Calculation of Higher Order Corrections to Positronium Energy Levels

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Summary

Collaborators

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Ben Akers
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Conor Larison

Acknowledgments

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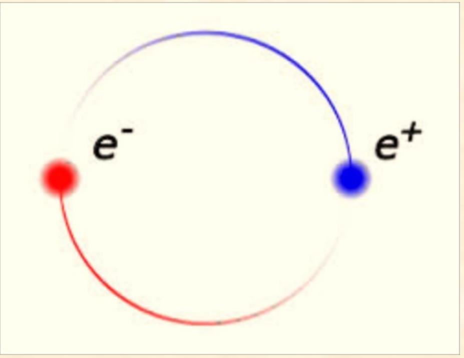
Franklin & Marshall College

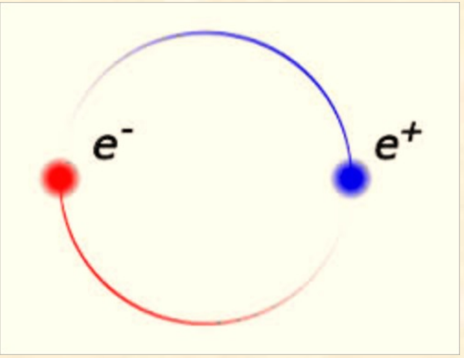


Why Study Positronium?

Positronium is intrinsically interesting. It is the simplest bound system. It's constituents are structureless pointlike particles. Many fundamental aspects of quantum field theory enter into its description. It differs from other exotic atoms in having large recoil effects, little sensitivity to hadronic physics, and in being subject to real and virtual annihilation.

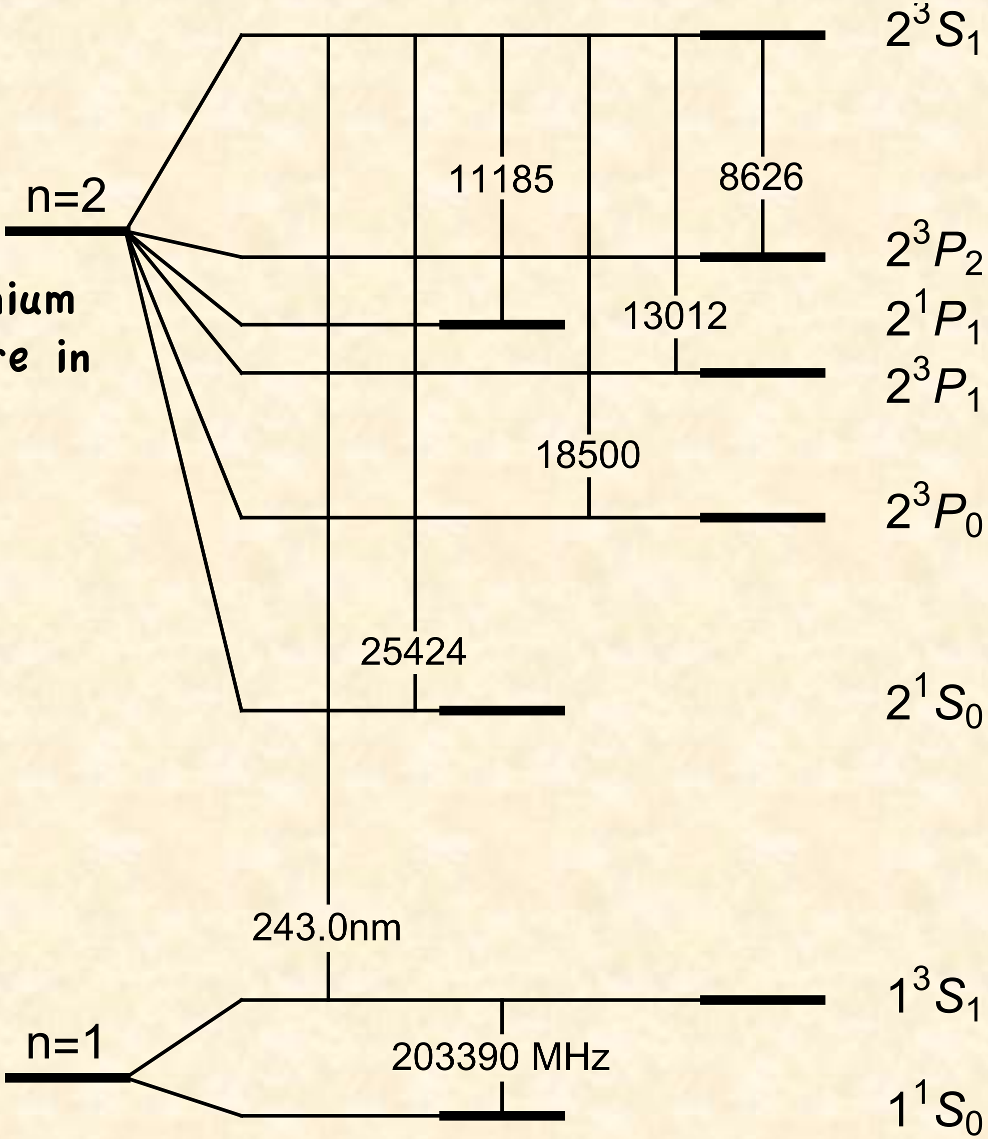
Positronium is accessible both to high precision experiments and to detailed calculations, so its study allows for a stringent test of the theory of bound states in QED (quantum electrodynamics) and quantum field theory generally.

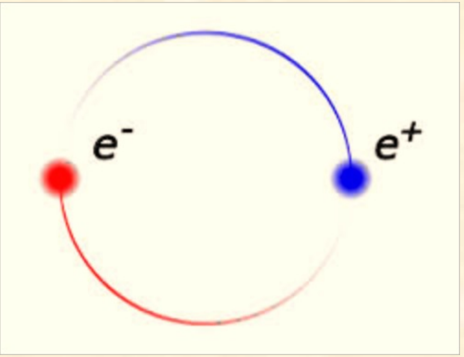
Positronium is ideal for tests of fundamental symmetries and is useful in searches for "new physics".



Positronium Spectrum

The n=1 and n=2 levels of positronium are shown. (Transition energies are in units of MHz.)





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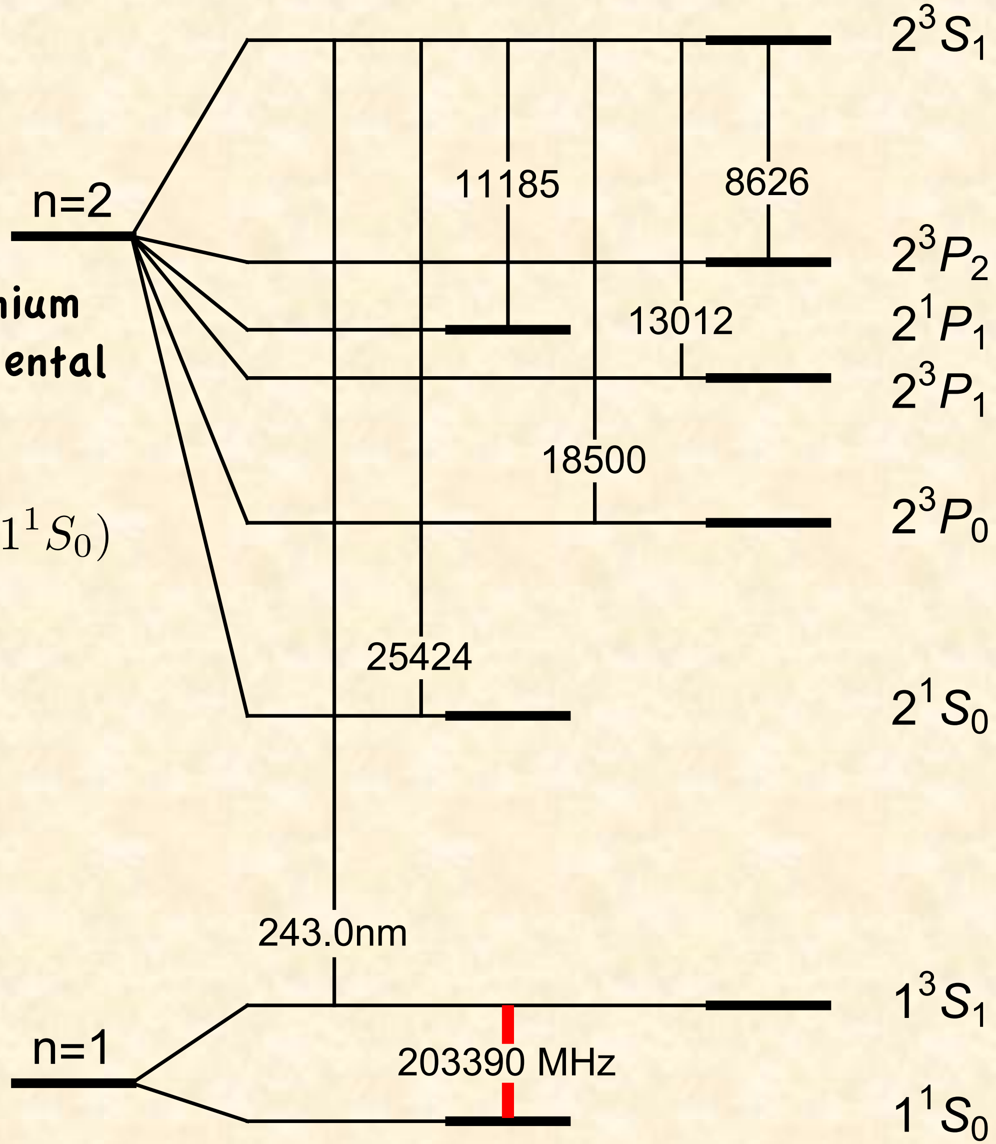
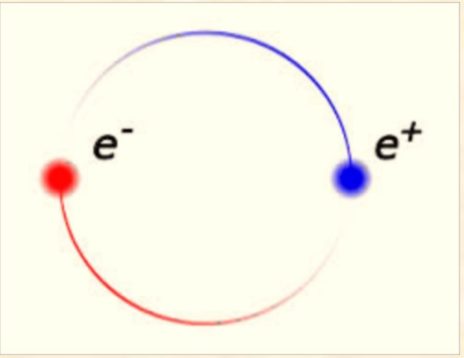
The n=1 and n=2 levels of positronium are shown. Transitions of experimental interest are the

(1) n=1 hyperfine splitting $(1^3S_1 - 1^1S_0)$





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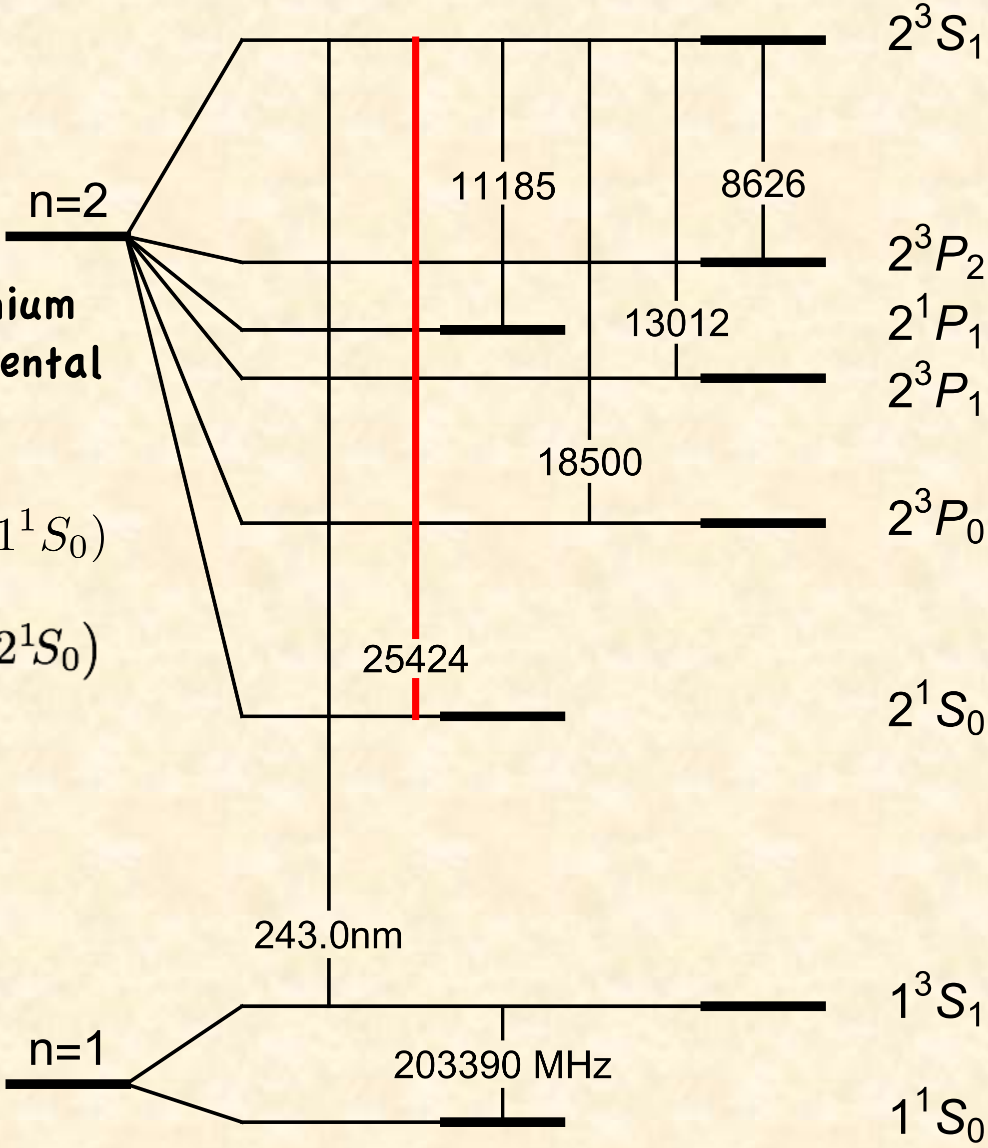

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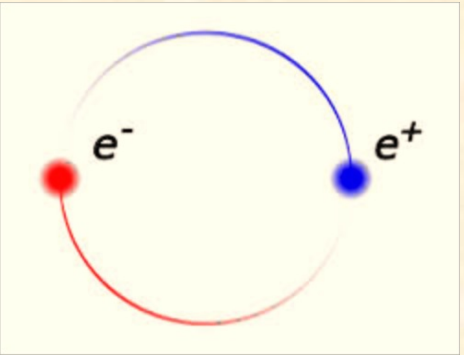

