Two-loop QED corrections to the bound-electron $g$ factor involving the magnetic loop

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Basics

There are twenty-nine (29) different diagrams \([1, 2, 3]\) that contribute to the bound-electron $g$ factor at the two-loop level. Four diagrams feature the so-called magnetic loop, which vanishes at the free-loop level.

The one-loop magnetic loop correction has been calculated \([4, 5]\). In the process, Lee et al. derived an analytical expression for the light-by-light scattering amplitude of a low-energy photon in the Coulomb field, which we use here.

We focus on the simplest case of the $1s$ state.

Electric loop-magnetic loop correction

The treatment is based on the simpler case of the one-loop magnetic loop correction \([4, 5]\), with a modified electronic current. The electric loop-corrected wave function is computed numerically at the Uehling approximation (i.e. lowest-order within the vacuum polarisation loop). The Uehling potential is well known and the correction to the wave functions can be computed with relative ease.

Magnetic loop-after-loop correction

We also start from the one-loop magnetic loop correction \([4, 5]\), but with an extra vacuum polarisation loop computed at the free-loop (Uehling) approximation. The vacuum polarisation-dressed photon propagator is well known \([2]\).

Non-vertex self-energy-magnetic loop correction

This diagram is split \([6]\) between a reducible contribution (the intermediate state of the bound electron between the self-energy loop and the magnetic loop is the reference state $1s$) and an irreducible contribution (all other allowed intermediate states).

The treatment of the reducible correction is based on previously computed diagrams:

\[
\Delta g_{SE-ML}(\gamma) = \Delta g_{ML}(\gamma) \Delta g_{SE}(\gamma),
\]

where $\Delta g_{ML}(\gamma)$ is the one-loop magnetic loop correction \([4, 5]\), $\Delta g_{SE}(\gamma)$ is the reducible one-loop self-energy correction \([6]\), and $g_{\gamma}$ is the Dirac value of the bound $g$ factor.

The treatment of the irreducible correction is similar to that of the electric loop-magnetic loop correction. The electric loop-corrected wave function is replaced by the self-energy-corrected wave function, the computation of which is a major numerical challenge \([7]\).

Vertex self-energy+magnetic loop correction

The vertex correction is split \([6]\) between a zero-potential term $\Delta g_{SE-ML}^{(0)}$ (wherein the electron does not interact with the Coulomb field of the nucleus under the self-energy loop) and a many-potential term $\Delta g_{SE-ML}^{(irr)}$ (which is a sum over all strictly positive numbers of interactions with the Coulomb field under the loop).

The zero-potential term is ultraviolet-divergent, before renormalisation, which is carried out by using the renormalised vertex function $\Gamma_{\gamma}$:

\[
\Delta g_{SE-ML}^{(0)} = \frac{2m_e}{m} \left[ \frac{dp}{(2\pi)^3} \right] \left( \frac{dp^{'}}{(2\pi)^3} \right) \psi(p) \Gamma_{\gamma}(p, p^{'}) \cdot A_{\text{ML}}(p - p^{'}) \psi(p^{'}),
\]

where $m = \pm 1/2$ is the projection of the total ($=$ spin) electron angular momentum, $B$ is the (homogeneous, constant) external magnetic field, $\psi$ is the bound wave function of the $1s$ state, and $A_{\text{ML}}$ is the light-by-light-scattered vector potential, expressed in terms of the light-by-light-scattering amplitude \([5]\). We compute integrals of products of up to five spherical harmonics over two different solid angles. We are left with a triple spatial integral (2 radial, 1 angular) to be performed numerically, on top of a numerical 1-Feynman parameter integral.

The many-potential term is computed numerically similarly to the corresponding term in the one-loop self-energy correction \([6]\). We exploit the fact that the magnetic loop preserves the angular structure of the external vector potential.

Some preliminary results

Upcoming experiments have been announced on heavy hydrogenlike ions at HITRAP \([8]\) and ALPHATRAP \([9]\). The corrections computed here increase strongly with increasing $Z$. For experimental relevance we present results for two high-$Z$ hydrogenlike ions ($\text{Xe}^{13+}$ and $\text{Pb}^{121+}$).

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Delta g_{SE-ML}^{(0)}$</th>
<th>$\Delta g_{SE-ML}^{(irr)}$</th>
<th>$\Delta g_{SE-ML}^{(0)} - \Delta g_{SE-ML}^{(irr)}$</th>
<th>$\Delta g_{SE-ML}^{(0)}$</th>
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<tbody>
<tr>
<td>54</td>
<td>1.4414(26) $\times 10^{-4}$</td>
<td>6.099(1) $\times 10^{-12}$</td>
<td>$-5.34(5) \times 10^{-13}$</td>
<td>1.7572(25) $\times 10^{-\dagger}$</td>
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<tr>
<td>82</td>
<td>0.998(1) $\times 10^{-3}$</td>
<td>6.81(5) $\times 10^{-4}$</td>
<td>$-5.57(7) \times 10^{-13}$</td>
<td>9.45(23) $\times 10^{-3}$</td>
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