Two-Loop Self-Energy Corrections To The g-Factor Of Bound Electrons

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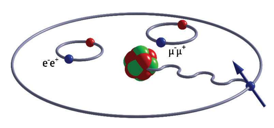
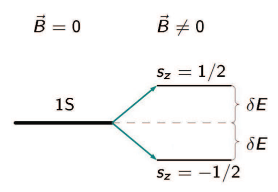
Zeeman effect and g-factor

Magnetic moment $\boldsymbol{\mu}$ of electron

$$\boldsymbol{\mu} = g\frac{e\boldsymbol{J}}{2m_e}$$



e: electron charge, m_e: electron mass, $\boldsymbol{J}$: electron angular momentum, g: electron g-factor [1]
Bound electron: energy shift δE of ground state $|1s\rangle$ with magnetic quantum number m_j in external magnetic field B:



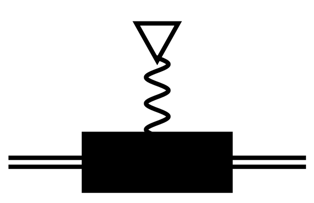
$$\delta E = -\langle 1s|\boldsymbol{\mu}\boldsymbol{B}|1s\rangle = -m_j g\frac{e}{2m_e}B$$

δE computed using the Two-time Green Function method [2].

Applications

- precision tests of QED in the presence of strong electromagnetic background fields [3, 4]
- determination of the electron mass [5, 6]
- determination of the muon mass [7]
- determination of the fine-structure constant [8, 9]
- test of physics beyond the standard model [10]

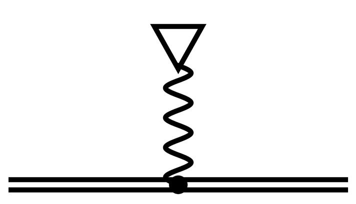
Theoretical description



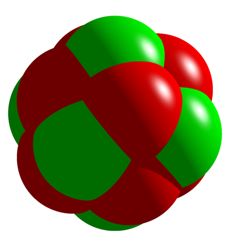
$$g = 2\left(C^{(0)} + C^{(2)}\left(\frac{\alpha}{\pi}\right) + C^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + \cdots\right)$$

$C^{(2n)} = C^{(2n)}_{\text{free}} + \delta C^{(2n)}_{\text{bound}}(Z\alpha) \,\widehat{=}$ sum of all Feynman diagrams with n closed loops [1]
Non-perturbative determination of $C^{(2n)}$: "Furry picture"

Leading g-factor diagram

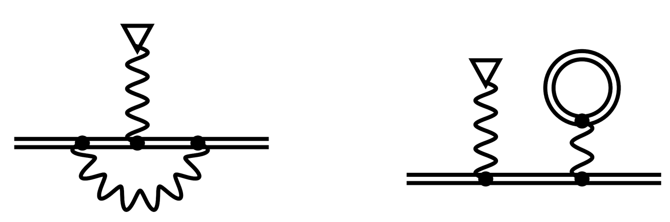


$$\Delta E_{\text{mag}} = -\frac{2}{3}i\int_0^\infty \mathrm{d}r\, r^2\, Bier\, f(r)g(r)$$
$$g_{\text{D}} = \frac{2}{3} + \frac{4}{3}\sqrt{1-(Z\alpha)^2}$$



- Finite nuclear size
- Finite nuclear mass (recoil) [11, 12]
- Nuclear deformation [13]
- Nuclear polarization [14]

Feynman diagrams with one loop



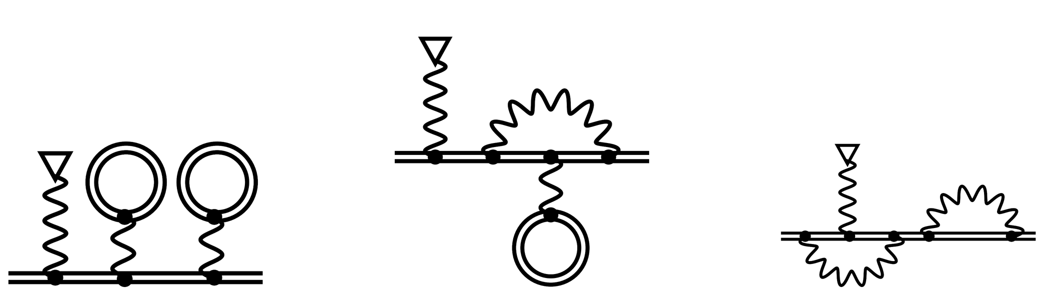
- Self-energy (SE): interaction of the electron with the photon field

$$g_{\text{SE}} = \frac{\alpha}{\pi}\left(1 + \frac{(Z\alpha)^2}{6}\right) + \cdots \qquad [15,\ 16]$$

