Theory of the g factor of few-electron ions

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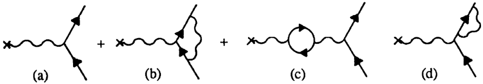
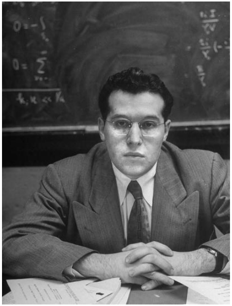
The g factor of the free electron

Interaction energy of an electron with an external **magnetic field**





At the one-loop level, it is only corrected by the vertex diagram

$$\Delta E = -\langle \vec{\mu} \rangle \cdot \vec{B} \, ,$$

with the magnetic moment μ, the Bohr magneton $\mu_B = \frac{e\hbar}{2mc}$

$$\langle \vec{\mu} \rangle = 2 \left(1 + \frac{\alpha}{2\pi} \right) \mu_B \langle \vec{S} \rangle = g \mu_B \langle \vec{S} \rangle \, .$$

Thus the **g-factor of the free electron up to the one-loop order** is

$$g_{\text{free}} = 2 + \frac{\alpha}{\pi} \approx 2(1 + 0.00116141)$$

The α/π term is the Schwinger term (Schwinger, 1947)

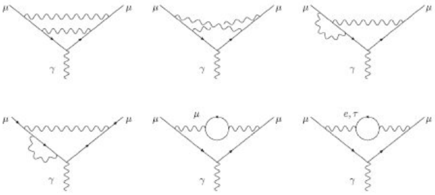
$$\mathbf{2} \quad + \quad \frac{\alpha}{\pi}$$

Dirac and Feynman at the Relativity conference, Jabłonna Palace, 1962

Dirac with his wife Margit Wigner and their children

Tagore promenade in Balatonfüred, with memorial trees planted by Dirac (1977), Feynman, Mandelbrot, Mössbauer, Pontecorvo, Prokhorov, Teller, etc.

Two-loop diagrams:



A. Peterman, Helv. Phys. Act **30**, 407 (1957);
C. M. Sommerfield, Ann. Phys. **5**, 26 (1958)

Three^{+}-loop diagrams:

S. Laporta, E. Remiddi, Phys. Lett. B **379**, 283 (1996) [3 loops, analytical]
T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. **109**, 111807 (2012) [numerical]
S. Laporta, Phys. Lett. B **772**, 232 (2017) [4 loops, semi-analytical, 1100 digits given]

Current best experimental value:

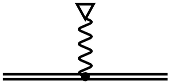
$$g_{\mathrm{exp}} = 2.002\,319\,304\,361(6)$$

(rel. accuracy=$3 \cdot 10^{-12}$)
Accurate value for the **fine-structure constant**: g_{exp} and corresponding multi-loop free-electron QED calculations

D. Hanneke, S. Fogwell, and G. Gabrielse, Phys. Rev. Lett. **100**, 120801 (2008)

The bound-electron g factor

For a Coulomb potential, the Dirac g-factor for the 1s state (G. Breit, 1928):



$$g_{\mathrm{D}} = \frac{2}{3}\left(1 + 2\sqrt{1-(Z\alpha)^2}\right) = 2 - \frac{2}{3}(Z\alpha)^2 - \frac{1}{6}(Z\alpha)^4 + \ldots$$

Double line: Coulomb-Dirac (wave function or) propagator:

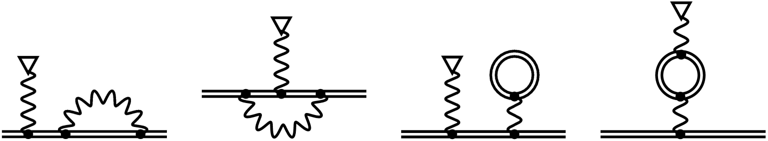
with an arbitrary number of interactions with the (strong) Coulomb potential $V(r)$ of the nucleus (Furry picture). Solution of the Dirac eq.:

$$\left(E + i\boldsymbol{\alpha}\nabla - m\gamma^0 - V(r)\right) G(\mathbf{x}, \mathbf{y}, E) = \delta(\mathbf{x} - \mathbf{y})$$

