the corresponding integral restricted to a sector we obtain

$$\int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)\theta(x_1 > x_2 > x_3)}{[x_1 A_1 + x_2 A_2 + x_3 A_3]^3} = \frac{1}{A_1(A_1 + A_2)(A_1 + A_2 + A_3)} \leq \frac{1}{A_1^3}.$$

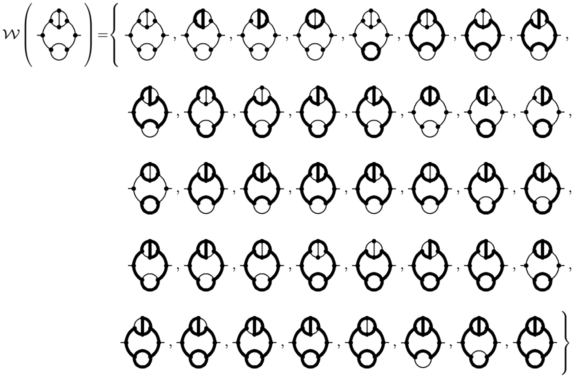
- Thus each sector corresponds to an ordering of the magnitude of the propagators $1/A_j$ just as the Henge decomposition does.

Equivalence to Bogoliubov's Definition

- A spinney is a covering of a graph by a set of disjoint 1PI subgraphs.
- Single vertices are allowed as elements of spinneys: in other words, all the vertices of a graph are included in a spinney, but not necessarily all of the edges.
- The wood $\mathcal{W}(\mathcal{G})$ is the set of all spinneys for a graph $\mathcal{G}$.
- Every henge is a spinney, but not vice versa.
- We shall use the notation $I\left(\mathcal{G}/\mathcal{S} \star \prod_{\Theta\in\mathcal{S}} f(\Theta)\right)$ to mean the Feynman integral for the graph $\mathcal{G}/\mathcal{S}$ where the function $f(\Theta)$ is the Feynman rule for the "effective vertex" Θ.
- The proper wood $\bar{\mathcal{W}}(\mathcal{G})$ is just the wood with the spinney $\mathcal{S} = \mathcal{G}$ omitted.

Example of a Wood

The following is an example from ϕ^3 theory



Bogoliubov's Definition

- Bogoliubov's definition of the R operation is

$$\bar{\mathrm{R}}_B I(\mathcal{G}) \equiv \sum_{\mathcal{S}\in\bar{\mathcal{W}}(\mathcal{G})} I\left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma\in\mathcal{S}} -\mathrm{K}\bar{\mathrm{R}}_B I(\Gamma)\right),$$

$$\mathrm{R}_B I(\mathcal{G}) \equiv (1-\mathrm{K})\bar{\mathrm{R}}_B I(\mathcal{G}) = \sum_{\mathcal{S}\in\mathcal{W}(\mathcal{G})} I\left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma\in\mathcal{S}} -\mathrm{K}\bar{\mathrm{R}}_B I(\Gamma)\right).$$

- This is easily shown *(caveat emptor)* to be be equivalent to our definition.
- The definition can be made even more explicit and less recursive using Zimmermann's forest notation: however it is easier to construct proofs and write programs to automate renormalization using recursive definitions.
- In Bogoliubov's form it is manifest that $[\partial, \mathrm{R}] = 0$, because
  - $[\partial, \mathrm{K}] = 0$.
  - The definition of R is purely graphical, and the graphical structure is not changed by differentiation.

Equivalence to Counterterms

- We shall show that the subtractions made by the R operation are equivalent to the addition of counterterms to the action. As this is a purely combinatorial proof it is convenient to use the generating functional

$$Z(J) = \int d\phi \, e^{-S(\phi)+J\phi} = \exp\left[-S_I\left(\tfrac{\delta}{\delta J}\right)\right] e^{\frac{1}{2}J\Delta J} Z(0),$$

  where $S(\phi) = \frac{1}{2}\phi\Delta^{-1}\phi + S_I(\phi)$.
- Perturbation theory may be viewed as an expansion in the number of vertices in a graph,

$$\begin{aligned} Z(J) &\sim \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left[S_I\left(\tfrac{\delta}{\delta J}\right)\right]^n e^{\frac{1}{2}J\Delta J} Z(0) \\ &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} I\big(\mathcal{G}_n(J)\big) Z(0); \end{aligned}$$

  where the last sum is over all graphs $\mathcal{G}_n$ containing exactly n vertices and which have J attached to their external legs.

