Circular Hadron Collider Beam Dynamics I

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Outline

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  - Review
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- Part II
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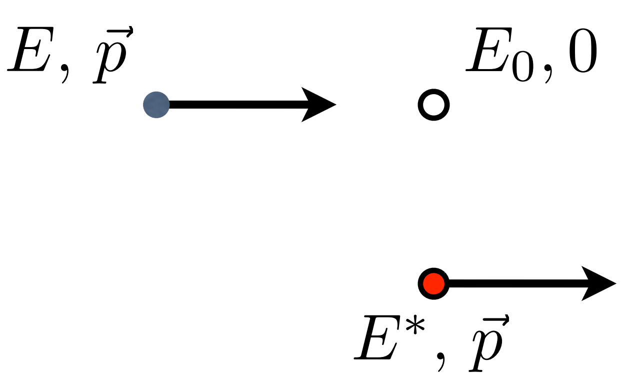
  - Space Charge and the Beam-Beam Interaction
  - Emittance Control and Luminosity
  - Effects due to Synchrotron Radiation
  - Optimization of Luminosity

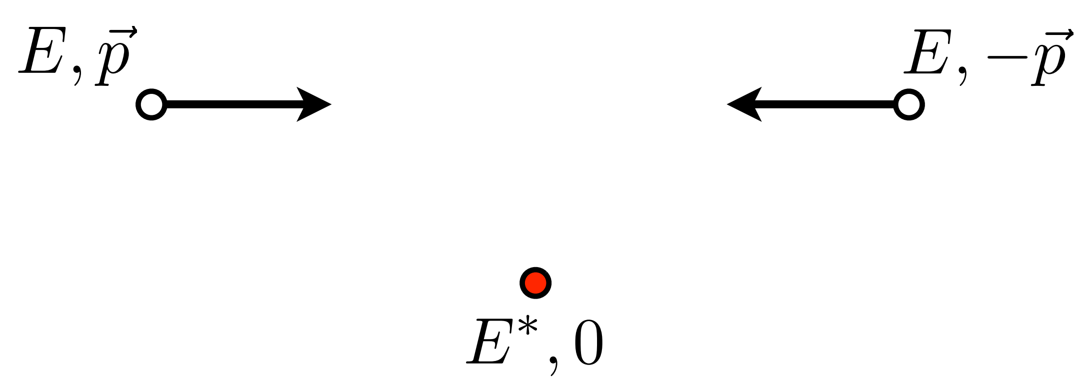
Fixed Target Energy vs. Collider Energy

- Beam/target particles: $E_0 \equiv m_p c^2$

<u>Fixed Target</u>



<u>Collider</u>



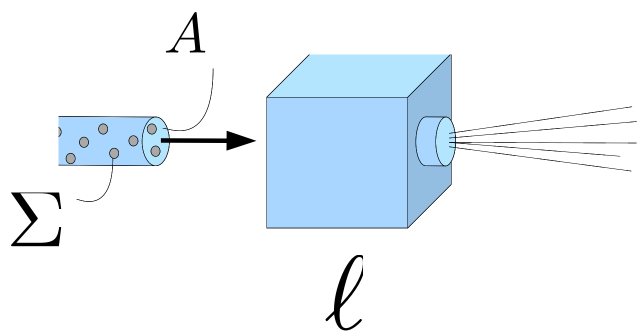
$$\begin{aligned} E^{*2} = (m^* c^2)^2 + (pc)^2 &= [\ E_0 + E\ ]^2 \\ &= E_0^2 + 2E_0 E + (E_0^2 + (pc)^2) \\ m^* c^2 &= \sqrt{2}\ E_0\ [1 + \gamma_{FT}]^{1/2} \end{aligned}$$