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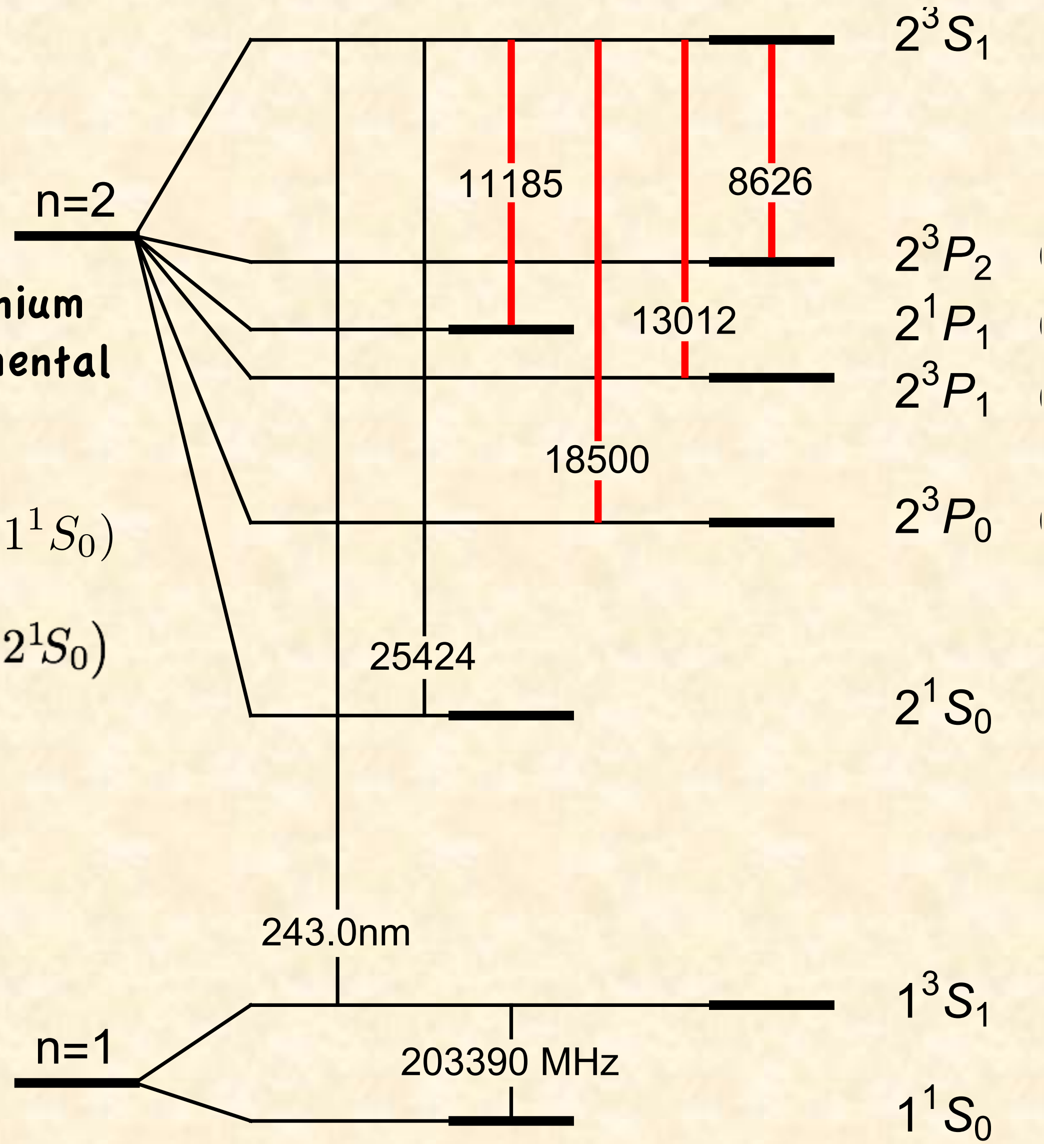
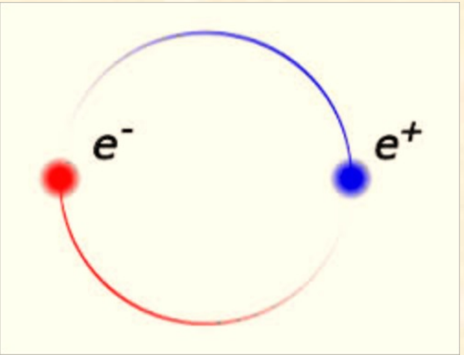
(3) n=2 fine structure $(2^3S_1 - 2P)$





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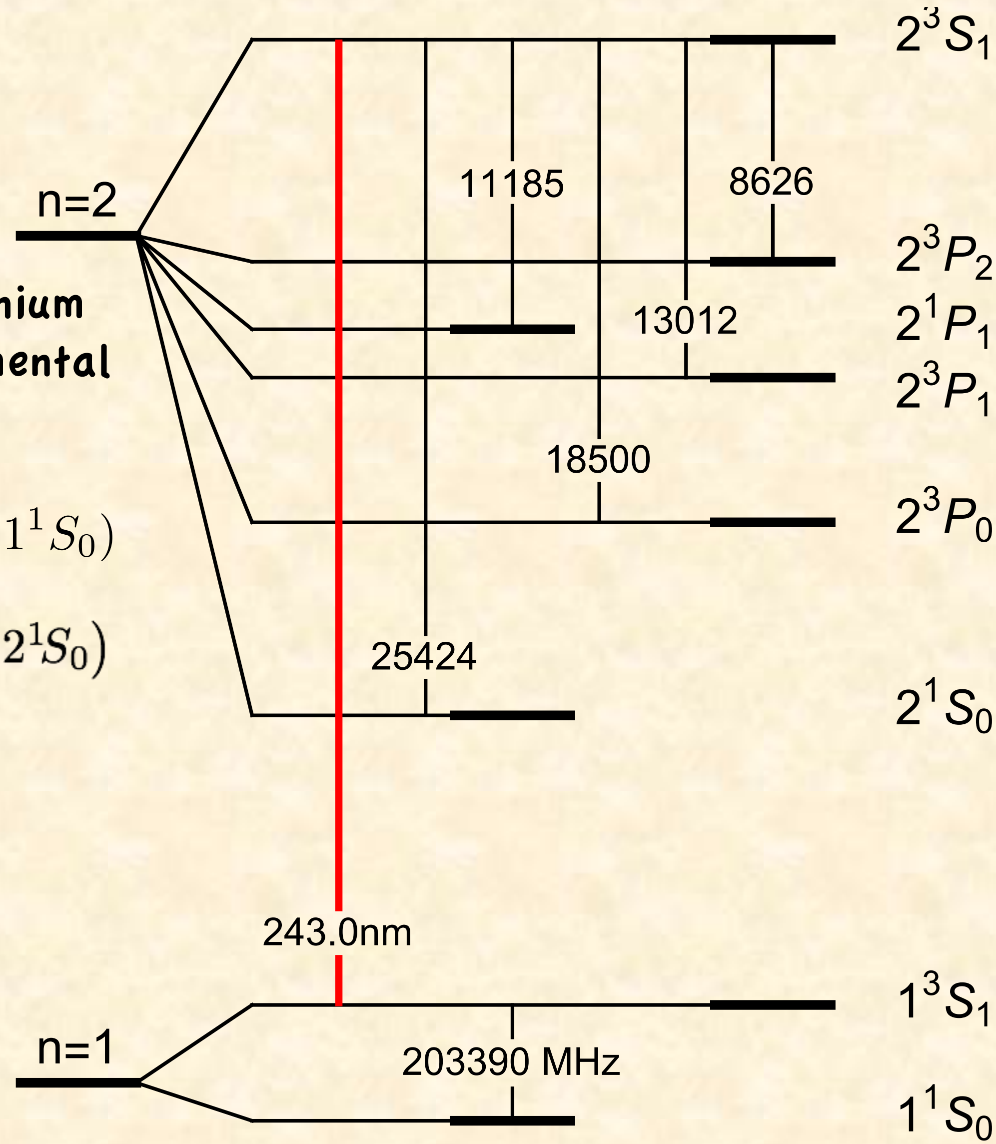
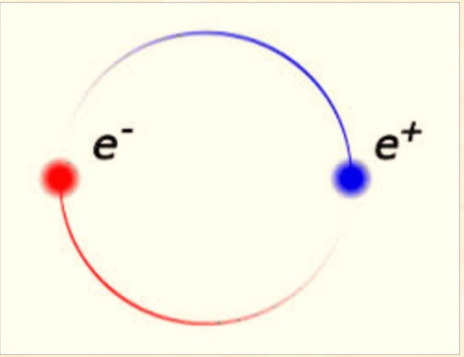
(4) 1S-2S transition $(1^3S_1 - 2^3S_1)$





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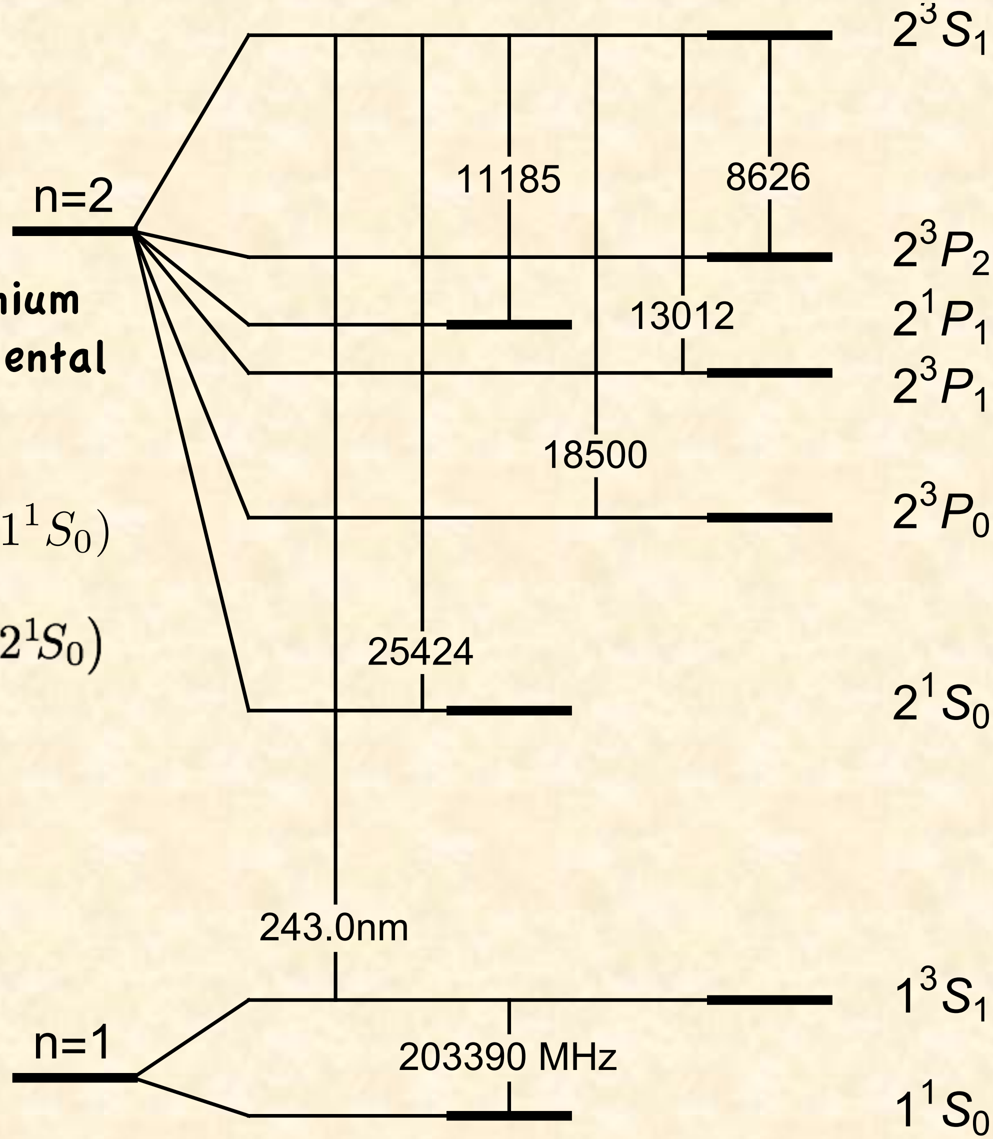
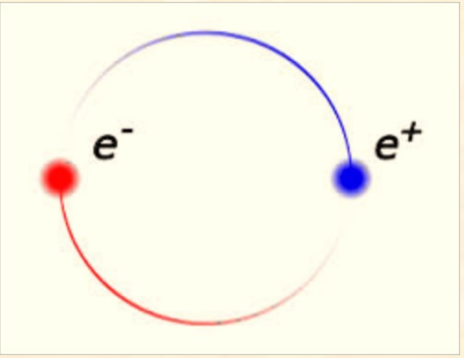
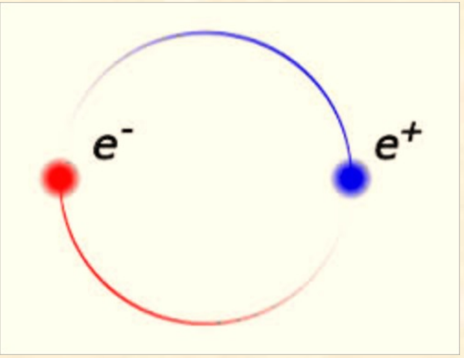
All of the measurements have uncertainties on the order of 1MHz





Measured Transitions

Transition	Common Name	Natural Linewidth	Expt. Uncert.
$1^3S_1 - 1^1S_0$	n=1 hyperfine	≈1300MHz	≈1MHz
$2^3S_1 - 2P$	n=2 fine structure	≈50MHz	≈2MHz
$1^3S_1 - 2^3S_1$	1S-2S	≈1.3MHz	≈3MHz



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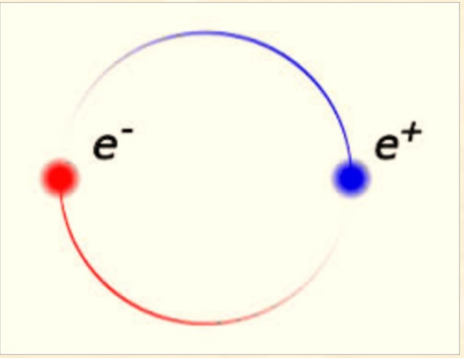
1S-2S ΔE=1233607216.4(3.2) MHz (2.6ppb)
seems to have the greatest potential for improvement.