Generating Functional for Renormalized Graphs

- We define the renormalized generating functional as

$$\begin{aligned} \mathrm{R}Z(J) &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} \mathrm{R}I(\mathcal{G}_n) Z(0) \\ &= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\mathcal{G}_n} \sum_{\mathcal{S}\in\mathcal{W}(\mathcal{G}_n)} I\left(\mathcal{G}/\mathcal{S} \star \prod_{\Gamma\in\mathcal{S}} -\mathrm{K}\bar{\mathrm{R}}\Gamma\right) Z(0). \end{aligned}$$

Generating Functional for Renormalized Graphs

- Using the identity

$$\sum_{\mathcal{G}_n} \sum_{\mathcal{S}\in\mathcal{W}(\mathcal{G}_n)} \prod_{\Gamma\in\mathcal{S}} -\mathrm{K}\bar{\mathrm{R}}\Gamma = \sum_{\substack{r_0,\dots,r_n \\ r_0+\cdots+r_n=n}} \frac{n!}{\prod_{j=0}^n j!^{r_j} r_j!} \prod_{j=0}^n \left[\sum_{\mathcal{G}_j} -\mathrm{K}\bar{\mathrm{R}} I(\mathcal{G}_j) \right]^{r_j}$$

  where the last sum is over all graphs $\mathcal{G}_j$ with exactly j vertices, we obtain that $RZ(J)$ is

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\substack{r_0,\dots,r_n \\ r_0+\cdots+r_n=n}} \frac{n!}{\prod_{j=0}^n j!^{r_j} r_j!} \prod_{j=0}^n \left[\sum_{\mathcal{G}_j} -\mathrm{K}\bar{\mathrm{R}} I\left(\mathcal{G}_j\left(\tfrac{\delta}{\delta J}\right)\right) \right]^{r_j} e^{\frac{1}{2}J\Delta J} Z(0)$$

$$= \prod_{j=0}^{\infty} \sum_{r_j=0}^{\infty} \frac{1}{r_j!} \left[\frac{1}{j!} \sum_{\mathcal{G}_j} \mathrm{K}\bar{\mathrm{R}} I\left(\mathcal{G}_j\left(\tfrac{\delta}{\delta J}\right)\right) \right]^{r_j} e^{\frac{1}{2}J\Delta J} Z(0)$$

$$= \prod_{j=0}^{\infty} \exp\left[\frac{1}{j!} \sum_{\mathcal{G}_j} \mathrm{K}\bar{\mathrm{R}} I\left(\mathcal{G}_j\left(\tfrac{\delta}{\delta J}\right)\right) \right] e^{\frac{1}{2}J\Delta J} Z(0) = \exp\left[\mathrm{K}\bar{\mathrm{R}} e^{S_I\left(\frac{\delta}{\delta J}\right)} \right] e^{\frac{1}{2}J\Delta J} Z(0).$$

Generating Functional for Renormalized Graphs

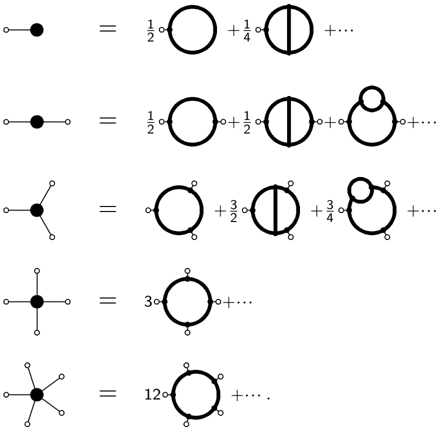
- We have shown that $RZ(J) = \int d\phi\, e^{-S_B(\phi)+J\phi}$, with the bare action S_B

$$S_B(\phi) = \tfrac{1}{2}\phi\Delta^{-1}\phi - \mathrm{K}\bar{\mathrm{R}}e^{S_I(\phi)}.$$

- Observe that there is no simple one to one correspondence between countergraphs and subtractions, but that the combinatorial factors arrange themselves correctly.
- The counterterms are monomials in the bare action, and we draw the fields ϕ or functional derivatives $\frac{\delta}{\delta J}$ by open circles at the end of the amputated external legs.
- Such graphs are symmetric under interchange of their external legs.
- Appropriate combinatorial factors must be used for each graph.