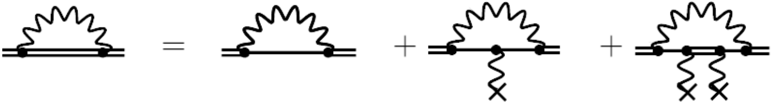
A number of corrections contribute to g_{th}:

$$g_{\text{th}} = g_{\text{D}} + \delta g_{1\text{L}} + \delta g_{2\text{L}} + \delta g_{\text{FS}} + \delta g_{\text{rec}} + \delta g_{\text{NP}} \,,$$

$\delta g_{1\text{L}}$ – one-loop QED: self-energy (SE) and vacuum polarization (VP),
$\delta g_{2\text{L}}$ – two-loop QED: SE-SE, VP-VP, SE-VP,
δg_{FS} – nuclear finite-size,
δg_{rec} – recoil,
δg_{NP} – nuclear polarizability,
...



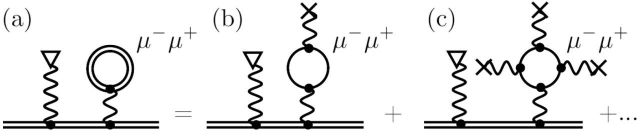
Bound-state QED in the Furry picture:
relate the diagrams to (known) free QED



self-energy function
$\Sigma(p)$
momentum space

vertex function
$\Gamma_\mu(p, p')$
momentum space

Coulomb-Dirac propagator
$G^{2+}(\mathbf{x}, \mathbf{y}, E)$
coordinate space



free-loop approximation
polarization function $\Pi(q^2)$
$\rightarrow$ Uehling potential $V_\mathrm{U}(r)$

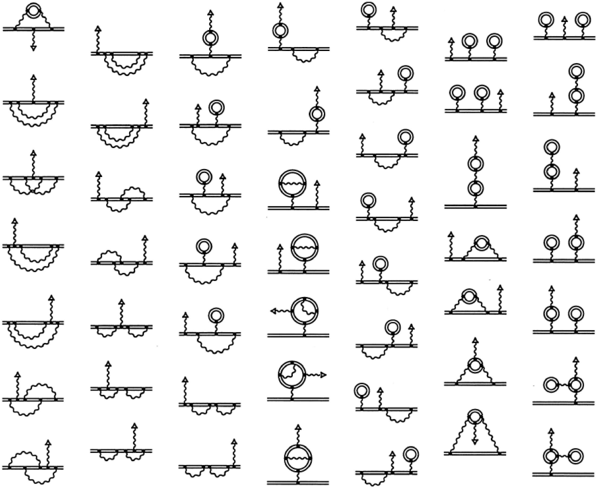
virtual
light-by-light
scattering

Results for $^{28}_{14}\mathrm{Si}^{13+}$ (from 2011)

Theory			
Dirac value		1.993 023 571 6	Breit 1928
Finite nuclear size		0.000 000 020 5	Dirac eq. num. and Karshenboim 2000
One-loop QED	$(Z\alpha)^0$	0.002 322 819 5	$\frac{\alpha}{\pi}$, Schwinger 1948
	$(Z\alpha)^2$	0.000 004 040 7	$\frac{\alpha}{\pi}\frac{(Z\alpha)^2}{6}$, Grotch 1970
	$(Z\alpha)^4$	0.000 001 244 6	Pachucki *et al.* 2004
	h.o. SE	0.000 000 542 8(3)	Yerokhin, Indelicato, Shabaev 2004, & Jentschura
	VP WK	0.000 000 032 6	Beier 2000
	VP magn.	0.000 000 002 5	Lee, Milstein, Terekhov, Karshenboim 2005
Two-loop QED	$(Z\alpha)^0$	-0.000 003 515 1	$\propto (\frac{\alpha}{\pi})^{2+}$, Sommerfield 1958, Kinoshita et al.
	$(Z\alpha)^2$	-0.000 000 006 1	Grotch 1970
	$(Z\alpha)^4$	-0.000 000 001 3	Pachucki, Czarnecki, Jentschura, Yerokhin 2005
	h.o.	0.000 000 000 0(17)	*ditto*
Recoil	m/M	0.000 000 206 1(1)	Shabaev, Yerokhin 2002
	rad-rec	-0.000 000 000 2	Grotch 1970
	$(m/M)^{2+}$	-0.000 000 000 1	Pachucki 2008
Total theory		1.995 348 958 0(17)	
Experiment (2011)		1.995 348 958 7(5)(3)(8)	(stat)(syst)(m_e)