Status of Energy Level Theory

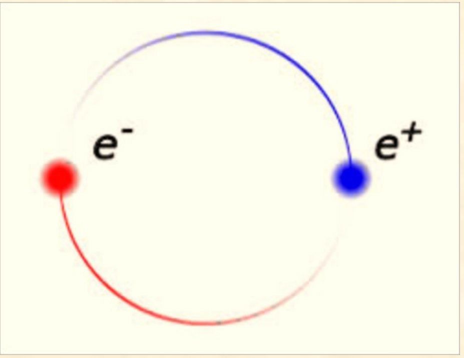
The theoretical formula for the energy levels of positronium states can be written as

$$\Delta E \quad = \quad m\Big\{C_{20}\alpha^2 + C_{40}\alpha^4 + C_{51}\alpha^5 L + C_{50}\alpha^5$$

$$+C_{61}\alpha^6 L + C_{60}\alpha^6 + C_{72}\alpha^7 L^2 + C_{71}\alpha^7 L + C_{70}\alpha^7 + \cdots\Big\}$$

where $L = \ln(1/\alpha)$. All terms through order $\alpha^7 \ln^2(1/\alpha)$ are known, as are the order $\alpha^7 \ln(1/\alpha)$ terms for the hyperfine interval. Estimates of the theoretical uncertainties are

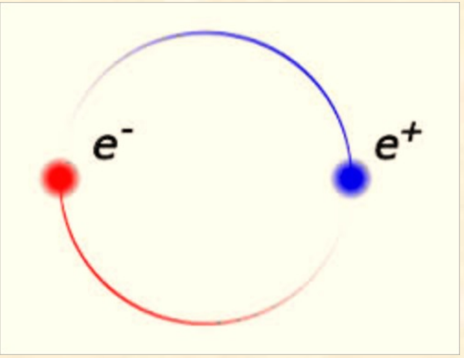
0.6 MHz	1S-2S
0.5 MHz	hfs
0.1 MHz	n=2 fine structure



Status of Energy Level Theory

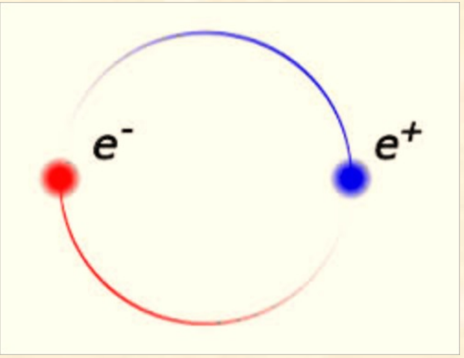
Contributions to positronium energy levels for n=1 (in MHz)

Order	Value	E_{av} $(1S)$	E_{HFS} $(1^3S_1 - 1^1S_0)$
$m\alpha^4$	350 377	38 322.493	204 386.630
$m\alpha^5 \ln(1/\alpha)$	12 580	3003.302	0
$m\alpha^5$	2557	−1018.784	−1005.497
$m\alpha^6 \ln(1/\alpha)$	91.8	2.869	19.125
$m\alpha^6$	18.7	3.000	−7.330
$m\alpha^7 \ln^2(1/\alpha)$	3.30	−1.091	−0.918
$m\alpha^7 \ln(1/\alpha)$	0.67	···	−0.323
$m\alpha^7$	0.14	···	···



Method of Calculation

1. Use Non-Relativistic QED (NRQED) and dimensional regularization
2. Obtain all required matching coefficients
3. Describe two-body bound states using the NRQED Bethe-Salpeter equation. Energies appear as poles in the Green function.
4. Build a perturbation scheme based on an exact lowest-order solution to the NRQED Bethe-Salpeter equation
5. Use the "method of regions" to identify contributions at various powers of the expansion parameter α
6. Express all contributions in terms of expectation values of various operators in states of the D-dimensional non-relativistic Schrödinger-Coulomb equation

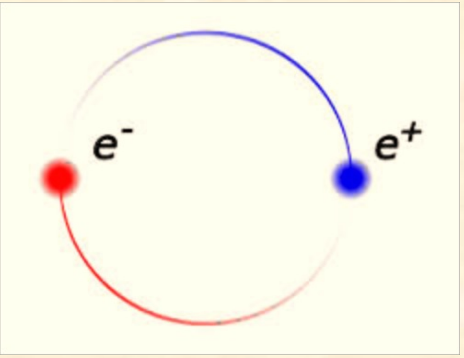


Lagrangian of the Effective Quantum Field Theory NRQED (Non-Relativistic QED)

We will replace the usual QED Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi + \text{photon terms}$$

by



Lagrangian of the Effective Quantum Field Theory NRQED (Non-Relativistic QED)

$$\mathcal{L} = \psi^\dagger \Big\{ iD_t + \frac{\vec{D}^{\,2}}{2m} + \frac{\vec{D}^{\,4}}{8m^3} + c_F \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + c_D \frac{q}{2m^2} [\vec{\nabla} \cdot \vec{E}]$$

$$+ c_S \frac{iq}{8m^2} \vec{\sigma} \cdot \left(\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right) + \cdots \Big\} \psi$$

$$+ \text{positron terms}$$

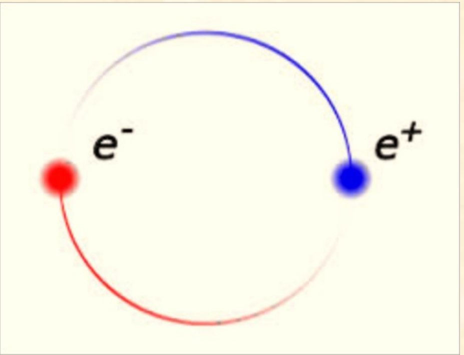
$$+ \text{four–fermion contact terms}$$

$$+ \text{photon terms}$$

with

$$D_t = \frac{\partial}{\partial t} + iqA^0, \quad \vec{D} = \vec{\nabla} - iq\vec{A}, \quad \vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The electron charge and mass are q and m



NRQED Feynman Rules

Propagation Factors:

Fermion

$$\frac{i}{p_0 - \frac{\vec{p}^{\,2}}{2\,m} + \mathrm{i}\epsilon}$$

Relativistic Kinetic Energy

$$\frac{i\,\vec{p}^{\,4}}{8\,m^3}$$

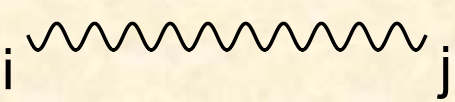
Coulomb Photon

$$\frac{i}{\vec{k}^{\,2}}$$

Coulomb Vacuum Polarization

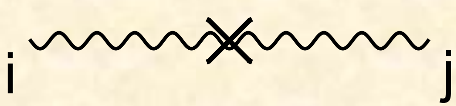
$$c_{VP}\,\frac{i\,\vec{k}^{\,2}\,k^2}{m_e^2}$$

Transverse Photon

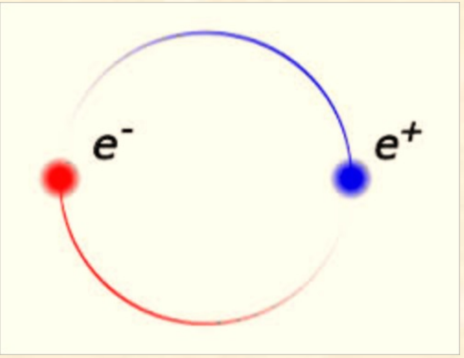


$$\frac{\mathrm{i}\delta_{ij}^{T}\,(\vec{k}\,)}{k^2 + \mathrm{i}\epsilon}$$

Transverse Vacuum Polarization



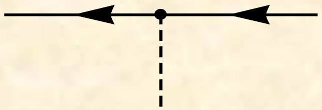
$$c_{VP}\,\frac{\mathrm{i}k^4}{m_e^2}\,\delta_{ij}$$



NRQED Feynman Rules

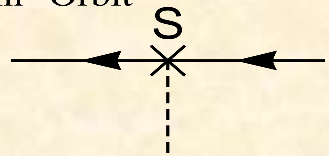
Interaction Vertices:

Coulomb



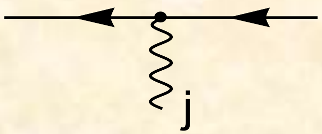
$-iq$

Spin–Orbit



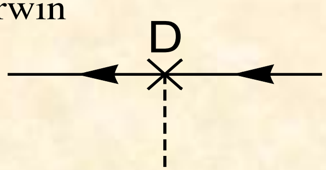
$c_S \frac{q}{4m^2} (\vec{p}_2 \times \vec{p}_1) \cdot \vec{\sigma}$

Convection



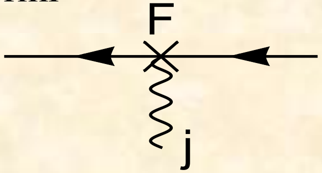
$\frac{iq}{2m} (p_2 + p_1)_j$

Darwin



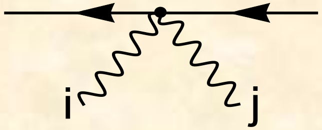
$c_D \frac{iq}{8m^2} (\vec{p}_2 - \vec{p}_1)^2$

Fermi



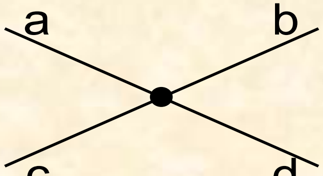
$c_F \frac{q}{2m} [(\vec{p}_2 - \vec{p}_1) \times \vec{\sigma}]_j$

Seagull



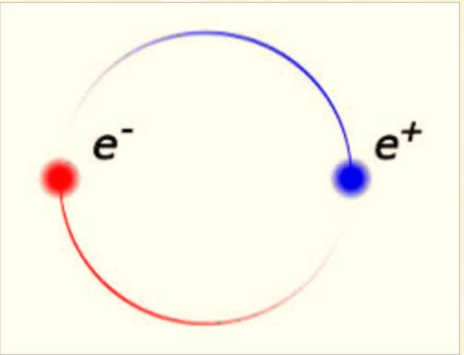
$\frac{-iq^2}{m} \delta_{ij}$

Contact



$\frac{d_s}{m^2} \delta_{ab} \delta_{cd} + \frac{d_v}{m^2} \vec{\sigma}_{ab} \cdot \vec{\sigma}_{cd}$

$+ \cdots$



NRQED Feynman Rules

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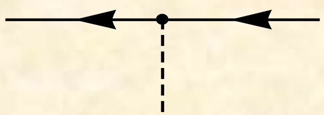

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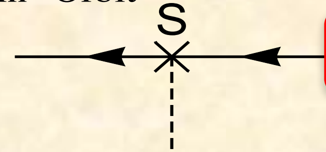
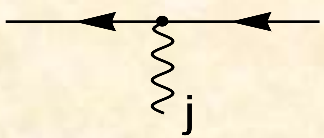
Spin–Orbit



$$c_S \frac{q}{4\,m^2} (\vec{p}_2 \times \vec{p}_1) \cdot \vec{\sigma}$$

Convection



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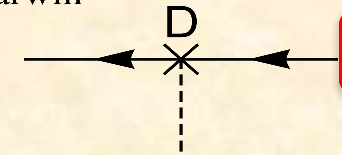

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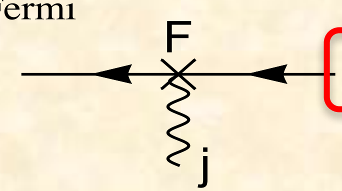
Fermi



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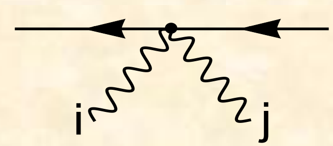
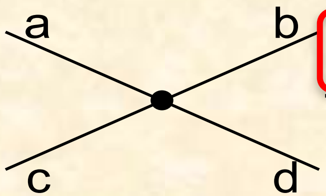
Seagull

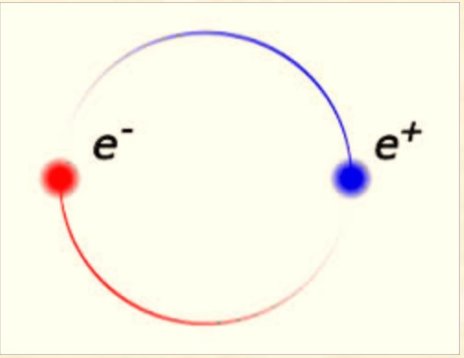


$$\frac{-iq^2}{m} \delta_{ij}$$

Contact



$$\frac{d_s}{m^2} \delta_{ab} \delta_{cd} + \frac{d_v}{m^2} \vec{\sigma}_{ab} \cdot \vec{\sigma}_{cd} \qquad + \cdots$$



Lagrangian of NRQED

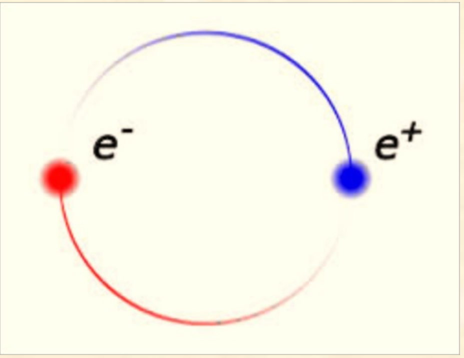
$$\mathcal{L} = \psi^\dagger \Big\{ iD_t + \frac{\vec{D}^{\,2}}{2m} + \frac{\vec{D}^{\,4}}{8m^3} + c_F \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + c_D \frac{q}{2m^2} [\vec{\nabla} \cdot \vec{E}]$$

$$+ c_S \frac{iq}{8m^2} \vec{\sigma} \cdot \left(\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right) + \cdots \Big\} \psi$$