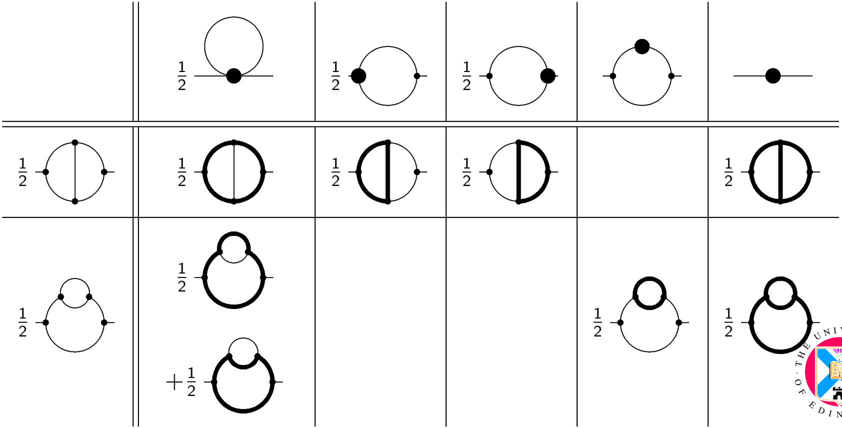
Examples of Counterterms

Some of the counterterms in ϕ^3 theory in d dimensions are



Correspondence Between Subtractions and Counterterms

The countergraphs built using these counterterms correspond to subtractions in the following non-trivial way:



Power Counting — Graph Theoretic Properties

- Consider a connected Feynman diagram $\mathcal{G}$ in a D dimensional field theory with an arbitrary polynomial action.
- Let it have I_a lines of type a, V_b vertices of type b, and E_a external legs of type a.
- Let n_{ab} be the number of lines of type a which are attached to vertex b, d'_b be the degree of this vertex, and d_a be the degree of lines of type a.
- Every line has to end on an appropriate vertex

$$\sum_b n_{ab} V_b = E_a + 2I_a, \qquad \forall a.$$

- We require exactly $V - 1$ lines to connect V vertices into a tree; every extra line produces a loop. Hence

$$L = I - V + 1 = \sum_a I_a - \sum_b V_b + 1.$$

Overall Degree of Divergence

- The overall degree of the graph can be obtained by counting,

$$\deg \mathcal{G} = LD + \sum_b V_b d'_b + \sum_a I_a d_a.$$

- Eliminate L and I_a from these equations to obtain

$$\deg \mathcal{G} = \sum_b V_b \left[\frac{1}{2} \sum_a \{ n_{ab}(D + d_a) \} + d'_b - D \right] - \frac{1}{2} \sum_a E_a(d_a + D) + D.$$

Field and Monomial Dimensions

- The dimension of the field ϕ_a is defined such that the dimension of its kinetic term in the action vanishes; that is, $\dim \phi_a \equiv \frac{1}{2}(d_a + D)$.
- The dimension of the monomial $\mathcal{V}_b$ in the action corresponding to the vertex of type b may be defined to be

$$\dim \mathcal{V}_b \equiv \sum_a n_{ab} \dim(\phi_a) + d'_b - D.$$

- This gives

$$\dim \mathcal{V}_b = \frac{1}{2} \sum_a n_{ab}(d_a + D) + d'_b - D.$$

- We thus obtain

$$\boxed{\deg \mathcal{G} = \sum_b V_b \dim \mathcal{V}_b - \sum_a E_a \dim \phi_a + D.}$$

Power Counting Results

- The theory is superrenormalizable, that is has only a finite number of overall divergent graphs, if the coefficients of V_b are negative: $\dim \mathcal{V}_b < 0 \quad (\forall b)$.
- The theory is renormalizable, that is only a finite number of Green's functions are overall divergent, if the none of the coefficients of V_b are positive, $\dim \mathcal{V}_b \leq 0 \quad (\forall b)$, and all the coefficients of E_a are positive, $\dim \phi_a > 0 \quad (\forall a)$.
- In general, all local monomials of dimension ≤ 0 will be required as counterterms.
- If the regulator and renormalization conditions preserve a symmetry then only symmetric counterterms will be required.
- If the symmetry is softly broken, i.e., by monomials of dimension < 0, then only counterterms of equal or lower dimension are required (Symanzik).