- Vacuum polarization (VP): creation of a virtual charged particle-antiparticle pair

$$g_{\text{VP}} = -\frac{16}{15}\left(\frac{\alpha}{\pi}\right)(Z\alpha)^4\left(\frac{m_e}{m_{\text{loop}}}\right)^2 + \cdots \qquad [17,\ 18]$$

Feynman diagrams with two loops



- Diagrams with two VP loops (VPVP) [19, 20]
- Diagrams with one SE and one VP loop (SEVP) [20]
- Diagrams with two SE loops (SESE) [21]

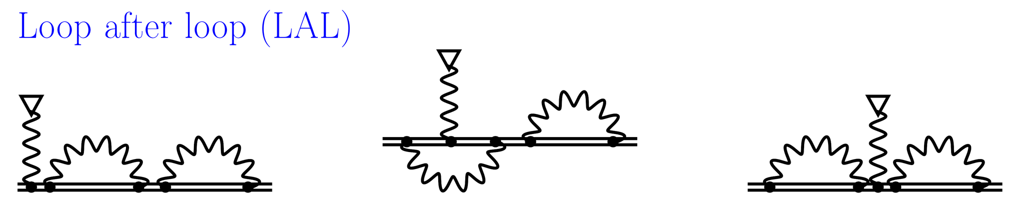
Magnetic loop VP diagrams: see poster by V. Debierre

Two-loop SESE corrections

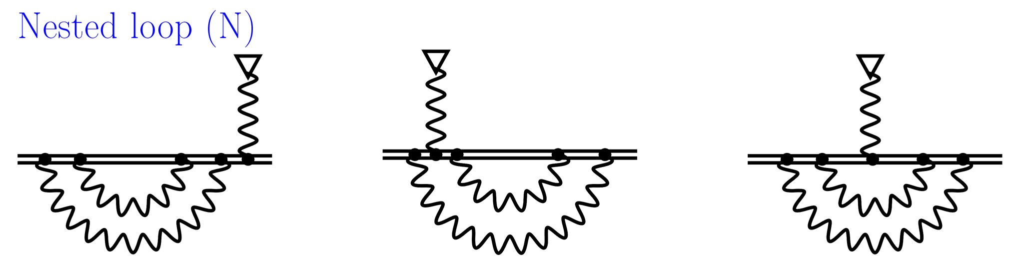
Theoretical uncertainty of g-factor dominated by uncalculated higher-order SESE correction

SESE diagrams

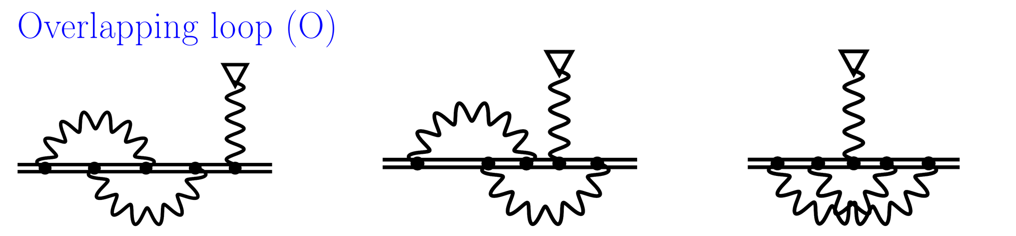
Loop after loop (LAL)



Nested loop (N)



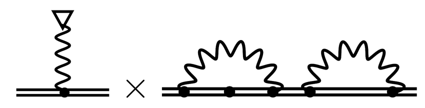
Overlapping loop (O)



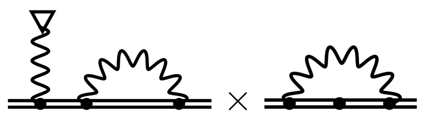
LAL, Irreducible and LAL, Reducible

- Electron propagator between SE loop and magnetic interaction: $G_{\text{D}} = \sum_n \frac{|n\rangle\langle n|}{E_{1s}-E_n}$
  $\Rightarrow$ Split into "irreducible" ($E_n \neq E_{1s}$) and "reducible" ($E_n = E_{1s}$)
- In LAL, up to two such electron propagators
  $\Rightarrow$ Regroup LAL contributions into

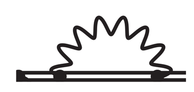
LAL, irred e.g.



LAL, red e.g.



Computation of LAL, irred: generalization of one-loop QED, with $|1s\rangle \longrightarrow |\delta_\Sigma 1s\rangle$



$$|\delta_\Sigma 1s\rangle = \sum_{n,n\neq 1s}\frac{|n\rangle\langle n|\Sigma|1s\rangle}{E_{1s}-E_n}$$

Renormalization

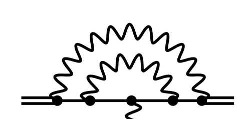
- LAL, irred: UV and IR finite
- LAL, red: UV and IR divergent
- N: UV and IR divergent
- O: UV and IR divergent

(LAL, red; N; O) $\Longrightarrow$ divergences cancel

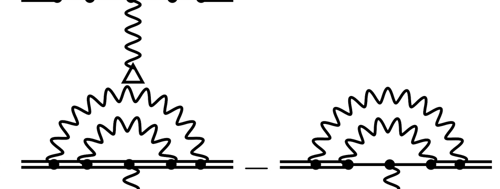
To deal with UV divergences, the N and O diagrams are divided into three terms [22].