- S. Sturm, A. Wagner, B. Schabinger *et al.*, Phys. Rev. Lett. **107**, 023002 (2011)
- Another application of H-like ions: determination of m_e, S. Sturm, F. Köhler, J. Zatorski *et al.*, Nature **506**, 467 (2014)
- Two-loop corrections of order $(Z\alpha)^5$ calculated: A. Czarnecki, M. Dowling, J. Piclum, R. Szafron, Phys. Rev. Lett. **120**, 043203 (2018), value: 1.995 348 958 109(584)

Two-loop corrections in a nonperturbative Coulomb field

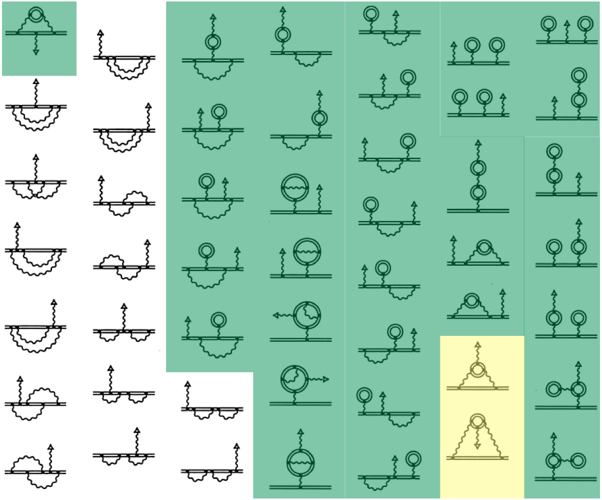


50 total diagrams

(29 inequivalent diagrams)

Slide courtesy of V. Debierre

Two-loop corrections in a nonperturbative Coulomb field



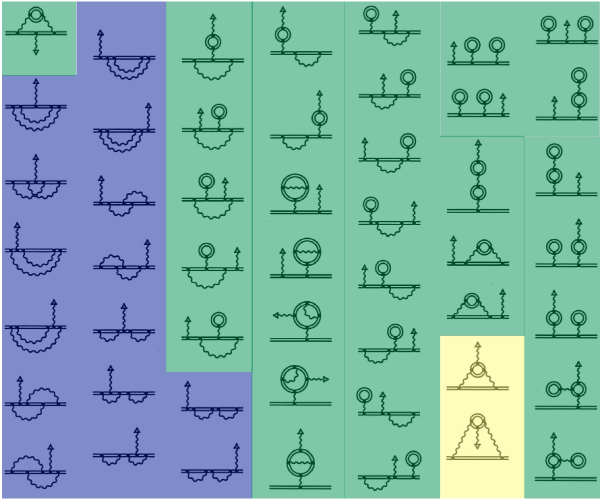
50 total diagrams

(29 inequivalent diagrams)

Diagrams with 0&1 self-energy loops → Treated in [V.A. Yerokhin, ZH, Phys. Rev. A **88**, 042502 (2013)]
(with free VP (e^-e^+) loops)
[A. Czarnecki, R. Szafron, Phys. Rev. A 94, 060501(R) (2016)]

Slide courtesy of V. Debierre

Two-loop corrections in a nonperturbative Coulomb field



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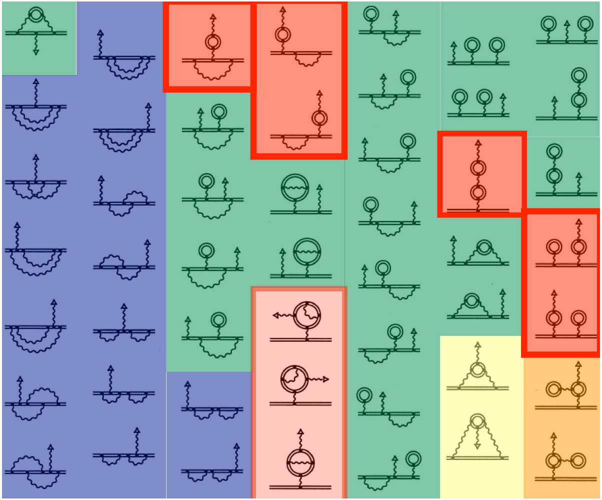
(with free VP (e^-e^+) loops)
[A. Czarnecki, R. Szafron, Phys. Rev. A 94, 060501(R) (2016)]

Diagrams with 2 self-energy loops → Calculation in progress
[B. Sikora, V. A. Yerokhin, N. S. Oreshkina *et al.*, arXiv:1804:05733]
See the POSTER of Bastian Sikora!