Operator Insertions

- Let $\Omega(\phi)$ be an operator which is local and polynomial in the field ϕ.
- Add a source term for Ω into the action,

$$Z(J, J') \equiv \int d\phi \, e^{-S(\phi)+J\phi+J'\Omega(\phi)}.$$

- The BPH theorem tells us that this theory can be renormalized by adding local counterterms of the form

$$S_I(\phi) - J'\Omega(\phi) + \Delta S(\phi, J') = -\mathrm{K}\bar{\mathrm{R}} \exp\left[S_I(\phi) - J'\Omega(\phi)\right].$$

- Expanding in powers of J' gives

$$-J'\Omega(\phi) + \Delta S(\phi, J') = \Delta S(\phi, 0) + J'\mathrm{K}\bar{\mathrm{R}}\left[e^{S_I(\phi)}\Omega(\phi)\right] + O(J'^2).$$

Operator Renormalization and Operator Product Expansion

- We may associate these counterterms with the operator to define a renormalized operator

$$\boxed{N(\Omega) \equiv -\mathrm{K}\bar{\mathrm{R}}\left[e^{S_I(\phi)}\Omega(\phi)\right].}$$

- Power counting tells us that

$$\deg \mathcal{G} = V_\Omega \dim \Omega + \sum_b V_b \dim \mathcal{V}_b - \sum_a E_a \dim \phi_a + D,$$

  where V_Ω are the number of Ω vertices in $\mathcal{G}$.
- As we are interested in a single insertion of Ω we only care about counterterms linear in J', and these get contributions only from diagrams $\mathcal{G}$ with $V_\Omega = 1$. Thus $\deg \mathcal{G} \leq \dim \Omega + D$, for a renormalizable theory, which means that we only get counterterms of dimension $\leq \dim \Omega$.
- Analogous arguments easily establish the operator product expansion.

Symanzik Improvement

- Zimmermann observed that if we oversubtract by removing more than $\deg \mathcal{G} + 1$ terms from the Taylor series in the external momenta then we reduce the cutoff dependence at the cost of introducing more counterterms.
- These extra counterterms are of higher dimension, but have explicit inverse powers of the momentum cutoff. It is easy to generalize Dyson's power-counting rules to take this into account by counting explicit cutoff factors as having dimension one.
- Following Symanzik we can improve the lattice action by such oversubtraction, but as the lattice Feynman rules explicitly depend upon the (inverse) cutoff a we must also subtract tree graphs.
- A simple modification of the induction hypothesis establishes that this procedure works to all orders in perturbation theory.
- The expansion in powers of a for the improved action is only an asymptotic series, so in general it does not permit us to keep cutoff effects small while making a larger.

Bounds on Cutoff Effects

- In our proof we added an arbitrarily small power ε to our bounds to handle logarithmic divergences correctly.
- If we refine our bounds on the integrals of bounding functions

$$\int dk\, k^n (\ln k)^r \leq c \begin{cases} k^{n+1}(\ln k)^r & n \neq -1 \\ k^{n+1}(\ln k)^{r+1} & n = -1 \end{cases}$$

  (hint: expand the previous bounds in powers of ε) then we can obtain slightly tighter bounds.
- This establishes that the cutoff effects for an L-loop graph with $O(a^s)$ Symanzik-improved actions are bounded by $a^{s+1}(\ln a)^L$.
- More precisely, the power of $\ln a$ is equal to the maximum number of nested subgraphs of dimension zero.

Decoupling Theorem

- The decoupling theorem is also an application of Zimmermann oversubtraction.
- Suppose we have a Lagrangian with light particles of mass m and heavy particles of mass $M \gg m$.
- Use bounds of form $\chi(k) = \max(k, m, M)$ for heavy particles and vertices that depend explicitly on M.
- We require tree graph subtractions, just as for Symanzik improvement (but here even in the continuum).
- For gauge symmetries we need Ward identities for 1LPI diagrams (presumably true for Symanzik improvement too).

Renormalization of the Schrödinger Functional

Work in progress *(caveat emptor)*, in collaboration with Stefan Sint.

- Coupling "constants" do not have to be constant; for example background field interactions such as coupling to a source $\int dx\, \phi(x)J(x)$.
- Impose (Dirichlet) boundary conditions by adding a wall interaction into the action, $(c = \pm 1 + O(\hbar))$

$$S = \int dx \left[\frac{1}{2}\phi(-\partial^2 + m^2)\phi + \frac{1}{4!}\lambda\phi^4 + c\phi(x_0 - 0)\delta'(x_0 - w)\phi(x_0 + 0)\right].$$

- Regulate this by smearing the wall into a narrow Gaussian $f(x)$.
- In momentum space we get the vertex $c\tilde{\phi}(-k)\tilde{\phi}(k+p)p_0\tilde{f}(p)$ where $\tilde{f}$ is a broad Gaussian. Point-splitting in x-space becomes an infinitesimal phase factor, which is preserved by renormalization.
- This has power-counting dimension 2 (p is an external momentum).
- In four dimensions the worst divergence is quadratic, so more than one insertion of c vertex is overall convergent.
- Single insertion of c vertex is proportional to $p_0\tilde{f}(p)$, so the counterterm lives on the wall.