- F-term: diagrams with only free internal electron lines (UV divergences)
- M-term: diagrams with bound internal electron lines (no UV divergences)
- P-term: diagrams with bound internal electron lines and a UV divergent subdiagram

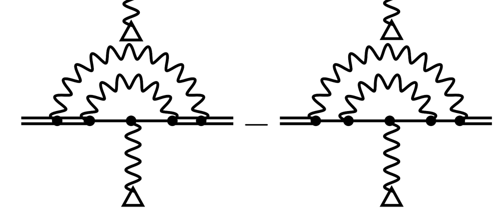
Example F-term:



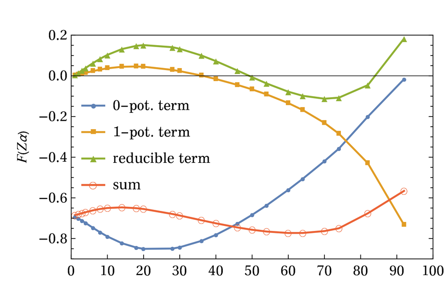
M-term:

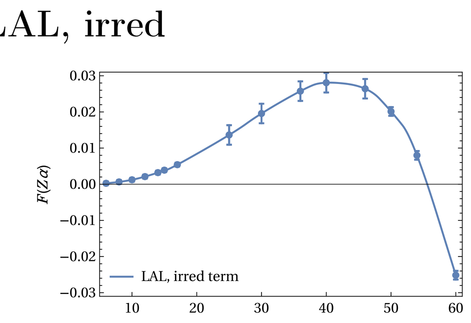


P-term:



Numerical results for F-term & LAL, irred





$$\Delta g = \left(\frac{\alpha}{\pi}\right)^2 F(Z\alpha)$$

- Zero-potential: two-loop diagrams with free internal electron lines, no interaction with the nuclear potential
- One-potential: two-loop diagrams with free internal electron lines, one interaction with the nuclear potential
- Reducible: products of two one-loop functions ($\widehat{=}$ part of LAL, red)

Our numerical results for the F-term converge to the free-electron value $F(0) = -0.68833...$ for low Z. [21, 15]

Two-loop SESE – outlook

IR divergences, methods

- Summation of diagrams whose IR divergences cancel (g-factor, one-loop SE [16])
- Subtraction terms to cancel IR divergences (Lamb shift, two-loop SESE [22])

Numerical challenges

M-term
- $\int \mathrm{d}\omega_1 \int \mathrm{d}\omega_2 \int \mathrm{d}r_1 \cdots \int \mathrm{d}r_5 f(\cdots)$
- $\sum_{\kappa_1,\kappa_2} g_{\kappa_1,\kappa_2}$ (infinite summations)

P-term
- Numerical Fourier transform
- $\sum_{\kappa_1} g_{\kappa_1}$ (infinite summation)

Consistency checks

- Comparison with perturbative determination of $C^{(4)}$ [15, 23]
- Comparison with SESE correction to Lamb shift [22]
- Further ideas would be highly appreciated ...

Possibilities after complete two-loop calculation

- Improved theoretical accuracy of bound-electron g-factor for high Z $\Longrightarrow$ Comparison with experimental g-factors (ALPHA-TRAP/ARTEMIS)
- Improvement of accuracy of α

Access to the muon mass

$$\left.\begin{array}{ll}\text{Larmor frequency:} & \omega_{\text{L}} = g\dfrac{e}{2m_\mu}B \\ \text{Cyclotron frequency:} & \omega_{\text{cycl}} = \dfrac{Q}{m_{\text{ion}}}B\end{array}\right\} \Longrightarrow m_\mu = \frac{g}{2}\frac{e}{Q}\frac{\omega_{\text{cycl}}}{\omega_{\text{L}}}m_{\text{ion}}$$

- $\frac{Q}{e}$ known exactly
- m_{ion} known very precisely
- $\frac{\omega_{\text{L}}}{\omega_{\text{cycl}}}$ from experiment
- g-factor from theory

Muonic ^{4}He$^+$:

- Small uncertainty of nuclear effects
- Sum of all considered terms:

$$g = 2.002\,195\,193\,4(20)_{\text{calc}}(50)_{\text{uncalc}}$$

$\longrightarrow$ possibility of improvement of muon mass accuracy by one order of magnitude [7]
$\longrightarrow$ alternative access to the controversial free-muon g factor with the subtraction of binding effects from theory [7]

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