Slide courtesy of V. Debierre

Two-loop corrections in a nonperturbative Coulomb field



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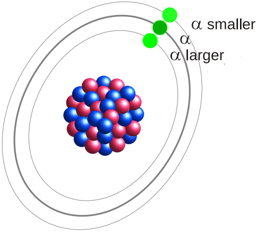
(with free VP (e^-e^+) loops)
[A. Czarnecki, R. Szafron, Phys. Rev. A 94, 060501(R) (2016)]

Also in progress: diagrams that vanished in the free VP loop approach & calculate lowest nonvanishing contribution
See the POSTER of Vincent Debierre!

Slide courtesy of V. Debierre

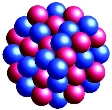
Towards the determination of the fine-structure constant from the bound-electron g-factor

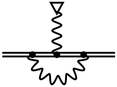
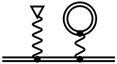
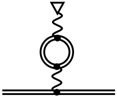


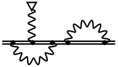
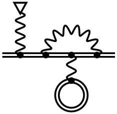
- In atoms/ions: Binding energies, wave functions and thus all properties depend on α – or, actually, $Z\alpha$
- Accurately determine the value α from atomic properties
  e.g. from the bound-electron g-factor – can be measured to very high accuracy
- Leading (Dirac) g-factor:
  $g_{\mathrm{D}} = {}^2\!/_3 \left(1 + 2\sqrt{1 - (Z\alpha)^2}\right)$

Determining α from the g-factor

Principle of determining α:

$g_{\text{exp}} \stackrel{!}{=} g_{\text{theo}} = {}^{2}\!/_{3} \left(1 + 2\sqrt{1-(Z\alpha)^2}\right)$ +  +

 +  +  +  +

 +  + ... $\Rightarrow$ solve for α

- **Different physics** to determine α than in the case of the *free-electron* g-factor: dominant dependence not from a radiative correction (α/π), but from the binding ($Z\alpha$)
- **Enhanced sensitivity** as compared to the *free-electron* g-factor

- **Problem:** nuclear parameters (e.g. $\langle r^2 \rangle$) are not known accurately
- **Solution:** weighted difference of H- and Li-like ions (same Z):

$$\delta_\Xi g = g(2s) - \Xi g(1s)\,,$$

with the weight Ξ theoretically chosen to suppress nuclear size effects

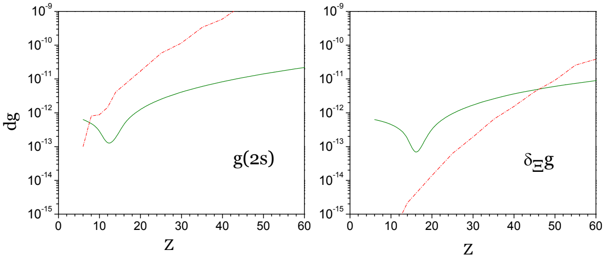
- Simplest approximation: $\Xi = \frac{1}{8} = 0.125$ – because: $|\psi_{ns}(r=0)|^2 \propto \frac{1}{n^3}$
- Accurate formula (incl. relativity, QED and $e^- - e^-$ interaction:

$$\Xi = 2^{-2\gamma-1}\left[1 + \frac{3}{16}(Z\alpha)^2\right]\left(1 - \frac{2851}{1000}\frac{1}{Z} + \frac{107}{100}\frac{1}{Z^2}\right)\,,$$

where $\gamma = \sqrt{1-(Z\alpha)^2}$

error due to $\delta\langle r^2\rangle$ + distr. $\rightarrow$

error due to present $\delta\alpha$ $\rightarrow$



$\Rightarrow$ α can be significantly improved

Earlier idea: weighted difference of **heavy** H- and B-like ions

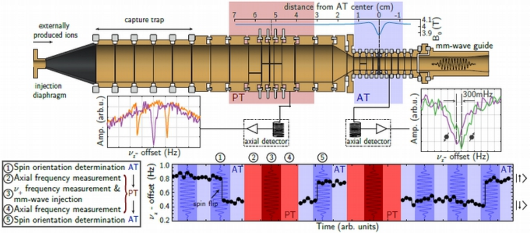
V. M. Shabaev, D. A. Glazov, N. S. Oreshkina *et al.*, Phys. Rev. Lett. **96**, 253002 (2006)