$+$ positron terms

$+$ four–fermion contact terms

$+$ photon terms



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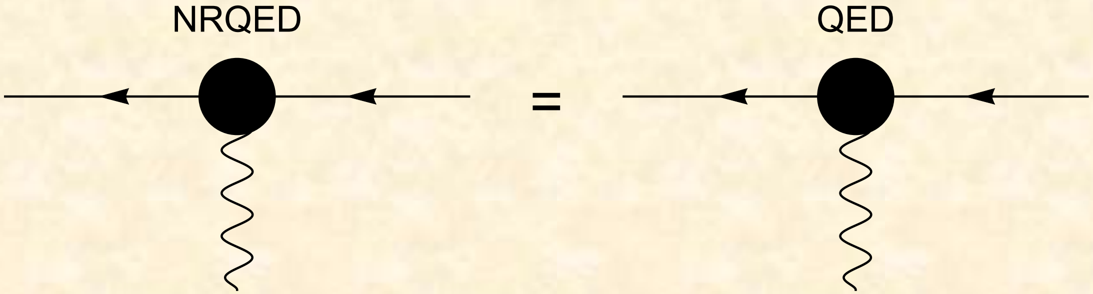
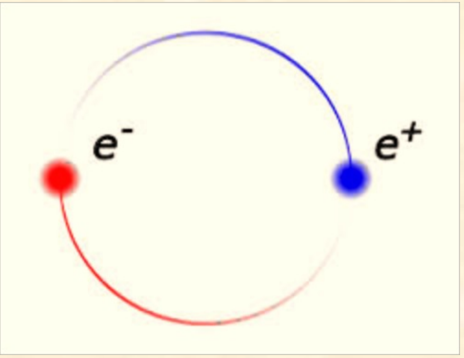


The matching coefficients have the values

(using dimensional regularization in $d = 4 - 2\epsilon$ dimensions)

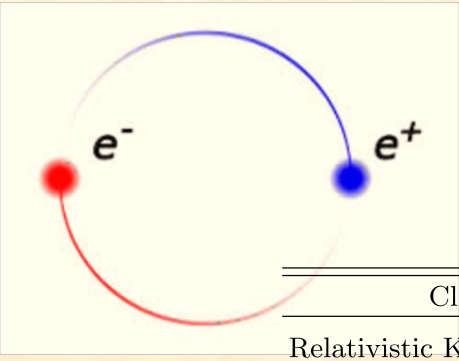
$$
\begin{aligned}
c_F &= 1 + \frac{\alpha}{2\pi} + \cdots \\
c_D &= 1 + \frac{8\alpha}{3\pi}\left\{\frac{-1}{2\epsilon} + \ln\frac{m}{\mu}\right\} + \cdots \\
c_S &= 1 + \frac{\alpha}{\pi} + \cdots
\end{aligned}
$$



NRQED Lagrangian

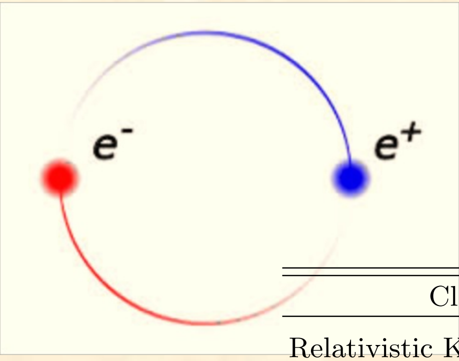
$$
\begin{aligned}
\mathcal{L} = \psi^\dagger \Big\{ & iD_t + c_2 \frac{\boldsymbol{D}^2}{2M} + c_4 \frac{\boldsymbol{D}^4}{8M^3} + c_F g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D g \frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + i c_S g \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})}{8M^2} + c_{W1} g \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} \\
& - c_{W2} g \frac{D^i \boldsymbol{\sigma}\cdot\boldsymbol{B} D^i}{4M^3} + c_{p'p} g \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} + i c_M g \frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} + c_{A1} g^2 \frac{\boldsymbol{B}^2 - \boldsymbol{E}^2}{8M^3} - c_{A2} g^2 \frac{\boldsymbol{E}^2}{16M^3} \\
& + c_{X1} g \frac{[\boldsymbol{D}^2, \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{D}]}{M^4} + c_{X2} g \frac{\{\boldsymbol{D}^2, [\boldsymbol{\partial}\cdot\boldsymbol{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + i c_{X4} g^2 \frac{\{\boldsymbol{D}^i, [\boldsymbol{E}\times\boldsymbol{B}]^i\}}{M^4} \\
& + i c_{X5} g \frac{D^i \boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E}-\boldsymbol{E}\times\boldsymbol{D})D^i}{M^4} + i c_{X6} g \frac{\epsilon^{ijk}\sigma^i D^j [\boldsymbol{\partial}\cdot\boldsymbol{E}] D^k}{M^4} + c_{X7} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{M^4} + c_{X8} g^2 \frac{[\boldsymbol{E}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{B}]}{M^4} \\
& + c_{X9} g^2 \frac{[\boldsymbol{B}\cdot\boldsymbol{\partial}\boldsymbol{\sigma}\cdot\boldsymbol{E}]}{M^4} + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} B^i]}{M^4} + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma}\cdot\boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma}\cdot\boldsymbol{E}\times[\partial_t \boldsymbol{E} - \boldsymbol{\partial}\times\boldsymbol{B}]}{M^4} + \mathcal{O}(1/M^5) \Big\} \psi .
\end{aligned}
$$

From R. J. Hill, G. Lee, G. Paz, and M. P. Solon
NRQED Lagrangian at order 1/M⁴
Phys. Rev. D 87, 053017 (2013)



Feynman Rules for NRQED

Class	Type	Label	Rule
Relativistic Kinetic Energy	$\psi^\dagger\psi$	$K4$	$\frac{i\vec{p}^{\,4}}{8m^3}$
		$K6$	$\frac{-i\vec{p}^{\,6}}{16m^5}$
Coulomb	$\psi^\dagger A^0\psi$		$-iq$
		D (Darwin)	$c_D\frac{iq}{8m^2}\vec{k}^{\,2}$
		C$X1$	$c_{X1}\frac{-iq}{m^4}\left(\vec{p}'^2-\vec{p}^{\,2}\right)^2$
		C$X2$	$c_{X2}\frac{-iq}{m^4}\left(\vec{p}'^2+\vec{p}^{\,2}\right)\vec{k}^{\,2}$
		C$X3$	$c_{X3}\frac{-iq}{m^4}\vec{k}^{\,4}$
Spin-orbit	$\psi^\dagger A^0\psi$	S (Spin-orbit)	$c_S\frac{q}{4m^2}\sigma_{ab}p'^a p^b$
		S$X5$	$c_{X5}\frac{-2q}{m^4}\vec{p}'\cdot\vec{p}\,\sigma_{ab}p'^a p^b$
		S$X6$	$c_{X6}\frac{q}{m^4}\vec{k}^{\,2}\sigma_{ab}p'^a p^b$
Convection	$\psi^\dagger A^i\psi$		$\frac{iq}{2m}P^i$
		con$K4$	$\frac{-iq}{8m^3}\left(\vec{p}'^2+\vec{p}^{\,2}\right)P^i$
		con$M$	$c_M\frac{-iq}{8m^3}\vec{k}^{\,2}P^i$
		con$X1$	$c_{X1}\frac{iq}{m^4}\left(\vec{p}'^2-\vec{p}^{\,2}\right)k^0P^i$
Fermi	$\psi^\dagger A^i\psi$	F (Fermi)	$c_F\frac{q}{2m}\sigma_{ij}k^j$
		F$W1$	$c_{W1}\frac{-q}{8m^3}\left(\vec{p}'^2+\vec{p}^{\,2}\right)\sigma_{ij}k^j$
		F$W2$	$c_{W2}\frac{q}{4m^3}\vec{p}'\cdot\vec{p}\,\sigma_{ij}k^j$
		F$p'p$	$c_{p'p}\frac{-q}{16m^3}\vec{P}^{\,2}\sigma_{ij}k^j$
P-Fermi	$\psi^\dagger A^i\psi$	PFS	$c_S\frac{-q}{8m^2}k^0\sigma_{ij}P^j$
		PF$p'p$	$c_{p'p}\frac{q}{16m^3}\left(\vec{p}'^2-\vec{p}^{\,2}\right)\sigma_{ij}P^j$
		PF$X5$	$c_{X5}\frac{q}{m^4}\vec{p}'\cdot\vec{p}\,k^0\sigma_{ij}P^j$
Spin-convection	$\psi^\dagger A^i\psi$	sc$p'p$	$c_{p'p}\frac{-q}{8m^3}\sigma_{ab}p'^a p^b P^i$



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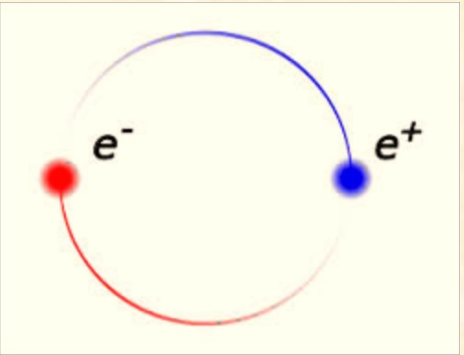
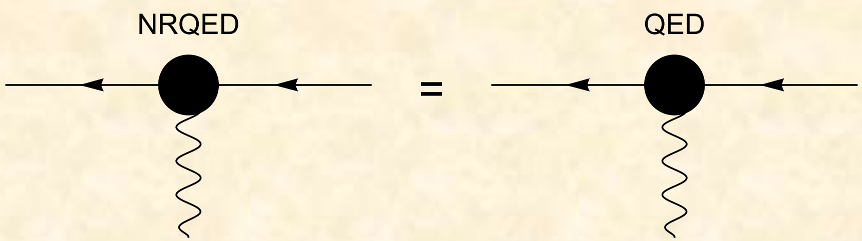
		D (Darwin)	$c_D \frac{iq}{8m^2}\vec{k}^{\,2}$
		$\mathrm{C}X1$	$c_{X1}\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)^2$
		$\mathrm{C}X2$	$c_{X2}\frac{-iq}{m^4}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\vec{k}^{\,2}$
		$\mathrm{C}X3$	$c_{X3}\frac{-iq}{m^4}\vec{k}^{\,4}$
Spin-orbit	$\psi^\dagger A^0\psi$	S (Spin-orbit)	$c_S\frac{q}{4m^2}\sigma_{ab}p'^a p^b$
		$\mathrm{S}X5$	$c_{X5}\frac{-2q}{m^4}\vec{p}^{\,\prime}\cdot\vec{p}\,\sigma_{ab}p'^a p^b$
		$\mathrm{S}X6$	$c_{X6}\frac{q}{m^4}\vec{k}^{\,2}\sigma_{ab}p'^a p^b$
Convection	$\psi^\dagger A^i\psi$		$\frac{iq}{2m}P^i$
		$\mathrm{con}K4$	$\frac{-iq}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)P^i$
		$\mathrm{con}M$	$c_M\frac{-iq}{8m^3}\vec{k}^{\,2}P^i$
		$\mathrm{con}X1$	$c_{X1}\frac{iq}{m^4}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)k^0P^i$
Fermi	$\psi^\dagger A^i\psi$	F (Fermi)	$c_F\frac{q}{2m}\sigma_{ij}k^j$
		$\mathrm{F}W1$	$c_{W1}\frac{-q}{8m^3}\left(\vec{p}^{\,\prime 2}+\vec{p}^{\,2}\right)\sigma_{ij}k^j$
		$\mathrm{F}W2$	$c_{W2}\frac{q}{4m^3}\vec{p}^{\,\prime}\cdot\vec{p}\,\sigma_{ij}k^j$
		$\mathrm{F}p'p$	$c_{p'p}\frac{-q}{16m^3}\vec{P}^{\,2}\sigma_{ij}k^j$
P-Fermi	$\psi^\dagger A^i\psi$	PFS	$c_S\frac{-q}{8m^2}k^0\sigma_{ij}P^j$
		$\mathrm{PF}p'p$	$c_{p'p}\frac{q}{16m^3}\left(\vec{p}^{\,\prime 2}-\vec{p}^{\,2}\right)\sigma_{ij}P^j$
		$\mathrm{PF}X5$	$c_{X5}\frac{q}{m^4}\vec{p}^{\,\prime}\cdot\vec{p}\,k^0\sigma_{ij}P^j$
Spin-convection	$\psi^\dagger A^i\psi$	$\mathrm{sc}p'p$	$c_{p'p}\frac{-q}{8m^3}\sigma_{ab}p'^a p^b P^i$



Matching Coefficient Expansions



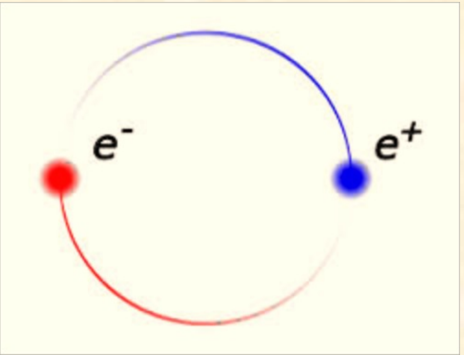
$$\Gamma_\mu^{\rm QED}(p_2,p_1) = \bar{u}(p_2)\left\{\gamma_\mu F_1(k^2) + \frac{i\sigma_{\mu\nu}k^\nu}{2m}F_2(k^2)\right\}u(p_1)$$

$$k^\mu = p_2^\mu - p_1^\mu$$

$$\overline{F}_i \equiv F_i(0), \qquad \overline{F}'_i \equiv \frac{dF_i}{d(q^2/m^2)}|_{q^2=0},$$

$$\overline{F}''_i \equiv \frac{d^2F_i}{d(q^2/m^2)^2}|_{q^2=0}.$$

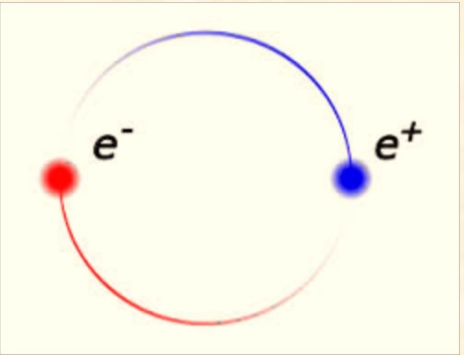
$$\begin{aligned}
c_F &= \overline{F}_1 + \overline{F}_2, \\
c_D &= \overline{F}_1 + 2\overline{F}_2 + 8\overline{F}'_1, \\
c_S &= \overline{F}_1 + 2\overline{F}_2, \\
c_{W1} &= \overline{F}_1 + \frac{1}{2}\overline{F}_2 + 4\overline{F}'_1 + 4\overline{F}'_2, \\
c_{W2} &= \frac{1}{2}\overline{F}_2 + 4\overline{F}'_1 + 4\overline{F}'_2, \\
c_{p'p} &= \overline{F}_2, \\
c_M &= \frac{1}{2}\overline{F}_2 + 4\overline{F}'_1, \\
c_{X1} &= \frac{5}{128}\overline{F}_1 + \frac{1}{32}\overline{F}_2 + \frac{1}{4}\overline{F}'_1, \\
c_{X2} &= \frac{3}{64}\overline{F}_1 + \frac{1}{16}\overline{F}_2, \\
c_{X3} &= \frac{1}{8}\overline{F}'_1 + \frac{1}{4}\overline{F}'_2 + \frac{1}{2}\overline{F}''_1, \\
c_{X5} &= \frac{3}{32}\overline{F}_1 + \frac{1}{8}\overline{F}_2, \\
c_{X6} &= -\frac{3}{32}\overline{F}_1 - \frac{1}{8}\overline{F}_2 - \frac{1}{4}\overline{F}'_1 - \frac{1}{2}\overline{F}'_2
\end{aligned}$$