- V. A. Yerokhin, E. Berseneva, Z. H., I. I. Tupitsyn, C. H. Keitel, Phys. Rev. Lett. **116**, 100801 (2016); Phys. Rev. A **94**, 022502 (2016)
- V. A. Yerokhin, C. H. Keitel, Z. H., J. Phys. B **46**, 245002 (2013)

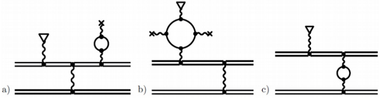
Recent progress: the g factor of boronlike $^{40}_{18}\mathrm{Ar}^{13+}$

Boronlike ground state: $1s^2 2s^2 2p\ ^2P_{1/2}$

The recently constructed ALPHATRAP Penning-trap setup:



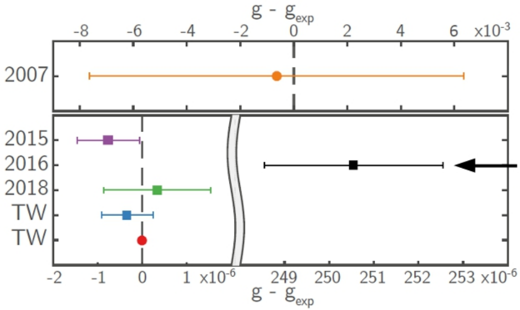
Challenge for theory: exchange photons, i.e. electron correlation and QED screening



⇒ See the talk of Alex Egl (Heidelberg) on the experiment in this session

Contribution	Value	Ref.
Dirac value	0.663 777 45	Zapryagaev, 1979; Karshenboim *et al.*, 2005
Electron correlation:		
one-photon exchange, $(1/Z)^1$		following V. M. Shabaev *et al.*, 2002
Coulomb	0.000 737 61	
Breit	-0.000 078 42	
frequency dependence	-0.000 001 67	
higher orders, $(1/Z)^{2+}$		
CI w/neg. continuum	-0.000 007 5(4)	A. A. Shchepetnov *et al.*, 2015
Mass shift	-0.000 009 09(19)	D. A. Glazov *et al.*, 2018
One-loop QED:		
self-energy, $(1/Z)^0$	-0.000 768 372 3(3)	
$(1/Z)^{1+}$	-0.000 000 98(15)	
vacuum polarization		
electric loop, $(1/Z)^0$	-4.187×10^{-10}	
$(1/Z)^1$	$6.531(2) \times 10^{-9}$	
magnetic loop, $(1/Z)^0$	4.131×10^{-10}	
$(1/Z)^1$	-1.341×10^{-10}	
Two-loop QED, $(Z\alpha)^0$	0.000 001 2(1)	H. Grotch and R. Kashuba, 1973
Total theory	0.663 648 2(5)	
	0.663 648 08(58)*	St. Petersburg, SPBU
Experiment	0.663 648 455 32(93)	

⇒ See also the talk of Daniel Maison (St. Petersburg) on many-body calculations later in this session



R. Soria Orts, J. R. Crespo López-Urrutia, H. Bruhns, A. J. González Martinez, Z.H., U. D. Jentschura, C. H. Keitel, A. Lapierre, H. Tawara, I. I. Tupitsyn, J. Ullrich, and A. V. Volotka, Phys. Rev. A 76, 052501 (2007), **EBIT experiment**

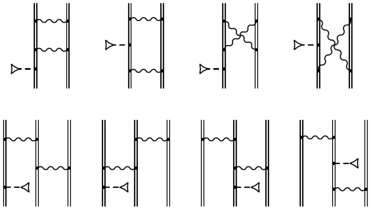
A. A. Shchepetnov, D. A. Glazov, A. V. Volotka, V. M. Shabaev, I. I. Tupitsyn, G. Plunien, J. Phys. Conf. Ser. **583**, 012001 (2015)

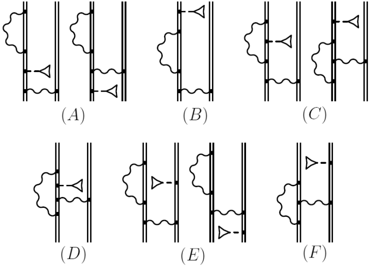
J. P. Marques, P. Indelicato, F. Parente, J. M. Sampaio, J. P. Santos, Phys. Rev. A **94**, 042504 (2016)