Matching Coefficient Expansions

Coefficient	$O(\alpha^0)$	$O(\alpha^1)$	$O(\alpha^2)$	$O(\alpha^3)$
c_F	1	$\frac{1}{2}$	$\frac{3}{4}\zeta_3-3\zeta_2\ln 2+\frac{1}{2}\zeta_2+\frac{197}{144}$	$-\frac{215}{24}\zeta_5+\frac{83}{12}\zeta_3\zeta_2-\frac{239}{24}\zeta_4-\frac{25}{3}\zeta_2\ln^2 2+\frac{100}{3}a_4$ $+\frac{25}{18}\ln^4 2+\frac{139}{18}\zeta_3-\frac{596}{3}\zeta_2\ln 2+\frac{17101}{135}\zeta_2+\frac{28259}{5184}$
c_D	1	$-\frac{8}{3}\Xi$	$-\frac{9}{2}\zeta_3+18\zeta_2\ln 2-\frac{40}{9}\zeta_2-\frac{1523}{324}$	$\frac{85}{12}\zeta_5-\frac{121}{6}\zeta_3\zeta_2+\frac{1591}{18}\zeta_4-\frac{956}{45}\zeta_2\ln^2 2-\frac{1136}{9}a_4$ $-\frac{142}{27}\ln^4 2-\frac{791}{12}\zeta_3+\frac{23791}{45}\zeta_2\ln 2-\frac{249767}{810}\zeta_2+\frac{88409}{11664}$
c_S	1	1	$\frac{3}{2}\zeta_3-6\zeta_2\ln 2+\zeta_2+\frac{197}{72}$	$-\frac{215}{12}\zeta_5+\frac{83}{6}\zeta_3\zeta_2-\frac{239}{12}\zeta_4-\frac{50}{3}\zeta_2\ln^2 2+\frac{200}{3}a_4$ $+\frac{25}{9}\ln^4 2+\frac{139}{9}\zeta_3-\frac{1192}{3}\zeta_2\ln 2+\frac{34202}{135}\zeta_2+\frac{28259}{2592}$
c_{W1}	1	$-\frac{4}{3}\Xi+\frac{1}{12}$		
c_{W2}	0	$-\frac{4}{3}\Xi+\frac{1}{12}$		
$c_{p'p}$	0	$\frac{1}{2}$		
c_M	0	$-\frac{4}{3}\Xi-\frac{1}{4}$		
c_{X1}	$\frac{5}{128}$	$-\frac{1}{12}\Xi-\frac{1}{64}$		
c_{X2}	$\frac{3}{64}$	$\frac{1}{32}$		
c_{X3}	0	$-\frac{11}{120}\Xi-\frac{13}{320}$		
c_{X5}	$\frac{3}{32}$	$\frac{1}{16}$		
c_{X6}	$-\frac{3}{32}$	$\frac{1}{12}\Xi-\frac{7}{96}$		

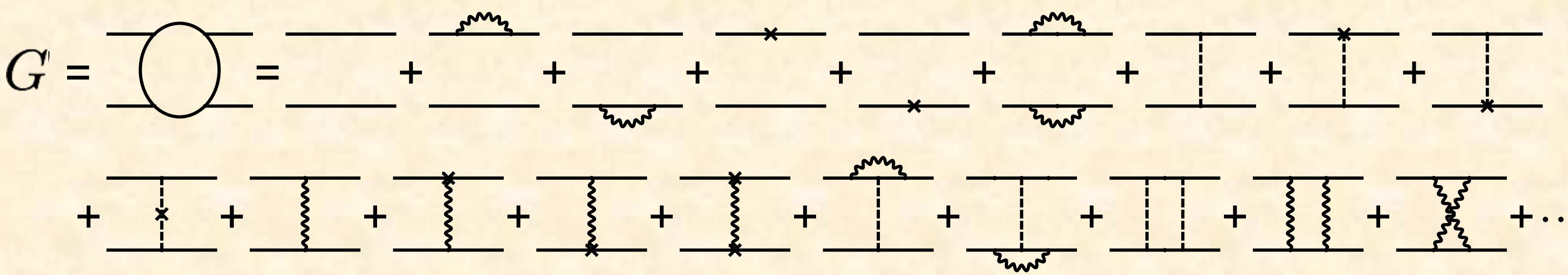
$$\Xi=\frac{1}{2\epsilon}-\ln\left(\frac{m}{\mu}\right)$$

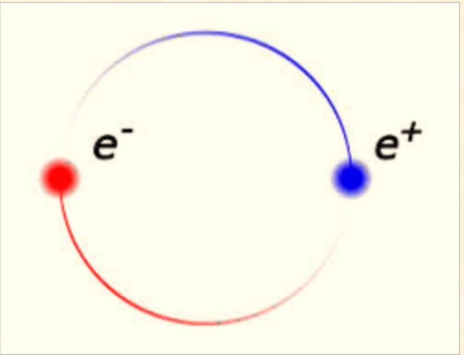


Method of Calculation

Energy levels are found as the positions of poles in the electron-positron 2-to-2 NRQED Green function G:

$$G(E; p_2, p_1) \to \frac{i \sum_k \Psi_{nak}(p_2) \bar{\Psi}_{nak}(p_1)}{E - E_{na}}$$





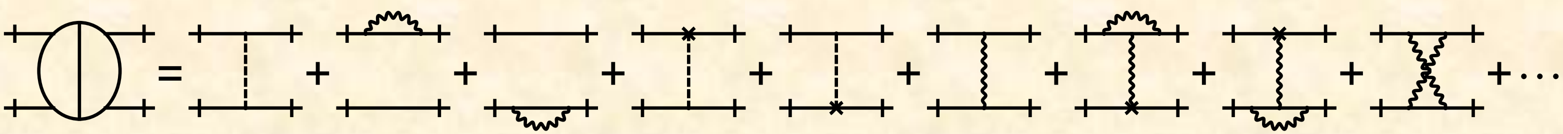
Method of Calculation

G satisfies the NRQED Bethe-Salpeter equation

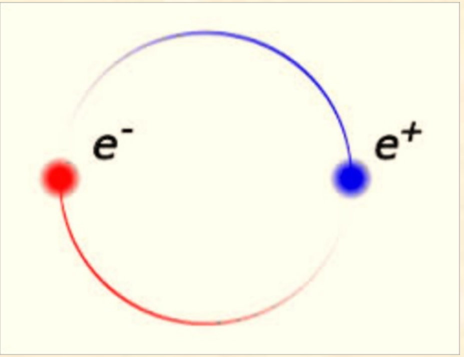
$$G = S + SKG$$

Where S is the product of a free electron and a free positron propagator and K is the two-particle irreducible electron-positron to electron-positron kernel

$S =$

$K =$ 

where the slash at the end of each fermion line indicates that the fermion line is not actually present in K but is shown only for clarity.



Method of Calculation

We want to base a perturbation scheme on an exactly-soluble lowest order problem:

$$G_0 = S + SK_0G_0$$

Where $K_0(E;p_2,p_1) = -iV(\vec{p}_2 - \vec{p}_1),$ $\qquad V(\vec{k}\,) = -\frac{4\pi\alpha\bar{\mu}^{2\epsilon}}{\vec{k}^2}$

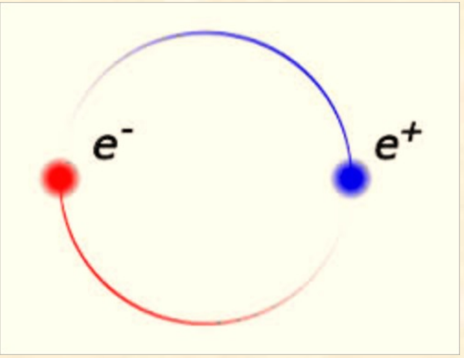
represents the exchange of a Coulomb photon. The solution for G_0 can be expressed in terms of solutions to the usual Schrodinger-Coulomb (SC) equation (but in D=3-2ε dimensions):

$$G_0 = \sum_{n,\ell,m} \frac{\psi^*_{0;n\ell m}\psi_{0;n\ell m}}{E - E_{0;n\ell}}$$

The perturbative scheme uses the perturbing kernel $\delta K = K - K_0$

$$\Delta E = (\delta K) + (\delta K \hat{G}_0 \delta K) + \left(\frac{d}{dE}\delta K|_{E_0}\right) + \cdots$$

with $(\delta K) = \int \frac{d^dp_2}{(2\pi)^d}\frac{d^dp_1}{(2\pi)^d}\bar{\Psi}(E;p_2)\delta K(E;p_2,p_1)\Psi(E;p_1)$, etc.



Hydrogen Atom in D=3-2ε Dimensions

The lowest order wave functions and energies satisfy the Schrödinger-Coulomb equation in D=3-2ε dimensions:

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x}\,) + V(r)\psi(\vec{x}\,) = E\psi(\vec{x}\,)$$

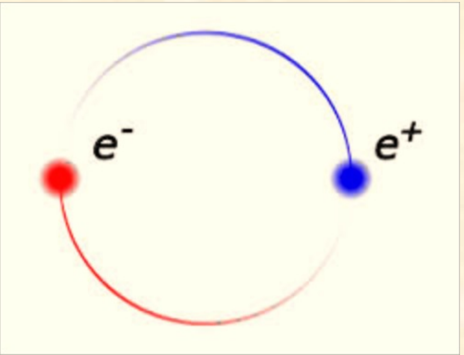
where V(r) is the D-dimensional Fourier transform of the Coulomb exchange interaction

$$V(r) = \int \frac{d^D k}{(2\pi)^D}\,\frac{-4\pi\alpha\bar{\mu}^{2\epsilon}}{\vec{k}^{\,2}} = -\frac{\Gamma(D/2-1)\bar{\mu}^{2\epsilon}\alpha}{\pi^{D/2-1}r^{D-2}}$$

and

$$\vec{\nabla}^2 = \left(\frac{\partial}{\partial r}\right)^2 + \frac{D-1}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+D-2)}{r^2}$$

when acting on a radial function times a D-dimensional spherical harmonic.



Hydrogen Atom in D=3-2ε Dimensions

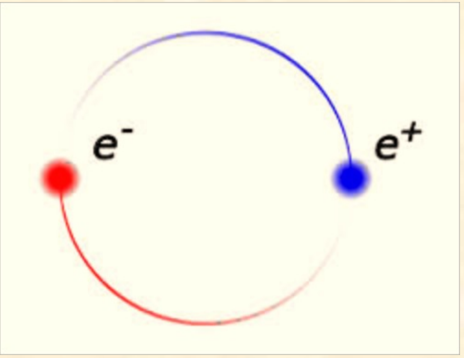
The radial function satisfies

$$\frac{1}{2m}\left\{-\left(\frac{\partial}{\partial r}\right)^2 - \frac{D-1}{r}\frac{\partial}{\partial r} + \frac{\ell(\ell+D-2)}{r^2}\right\} R(r) + V(r)R(r) = ER(r)$$

The short and long distance behavior can be factored out as in three dimensions:

$$R(r) = \phi\, \Omega_{D-1}^{1/2} \left(\frac{(n+\ell)!}{n(n-\ell-1)!}\right)^{1/2} \frac{\rho^\ell e^{-\rho/2}}{(2\ell+1)!} L(\rho)$$

Where ϕ is the “wave function at the origin”, Ω is a solid angle factor in D dimensions, ρ=2γr where E=-γ^2/(2m) and L(0)=1. L(ρ) generalizes the usual associated Laguerre polynomial.



Hydrogen Atom in D=3-2ε Dimensions

The equation for L(ρ) is

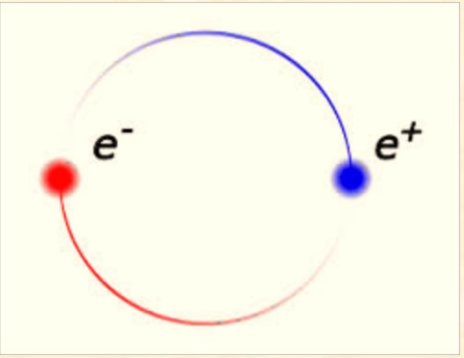
$$\left\{\left(\frac{\partial}{\partial\rho}\right)^2+\left(\frac{2(\ell+1-\epsilon)}{\rho}-1\right)\frac{\partial}{\partial\rho}-\frac{(\ell+1-\epsilon)}{\rho}+\frac{\bar{n}\rho^{2\epsilon}}{\rho}\right\}L(\rho)=0$$

with the series solution

$$\begin{aligned} L(\rho) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{jk}\rho^j\left(\bar{n}\rho^{2\epsilon}\right)^k \\ &= 1+\rho\left(a_{10}+a_{11}\bar{n}\rho^{2\epsilon}\right)+\rho^2\left(a_{20}+a_{21}\bar{n}\rho^{2\epsilon}+a_{22}\bar{n}^2\rho^{4\epsilon}\right)+\cdots \end{aligned}$$

where

$$a_{jk}=\frac{a_{j-1,k}(j+\ell+\epsilon[2k-1])-a_{j-1,k-1}}{(j+2\epsilon k)(j+2\ell+1+2\epsilon[k-1])}$$ with $$a_{0,0}=1,\quad a_{j,k}=0 \text{ unless } 0\le \mathrm{k}\le \mathrm{j}$$



Hydrogen Atom in D=3-2ε Dimensions

Perturbation Theory

Hydrogen with the potential $V_0(r) = -\frac{\alpha}{r}$ can be solved exactly, even in D=3-2ε dimensions. Using $V_0(r)$ as the starting point of perturbation theory with

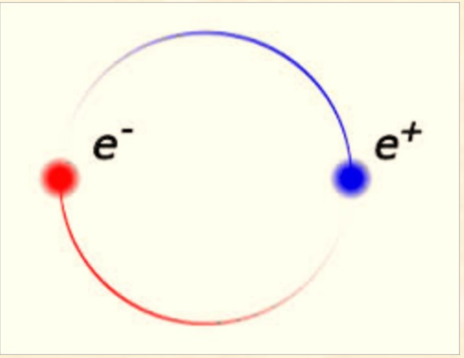
$$\delta V = V - V_0 = -\frac{\Gamma(D/2-1)\alpha\bar{\mu}^{2\epsilon}}{\pi^{D/2-1}r^{D-1}} + \frac{\alpha}{r} = -\frac{2\alpha}{r}\left(\ln(\mu r) + \gamma_E\right) + \cdots$$

one can find expansions for the energy levels and wave functions. These can be used to find expansions for the corrected principal quantum number $\bar{n}$ and the wave function at the origin ϕ. One finds

$$\bar{n} = n + \left(2\gamma_E - 2H_{n+\ell} - 1/n\right) n\epsilon + O(\epsilon^2) \qquad \left(H_n = \sum_{j=1}^{n} \frac{1}{j}\right)$$

for the state with quantum numbers n, ℓ; and (for the ground state)

$$\phi = \left(\frac{(m\alpha)^D}{\pi}\right)^{1/2} \left\{1 + \epsilon\left[3\ln\left(\frac{1}{\alpha}\right) - 2\zeta(2) + 4 + \frac{1}{2}\left(\ln\pi - \gamma_E\right)\right] + O(\epsilon^2)\right\}$$



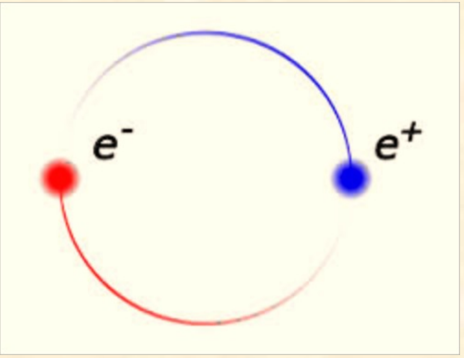
Hydrogen Atom in D=3-2ε Dimensions

What is it good for? The analysis shown here allows for simple evaluation of (possibly divergent) expectation values. As an example, consider the following expectation value that contributes at O(mα^6):

$$\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle_{n\ell} = \int_0^\infty dr\, r^{D-1} \left[\frac{dV}{dr}\right]^2 R_{n\ell}^2(r) \propto \int_0^\infty dr\, r^{-2+2\epsilon} R_{n\ell}^2(r)$$

This expectation value is divergent for S states when D=3, but with D=3-2ε we can use the knowledge of the small r behavior of the wave function to find

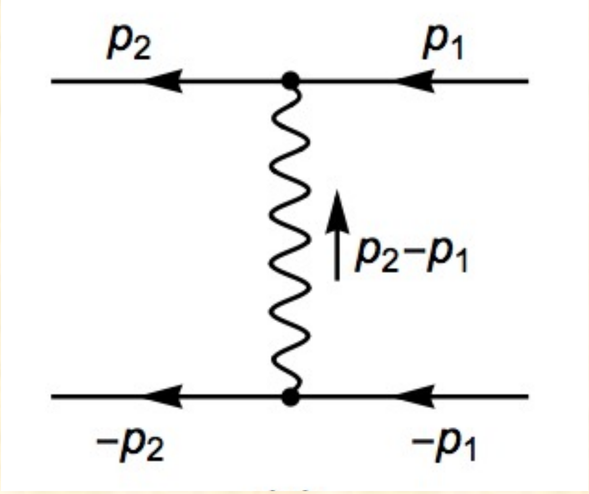
$$\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle_{n0} = \pi m \alpha^3 \phi^2 \bar{\mu}^{2\epsilon} \left\{ -\frac{2}{\epsilon} - 8\ln\left(\frac{\mu n}{2m\alpha}\right) + 8H_n + \frac{4}{3n^2} - \frac{4}{n} - \frac{16}{3} + O(\epsilon) \right\}$$

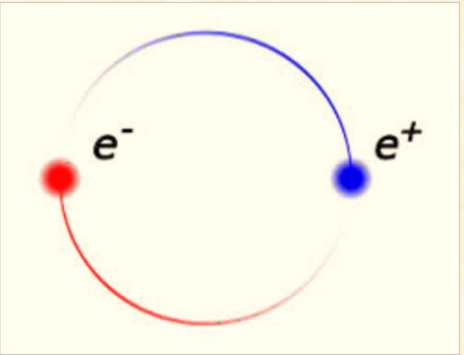


Example Calculation: Transverse Photon Exchange

The Feynman rules are used to write down the energy correction due to exchange of a single transverse photon with convection vertices:

$$\Delta E^{\mathrm{T;c,c}} = i\langle \delta K \rangle = i \int đ^d p_2 đ^d p_1 \, \bar{\Psi}(p_2) \, \frac{iq_1}{2m_1}(p_2+p_1)_i \, \frac{i\delta^T_{ij}(\vec{p}_2 - \vec{p}_1\,)}{(p_2-p_1)^2 + i\epsilon} \, \frac{iq_2}{2m_2}(-p_2-p_1)_j \, \Psi(p_1)$$





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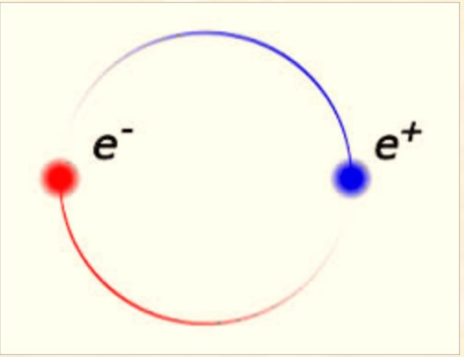
The components of this expression include:

(1) the lowest-order NRQED Bethe-Salpeter wave functions

$$\Psi(p) = \frac{i}{\xi_1 E + p_0 - \frac{\vec{p}^{\,2}}{2m_1} + i\epsilon} \, \frac{i}{\xi_2 E - p_0 - \frac{\vec{p}^{\,2}}{2m_2} + i\epsilon} (-i) \left(E - \frac{\vec{p}^{\,2}}{2m_r} \right) \psi(\vec{p})$$

where $\xi_i = m_i/(m_1+m_2)$, and $\psi(\vec{p})$ is the standard Schrödinger-Coulomb wave function (in D=3-2ε dimensions).



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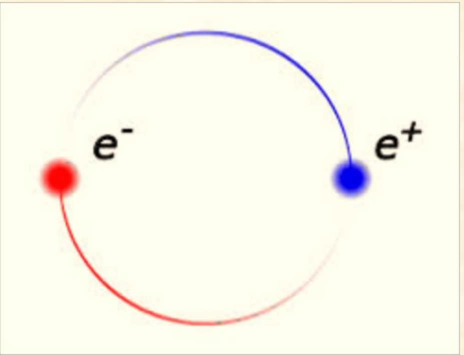
The components of this expression include:

(1) the Bethe-Salpeter wave functions
(2) the convection vertex factors



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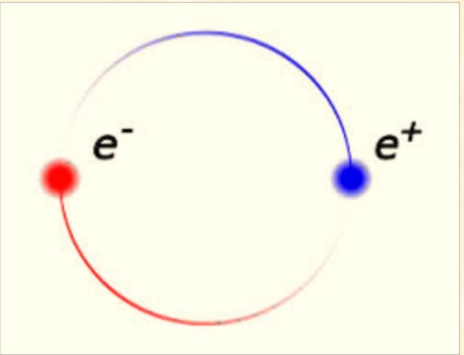
The components of this expression include:

(1) the Bethe-Salpeter wave functions
(2) the convection vertex factors
(3) the transverse photon propagator, where

$$\delta^{\mathrm{T}}_{ij}(\vec{k}\,) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^{\,2}} \quad , \quad (p_2 - p_1)^2 + i\epsilon = (p_{20} - p_{10})^2 - (\vec{p}_2 - \vec{p}_1\,)^2 + i\epsilon$$



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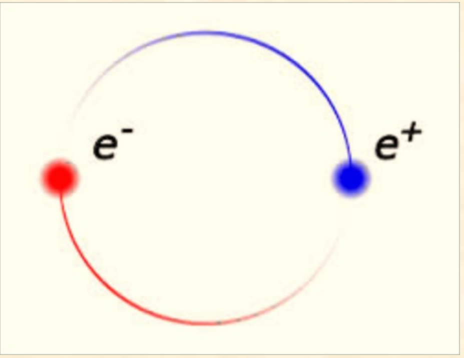
The first step in the evaluation of this term is to perform the energy integral using the residue theorem. The energy integral is

$$\begin{aligned} R(\vec{p}_2, \vec{p}_1) \; = \; & \int \frac{dp_{20}}{2\pi} \frac{dp_{10}}{2\pi} \Bigg[\left(\xi_1 E + p_{20} - \frac{\vec{p}_2^{\,2}}{2m_1} + i\epsilon_{11} \right) \left(\xi_2 E - p_{20} - \frac{\vec{p}_2^{\,2}}{2m_2} + i\epsilon_{12} \right) \\ & \times (p_{20} - p_{10} - q + i\epsilon_2)\,(p_{20} - p_{10} + q - i\epsilon_2) \\ & \times \left(\xi_1 E + p_{10} - \frac{\vec{p}_1^{\,2}}{2m_1} + i\epsilon_{31} \right) \left(\xi_2 E - p_{10} - \frac{\vec{p}_1^{\,2}}{2m_2} + i\epsilon_{32} \right) \Bigg]^{-1} \\ & \times \left(E - \frac{\vec{p}_2^{\,2}}{2m_r} \right) \left(E - \frac{\vec{p}_1^{\,2}}{2m_r} \right) \end{aligned}$$

with $q \equiv |\vec{p}_2 - \vec{p}_1|$



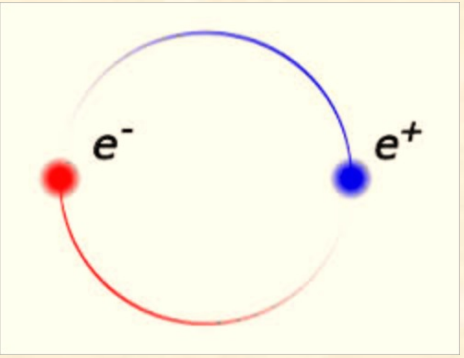
Example Calculation: Transverse Photon Exchange

The result for the energy contribution is

$$\Delta E^{\mathrm{T;c,c}} = -\frac{4\pi Z\alpha\bar{\mu}^{2\epsilon}}{m_1 m_2}\int \bar{d}^D p_2\, \bar{d}^D p_1\, \psi^\dagger(\vec{p}_2)\; p_{2i}p_{1j}\delta^{\mathrm{T}}_{ij}(\vec{p}_2-\vec{p}_1)\; R(\vec{p}_2,\vec{p}_1)\; \psi(\vec{p}_1)$$

where

$$R(\vec{p}_2,\vec{p}_1) = \frac{1}{2}\left[\frac{1}{q-E+\frac{\vec{p}_2^{\,2}}{2m_2}+\frac{\vec{p}_1^{\,2}}{2m_1}} + \frac{1}{q-E+\frac{\vec{p}_2^{\,2}}{2m_1}+\frac{\vec{p}_1^{\,2}}{2m_2}}\right]$$



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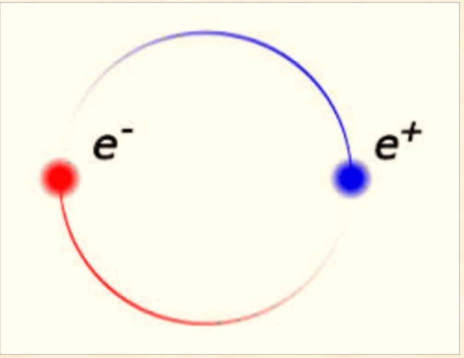
The energy integral $R(\vec{p}_2,\vec{p}_1)$ has contributions from two regions:

(1) soft (all momenta of order mα),

giving contributions at orders $m\alpha^4$, $m\alpha^5$, $m\alpha^6$, and $m\alpha^7$



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The result for the energy contribution is

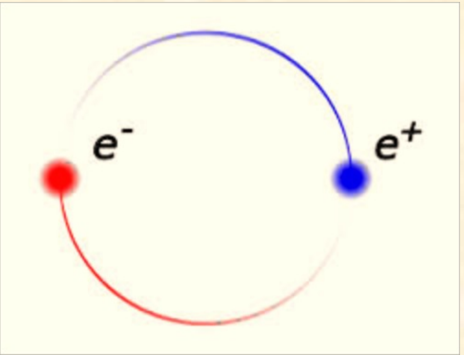
$$\Delta E^{\mathrm{T;c,c}} = -\frac{4\pi Z\alpha\bar{\mu}^{2\epsilon}}{m_1 m_2}\int \text{đ}^D p_2\, \text{đ}^D p_1\, \psi^\dagger(\vec{p}_2)\, p_{2i}p_{1j}\delta^{\mathrm{T}}_{ij}(\vec{p}_2-\vec{p}_1)\, R(\vec{p}_2,\vec{p}_1)\, \psi(\vec{p}_1)$$

The energy integral $R(\vec{p}_2,\vec{p}_1)$ has contributions from two regions:

(1) soft (all momenta of order mα), and

(2) <u>ultrasoft</u>, where $\vec{p}_2$ and $\vec{p}_1$ are soft but $\vec{p}_2-\vec{p}_1$ is ultrasoft (of order $m\alpha^2$),

giving a contribution to the Lamb-shift-like Salpeter term at order $m\alpha^5$, which contains the Bethe log, and additional Lamb-shift-like contributions at order $m\alpha^7$.