V. A. Agababaev, D. A. Glazov, A. V. Volotka, D. V. Zinenko, V. M. Shabaev, G. Plunien, J. Phys. Conf. Ser. **1138**, 012003 (2018)

- “This work” (TW), experiment+theory: I. Arapoglou, A. Egl, M. Höcker, T. Sailer, B. Tu, A. Weigel, R. Wolf, H. Cakir, V. A. Yerokhin, N. S. Oreshkina, V. A. Agababaev, A. V. Volotka, D. V. Zinenko, D. A. Glazov, Z.H., C. H. Keitel, S. Sturm, K. Blaum, arXiv:1906:0088 (2019)

- **Further improvement of the theory possible:**
  two-photon exchange and SE screening diagrams:





Done in Li-like ions:

- A. Wagner, S. Sturm, F. Köhler, D. A. Glazov, A. V. Volotka, G. Plunien, W. Quint, G. Werth, V. M. Shabaev, K. Blaum Phys. Rev. Lett. **110**, 033003 (2013)
- A. V. Volotka, D. A. Glazov, V. M. Shabaev *et al.*,, Phys. Rev. Lett. **103**, 033005 (2009)

Summary

- Accurate **test of QED** in strong fields
- Determining the **electron mass** with an order-of-magnitude improvement via the g-factor of C^{5+}
- New independent scheme for the improved determination of the **fine-structure constant** α in (near?) future from the g-factors of **light** or intermediate-Z **H- and Li-like** *or* **heavy H-and B-like** ions
- Evaluation of **two-loop corrections** to all orders in $Z\alpha$ is in progress
- Improvement of the accuracy of **electron correlation** and **QED screening** is needed – probably achievable with the combination of NRQED ($Z\alpha$ expansion, all-order in $1/Z$) and Furry-picture ($1/Z$ expansion, all-order in $Z\alpha$) methods?

Bedankt voor uw andacht!

Dziękuję za uwagę!

Grazie per l'attenzione!

Köszönöm a figyelmet!

Merci à tous pour votre attention!

Muchas gracias por su atención!

Mulţumesc pentru atenţie!

Obrigado pela atenção!

Спасибо за внимание!

Thank you for your attention!

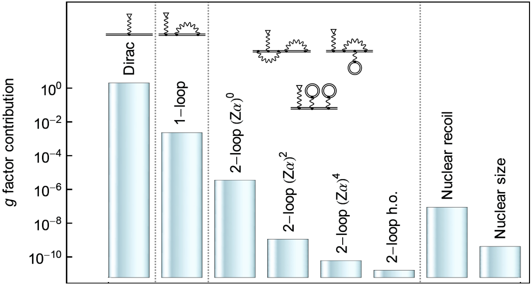
Vielen Dank für Ihre Aufmerksamkeit!

High-precision determination of the electron mass

The mass of the electron can be expressed by the mass and charge of the $^{12}C^{5+}$ ion, the experimentally measured cyclotron and Larmor frequencies, and the theoretical g-factor as

$$m_e = \frac{g}{2}\frac{e}{Q}\frac{\nu_c}{\nu_L}m_{\text{ion}}$$

- $e/Q = 1/6$;
- m_{ion} is known very well ($m_{^{12}\text{C atom}} \equiv 12$ u);
- ν_c/ν_L is measured very precisely;
- the g-factor is taken from theory



The resulting value $m_e = 0.000\,548\,579\,909\,069\,4(128)_{\text{stat}}(86)_{\text{sys}}(13)_{\text{theo}}$ u surpasses the earlier CODATA value by more than an order of magnitude and largely defines the new CODATA value

- S. Sturm, F. Köhler, J. Zatorski, A. Wagner, Z. H., G. Werth, W. Quint, C. H. Keitel, K. Blaum, Nature **506**, 467 (2014)
- F. Köhler, S. Sturm, A. Kracke, G. Werth, W. Quint, and K. Blaum, J. Phys. B **48**, 144032 (2015)
- J. Zatorski, B. Sikora, S. G. Karshenboim, S. Sturm, F. Köhler-Langes, K. Blaum, C. H. Keitel, Z. H., Phys. Rev. A **96**, 012502 (2017)