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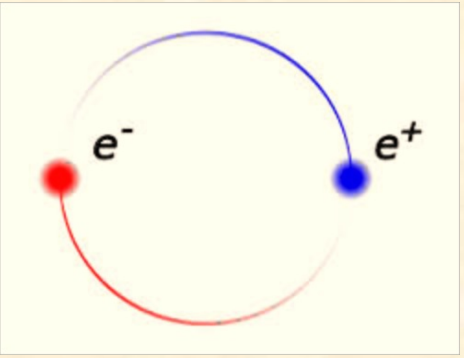
The result for the energy contribution is

$$\Delta E^{\mathrm{T;c,c}} = -\frac{4\pi Z\alpha\bar{\mu}^{2\epsilon}}{m_1 m_2} \int đ^D p_2\, đ^D p_1\, \psi^\dagger(\vec{p}_2)\, p_{2i} p_{1j} \delta^{\mathrm{T}}_{ij}(\vec{p}_2 - \vec{p}_1)\, R(\vec{p}_2, \vec{p}_1)\, \psi(\vec{p}_1)$$

The soft expansion of $R(\vec{p}_2, \vec{p}_1)$ has the form

$$\begin{aligned}
R(\vec{p}_2, \vec{p}_1) \;&=\; \frac{1}{2q^2}\left[\frac{1}{1-\frac{1}{q}\left(E-\frac{\vec{p}_2^{\,2}}{2m_2}-\frac{\vec{p}_1^{\,2}}{2m_1}\right)} + \frac{1}{1-\frac{1}{q}\left(E-\frac{\vec{p}_2^{\,2}}{2m_1}-\frac{\vec{p}_1^{\,2}}{2m_2}\right)}\right] \\
&= \frac{1}{q^2} + \frac{1}{2q^3}\left\{\left(E-\frac{\vec{p}_2^{\,2}}{2m_r}\right) + \left(E-\frac{\vec{p}_1^{\,2}}{2m_r}\right)\right\} \\
&+ \frac{1}{2q^4}\left\{\left(E-\frac{\vec{p}_2^{\,2}}{2m_r}\right)^2 + \left(E-\frac{\vec{p}_1^{\,2}}{2m_r}\right)^2 - \frac{(\vec{p}_2^{\,2}-\vec{p}_1^{\,2})^2}{2m_1 m_2}\right\} \\
&+ \frac{1}{2q^5}\left\{\left(E-\frac{\vec{p}_2^{\,2}}{2m_r}\right)^3 + \left(E-\frac{\vec{p}_1^{\,2}}{2m_r}\right)^3 - \frac{3(\vec{p}_2^{\,2}-\vec{p}_1^{\,2})^2}{4m_2 m_1}\left[\left(E-\frac{\vec{p}_2^{\,2}}{2m_r}\right) + \left(E-\frac{\vec{p}_1^{\,2}}{2m_r}\right)\right]\right\} + \cdots
\end{aligned}$$



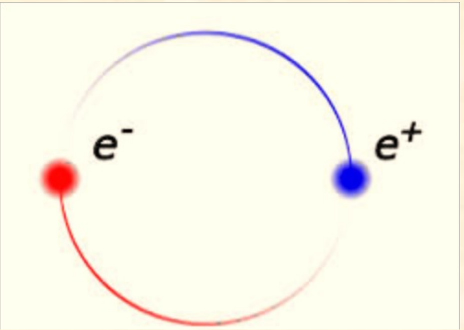
Example Calculation: Transverse Photon Exchange

Transverse photon exchange with convection vertices gives an energy contribution at order $m\alpha^6$ of

$$\Delta E^{\mathrm{T;c,c}}_{\mathrm{soft6}} = \frac{m_r^3 (Z\alpha)^6}{8 m_1 m_2} \left\{ \left(-\frac{2}{n^4} + \frac{5}{(\ell+1/2)n^3} - \frac{12}{n^3}\delta_{\ell=0} \right) + A \left(\frac{-n^2 - 5 + 7\ell(\ell+1)}{4(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5} \right) \right\} + \frac{\langle V^3 \rangle}{4 m_1 m_2} \left\{ -1 + A \right\}$$

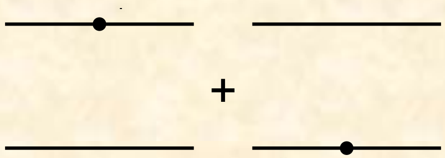
where $A = 1 - \frac{2m_r^2}{m_1 m_2}$ and

$$\langle V^3 \rangle = \begin{cases} \pi \phi_n^2 (Z\alpha)^3 \bar{\mu}^{2\epsilon} \left\{ -\frac{1}{\epsilon} - 4\ln\left(\frac{\mu}{2\gamma_n}\right) + 4H_n - \frac{2}{n} - 4 \right\} & \text{if } \ell = 0 \\ \frac{-m_r^3 (Z\alpha)^6}{\ell(\ell+1)(\ell+1/2)n^3} & \text{if } \ell > 0 \end{cases}$$

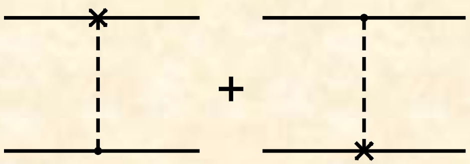


Results for energy contributions at O(mα⁶) for ℓ>0

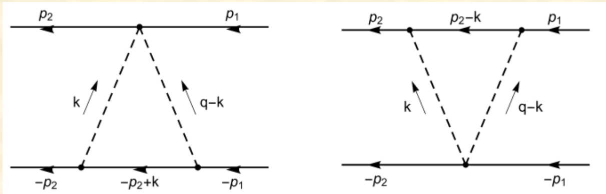
Class	Type	Energy contribution
Relativistic kinetic energy	KE_6	$\frac{1}{512}\left\{\frac{5}{n^6}+\frac{8n^2+7-12\ell(\ell+1)}{(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}$

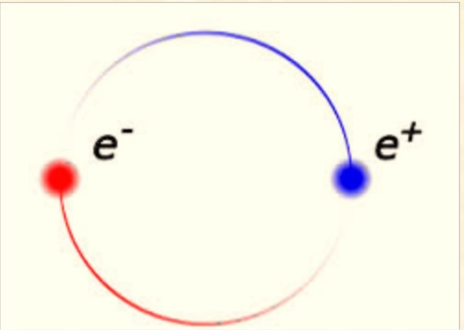


Class	Type	Energy contribution
Single Coulomb exchange	Darwin–Darwin	$\frac{n^2-1}{768n^5}\delta_{\ell=1}$
	Darwin–SO	$-\frac{n^2-1}{384n^5}\delta_{\ell=1}\langle\vec{L}\cdot\vec{S}\rangle$
	SO–SO	$\frac{1}{128}\left(\delta_{s=1}-3\delta_{s=0}\right)\left\{\frac{\xi(48n^2+9-28X)+(-n^2+X/3)}{8X(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}+\left(-15\xi+\frac{1}{2}\right)\frac{(n^2-1)}{9n^5}\delta_{\ell=1}\right\}$
	CX1	$-\frac{5}{512}\langle\overline{(V')^2}\rangle$
	CX2	0
	SX5	$\frac{3}{256}\left\{\frac{-3n^2-3/4+2\ell(\ell+1)}{\ell(\ell+1)(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}+\frac{2(n^2-1)}{3n^5}\delta_{\ell=1}\right\}\langle\vec{L}\cdot\vec{S}\rangle$
	SX6	$-\frac{(n^2-1)}{128n^5}\delta_{\ell=1}\langle\vec{L}\cdot\vec{S}\rangle$

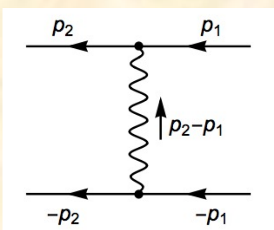
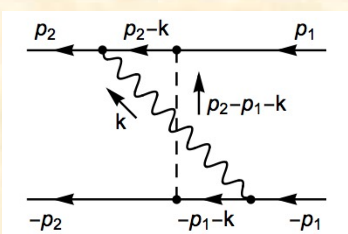


Class	Type	Energy contribution
Coulomb-Coulomb Λ and V	$CC\Lambda V$	$\frac{1}{64}c_{A1}\langle\overline{(V')^2}\rangle$





Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

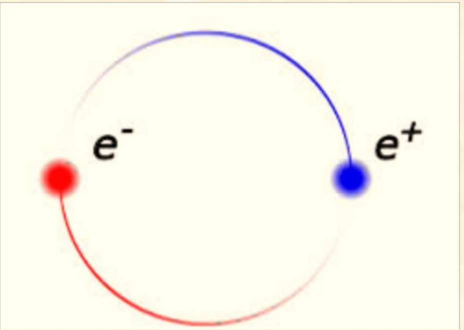
Class	Type	Energy contribution
Single transverse exchange	T;c,c	$\frac{1}{64}\left\{-\langle\overline{V^3}\rangle+\frac{-n^2-5+7\ell(\ell+1)}{8(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}-\frac{2}{n^4}+\frac{5}{(\ell+1/2)n^3}\right\}$
	T;c,F	$-\frac{1}{16}\left\{\langle\overline{V^3}\rangle-\frac{1}{4}\langle\overline{(V')^2}\rangle\right\}\langle\vec{L}\cdot\vec{S}\rangle$
	T;F,F	$-\frac{1}{192}\left\{\langle\overline{V^3}\rangle-\frac{1}{4}\langle\overline{(V')^2}\rangle\right\}\left\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\left(2\delta^0_{ab}+\delta^2_{ab}\right)\right\rangle$
	T;c,conK4	$\frac{1}{16}\left\{-\langle\overline{V^3}\rangle+\frac{1}{2n^6}+\frac{n^2+17/4-6\ell(\ell+1)}{4(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}$
	T;F,conK4	$-\frac{1}{64}\left\{\frac{3n^2+3/4-2\ell(\ell+1)}{\ell(\ell+1)(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}\langle\vec{L}\cdot\vec{S}\rangle$
	T;c,FW1	$-\frac{1}{64}\left\{\frac{3n^2+3/4-2\ell(\ell+1)}{\ell(\ell+1)(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}\langle\vec{L}\cdot\vec{S}\rangle$
	T;F,FW1	$-\frac{1}{128}\left\{\frac{3n^2+3/4-2\ell(\ell+1)}{\ell(\ell+1)(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}\left\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ab}\delta^2_{ab}\right\rangle$
	T;c,PFS	0
	T;F,PFS	0
Crossed Coulomb transverse	CT;c,c	$\frac{1}{64}\left\{\frac{11n^2+1-5\ell(\ell+1)}{4(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}+\frac{4}{n^4}-\frac{10}{(\ell+1/2)n^3}\right\}$
	CT;c,F	$-\frac{1}{8\ell(\ell+1)(\ell+1/2)n^3}\langle\vec{L}\cdot\vec{S}\rangle$
	CT;F,F	$\frac{1}{96}\left\{\langle\overline{V^3}\rangle-\frac{1}{4}\langle\overline{(V')^2}\rangle\right\}\left\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\left(2\delta^0_{ab}+\delta^2_{ab}\right)\right\rangle$
	CT;c,PFS	$\frac{3}{128}\langle\overline{(V')^2}\rangle\langle\vec{L}\cdot\vec{S}\rangle$
	CT;F,PFS	$\frac{1}{384}\langle\overline{(V')^2}\rangle\left\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\left\{\delta^0_{ab}-\delta^2_{ab}\right\}\right\rangle$

$$\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\delta^0_{ab}\rangle=\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\delta_{ab}\rangle=2(2\vec{S}^{\,2}-3)$$

$$\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\delta^2_{ab}\rangle=\langle\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}(\delta_{ab}-D\hat{x}_a\hat{x}_b)\rangle=6\xi$$

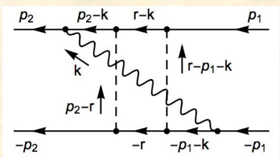
$$\xi=\xi(j,\ell,s)=\frac{2\vec{L}^{\,2}\vec{S}^{\,2}-3\vec{L}\cdot\vec{S}(2\vec{L}\cdot\vec{S}+3)}{3(4\vec{L}^{\,2}-3)}$$

$$\langle\vec{L}\cdot\vec{S}\rangle=\frac{1}{2}\left\{\vec{J}^2-\vec{L}^{\,2}-\vec{S}^{\,2}\right\}$$

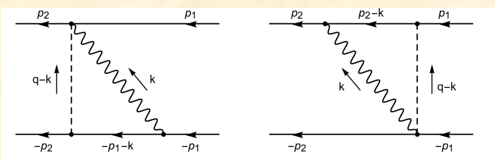


Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

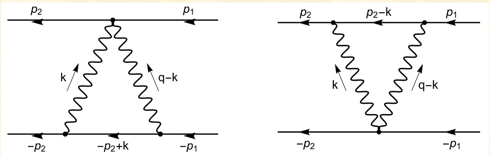
Class	Type	Energy contribution
Coulomb Coulomb transverse	CCT;c,c	$\frac{1}{64}\left\{\frac{5}{(\ell+1/2)n^3}-\frac{2}{n^4}\right\}$
	CCT;c,F	$\frac{1}{16\ell(\ell+1)(\ell+1/2)n^3}\langle\vec{L}\cdot\vec{S}\rangle$
	CCT;F,F	$\frac{1}{192\ell(\ell+1)(\ell+1/2)n^3}\left\langle\sigma_{ia}^{(1)}\sigma_{ib}^{(2)}\left(2\delta_{ab}^0+\delta_{ab}^2\right)\right\rangle$



Class	Type	Energy contribution
Coulomb-transverse Λ and V	CTΛV;c	$-\frac{1}{64}\langle\overline{(V')^2}\rangle\left\langle\vec{L}\cdot\vec{S}\right\rangle$
	CTΛV;F	$\frac{1}{192}\langle\overline{(V')^2}\rangle\left\langle\sigma_{ia}^{(1)}\sigma_{ib}^{(2)}(\delta_{ab}^0-\delta_{ab}^2)\right\rangle$

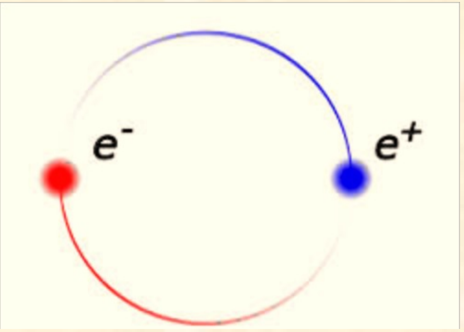


Class	Type	Energy contribution
transverse-transverse Λ and V	TTΛV;c,c	$\frac{1}{32}\left\{\frac{7n^2+2-5\ell(\ell+1)}{4(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}\right\}$
	TTΛV;c,F	$-\frac{1}{32}\langle\overline{(V')^2}\rangle\left\langle\vec{L}\cdot\vec{S}\right\rangle$
	TTΛV;F,F	$\frac{1}{32}\langle\overline{(V')^2}\rangle$



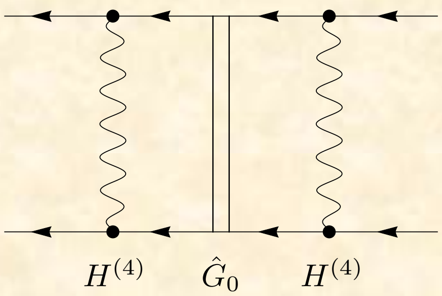
$$\langle\overline{V^3}\rangle = \frac{\langle V^3\rangle}{m_r^3(Z\alpha)^6} \qquad \langle V^3\rangle = \begin{cases}\pi\phi_n^2(Z\alpha)^3\bar{\mu}^{2\epsilon}\left\{-\frac{1}{\epsilon}-4\ln\left(\frac{\mu}{2\gamma_n}\right)+4H_n-\frac{2}{n}-4\right\} & \text{if } \ell=0\\ \frac{-m_r^3(Z\alpha)^6}{\ell(\ell+1)(\ell+1/2)n^3} & \text{if } \ell>0\end{cases}$$

$$\langle\overline{(V')^2}\rangle = \frac{\langle(V')^2\rangle}{m_r^4(Z\alpha)^6} \qquad \langle(V')^2\rangle = \begin{cases}\pi\bar{\phi}_n^2 m_r(Z\alpha)^3\bar{\mu}^{2\epsilon}\left\{-\frac{2}{\epsilon}-8\ln\left(\frac{\mu}{2\gamma_n}\right)+8H_n+\frac{4}{3n^2}-\frac{4}{n}-\frac{16}{3}\right\} & \text{if } \ell=0\\ \frac{3n^2-\ell(\ell+1)}{2\ell(\ell+1)(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}m_r^4(Z\alpha)^6 & \text{if } \ell>0\end{cases}$$

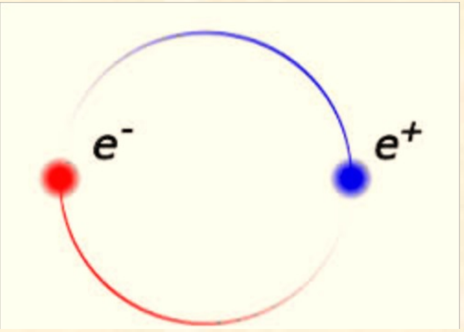


Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

Class	Type	Energy contribution
second-order perturbations	FF	$-\frac{45}{256n^6}+\frac{9(-20+27X)}{512(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}-\frac{3}{16(\ell+1/2)^2n^4}+\frac{36+183X-356X^2}{2048X(\ell-1/2)(\ell+1/2)^3(\ell+3/2)n^3}$
	FG	$\bar{\xi}\left\{\frac{9(15-22X)}{512X(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}+\frac{9}{32X(\ell+1/2)^2n^4}+\frac{3(-12-47X+132X^2)}{1024X^2(\ell-1/2)(\ell+1/2)^3(\ell+3/2)n^3}\right\}$
	GG	$\bar{\xi}^2\left\{\frac{27}{512X(\ell-1/2)(\ell+1/2)(\ell+3/2)n^5}-\frac{27}{256X^2(\ell+1/2)^2n^4}+\frac{27+45X-540X^2}{2048X^3(\ell-1/2)(\ell+1/2)^3(\ell+3/2)n^3}\right\}$ $+\frac{3}{2048\ell(\ell-1/2)(\ell+1/2)(\ell+3/2)^3}\left\{\frac{\ell(4\ell+3)}{n^5}-\frac{10}{n^3}\right\}\delta_{s=1}\delta_{j=\ell+1}$ $+\frac{3}{2048(\ell+1)(\ell-1/2)^3(\ell+1/2)(\ell+3/2)}\left\{-\frac{(\ell+1)(4\ell+1)}{n^5}+\frac{10}{n^3}\right\}\delta_{s=1}\delta_{\ell>1}\delta_{j=\ell+1}$

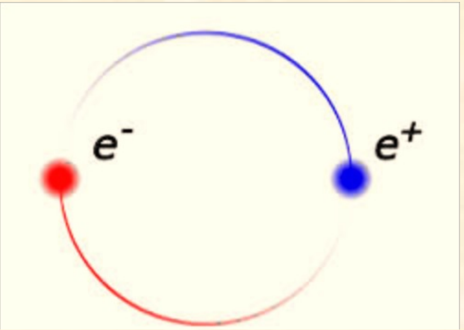


$$
\begin{aligned}
H^{(4)} &= H_F + H_G \\
H_F &= -\frac{1}{4m}H^2 + \frac{3}{4m}V(H-E_n) + \frac{3}{4m}(H-E_n)V + \frac{3E_n}{2m}V - \frac{5}{4m}V^2 + \frac{\vec{L}^{\,2}}{2m^2}\frac{V'}{r} \\
H_G &= \left\{\sigma^{(1)}_{ia}\sigma^{(2)}_{ib}\delta^2_{ab} + 6\vec{L}\cdot\vec{S}\right\}\frac{1}{4m^2}\frac{V'}{r} \\
\bar{\xi} &= \xi(j,\ell,s) + \langle \vec{L}\cdot\vec{S}\rangle
\end{aligned}
$$



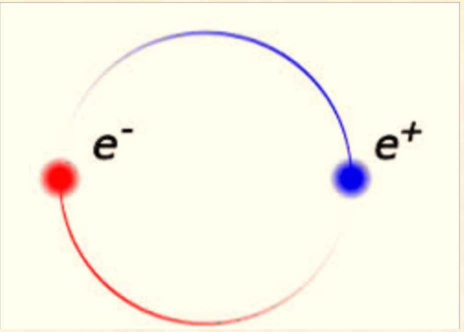
Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6\Bigg\{ & \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\xi(j,\ell,s) \\
& +\Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\langle\vec{L}\cdot\vec{S}\rangle \\
& +\Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\vec{S}^{\,2} \\
& +\Bigg[-\frac{69}{512n^6}+\frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3}{4(1+2\ell)^2n^4} \\
& +\frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3}\Bigg]\Bigg\},
\end{aligned}
$$



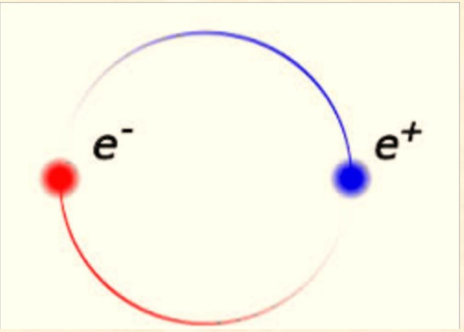
Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6 \Bigg\{ & \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\xi(j,\ell,s) \\
& + \Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\langle\vec{L}\cdot\vec{S}\rangle \\
& + \Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\vec{S}^{\,2} \\
& + \Bigg[-\frac{69}{512n^6} + \frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3}{4(1+2\ell)^2n^4} \\
& + \frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3}\Bigg]\Bigg\},
\end{aligned}
$$



Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6\Bigg\{ & \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\xi(j,\ell,s) \\
& +\Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\langle\vec{L}\cdot\vec{S}\rangle \\
& +\Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& +\frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\vec{S}^{\,2} \\
& +\Bigg[-\frac{69}{512n^6}+\frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3}{4(1+2\ell)^2n^4} \\
& +\frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3}\Bigg]\Bigg\},
\end{aligned}
$$



Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$
\begin{aligned}
\Delta E^{\ell>0} = m\alpha^6 \Bigg\{ & \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \xi(j,\ell,s) \\
& + \Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \langle \vec{L}\cdot\vec{S} \rangle \\
& + \Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \vec{S}^{\,2} \\
& + \Bigg[-\frac{69}{512n^6} + \frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3}{4(1+2\ell)^2n^4} \\
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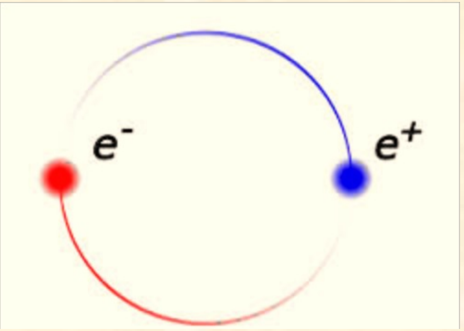
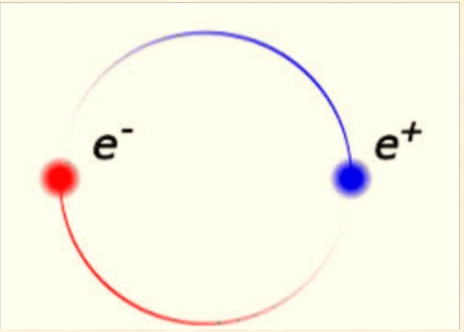
\end{aligned}
$$



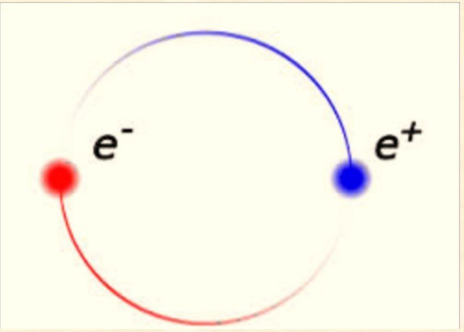
Results for energy contributions at O(mα⁶) for ℓ>0

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6\Bigg\{&\Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4}\\
&+\frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\xi(j,\ell,s)\\
&+\Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5}+\frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4}\\
&+\frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\langle\vec{L}\cdot\vec{S}\rangle\\
&+\Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4}\\
&+\frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3}\Bigg]\vec{S}^{\,2}\\
&+\Bigg[-\frac{69}{512n^6}+\frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5}-\frac{3}{4(1+2\ell)^2n^4}\\
&+\frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3}\Bigg]\Bigg\},
\end{aligned}
$$



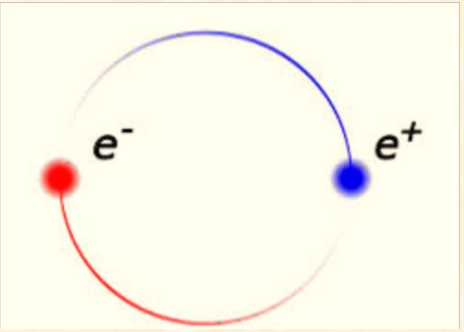
Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6 \Bigg\{ \Bigg[& -\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \xi(j,\ell,s) \\
+ \Bigg[& \frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \langle \vec{L}\cdot\vec{S} \rangle \\
+ \Bigg[& \frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \vec{S}^{\,2} \\
+ \Bigg[& -\frac{69}{512n^6} + \frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3}{4(1+2\ell)^2n^4} \\
& + \frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3} \Bigg] \Bigg\},
\end{aligned}
$$



Results for energy contributions at O(mα⁶) for ℓ>0

$$
\begin{aligned}
\Delta E^{\ell>0} \;=\; m\alpha^6 \Bigg\{ & \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \xi(j,\ell,s) \\
& + \Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \langle \vec{L}\cdot\vec{S} \rangle \\
& + \Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} \\
& + \frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \vec{S}^{\,2} \\
& + \Bigg[-\frac{69}{512n^6} + \frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3}{4(1+2\ell)^2n^4} \\
& + \frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3} \Bigg] \Bigg\},
\end{aligned}
$$



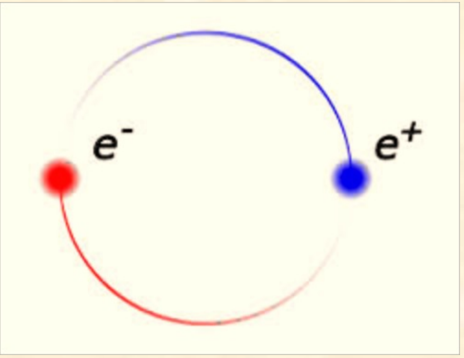
Results for energy contributions at $O(m\alpha^6)$ for $\ell>0$

$$\Delta E^{\ell>0} = m\alpha^6 \Bigg\{ \Bigg[-\frac{3\left(-33+68\ell+68\ell^2\right)}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-13\ell+15\ell^2+56\ell^3+28\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} + \frac{3\left(9+54\ell-39\ell^2-850\ell^3-1077\ell^4+2040\ell^5+5384\ell^6+4032\ell^7+1008\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \xi(j,\ell,s)$$

$$+ \Bigg[\frac{99-152\ell-152\ell^2}{64\ell(1+\ell)(-1+2\ell)(1+2\ell)(3+2\ell)n^5} + \frac{9\left(-3-7\ell+9\ell^2+32\ell^3+16\ell^4\right)}{32\ell^2(1+\ell)^2(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} + \frac{3\left(9+36\ell-45\ell^2-370\ell^3-369\ell^4+720\ell^5+1808\ell^6+1344\ell^7+336\ell^8\right)}{32\ell^3(1+\ell)^3(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \langle \vec{L}\cdot\vec{S} \rangle$$

$$+ \Bigg[\frac{1}{16(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3\left(-2+3\ell+3\ell^2\right)}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^2(3+2\ell)n^4} + \frac{-6-11\ell+99\ell^2+100\ell^3-250\ell^4-360\ell^5-120\ell^6}{16\ell^2(1+\ell)^2(-1+2\ell)^2(1+2\ell)^3(3+2\ell)^2n^3} \Bigg] \vec{S}^{\,2}$$

$$+ \Bigg[-\frac{69}{512n^6} + \frac{-17+20\ell+20\ell^2}{8(-1+2\ell)(1+2\ell)(3+2\ell)n^5} - \frac{3}{4(1+2\ell)^2n^4} + \frac{3+48\ell+64\ell^2+32\ell^3+16\ell^4}{16\ell(1+\ell)(-1+2\ell)(1+2\ell)^3(3+2\ell)n^3} \Bigg] \Bigg\},$$



Summary

Methods are in place that are adequate for the calculation of order $m\alpha^7$ corrections to the energy levels of positronium. These methods are being confirmed by calculation of lower order corrections. All $O(\alpha^4)$, $O(\alpha^5)$, and $O(\alpha^6)$ corrections have been found using the new approach. (The $O(\alpha^6)$ ones for $\ell>0$ are still being confirmed.) While many contributions at $O(\alpha^7)$ have already been obtained, much work remains to be done.

Thank you!

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Jens Zorn (1999)
"The short, rich life of positronium"