$$\begin{aligned} m^* c^2 &= 2E \\ &= 2E_0 \gamma_{coll} \end{aligned}$$

100,000 TeV FT synch. == 14 TeV LHC

Luminosity

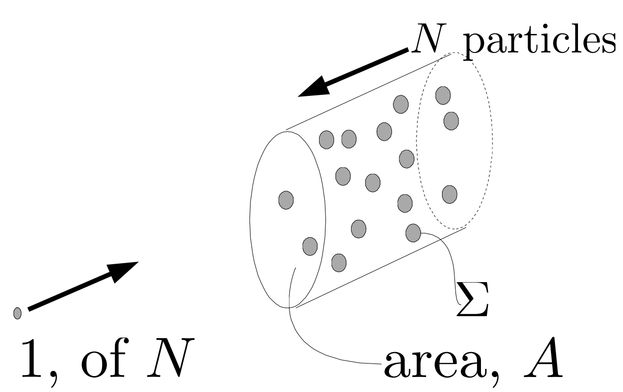
- *Fixed Target Experiment:*



$$\mathcal{R} = \left(\frac{\Sigma}{A}\right) \cdot \rho \cdot A \cdot \ell \cdot N_A \cdot \dot{N}_{beam}$$
$$= \rho N_A \ell \dot{N}_{beam} \cdot \Sigma$$
$$\equiv \mathcal{L} \cdot \Sigma$$

ex. : $\mathcal{L} = \rho N_A \ell \dot{N}_{beam} = 10^{24}/\mathrm{cm}^3 \cdot 100\ \mathrm{cm} \cdot 10^{13}/\mathrm{sec} = 10^{39}\mathrm{cm}^{-2}\mathrm{sec}^{-1}$

- *Bunched-Beam Collider:*



$$\mathcal{R} = \left(\frac{\Sigma}{A}\right) \cdot N \cdot (f \cdot N)$$
$$= \frac{f\ N^2}{A} \cdot \Sigma$$
$$\mathcal{L} \equiv \frac{f\ N^2}{A}$$

ex. : $\mathcal{L} = 10^9/25/\mathrm{sec} \cdot (10^{11})^2/(\pi(.002)^2)/\mathrm{cm}^2 = 10^{34}\mathrm{cm}^{-2}\mathrm{sec}^{-1}$

Integrated Luminosity

- Bunched beam is natural in collider that "accelerates" (more later)

$$\mathcal{L} = \frac{f_0 B N^2}{A}$$

f_0 = rev. frequency
B = no. bunches

- In ideal case, particles would be "lost" only due to "collisions"

$$B\dot{N} = -\mathcal{L} \, \Sigma \, n$$

(n = no. of detectors receiving luminosity $\mathcal{L}$)

- So, in this ideal case,

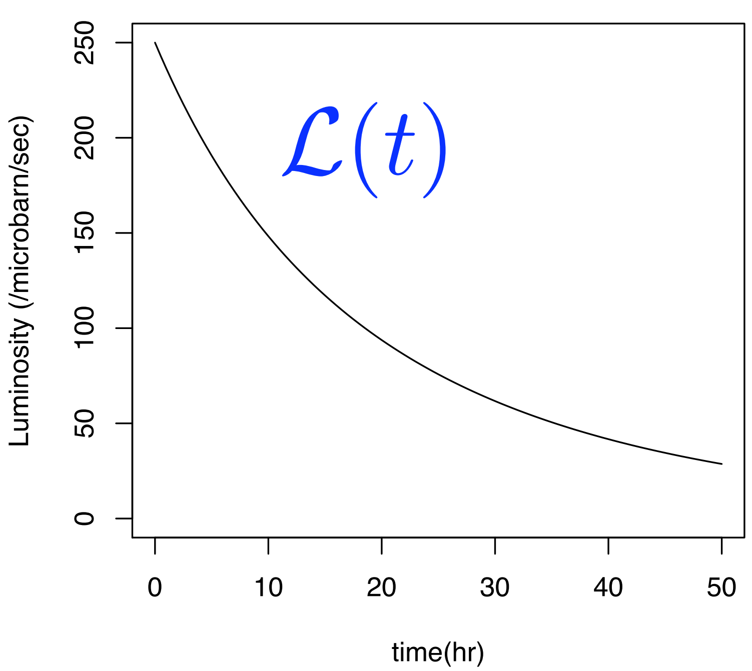
$$\mathcal{L}(t) = \frac{\mathcal{L}_0}{\left[1 + \left(\frac{n\mathcal{L}_0\Sigma}{BN_0}\right) t\right]^2}$$

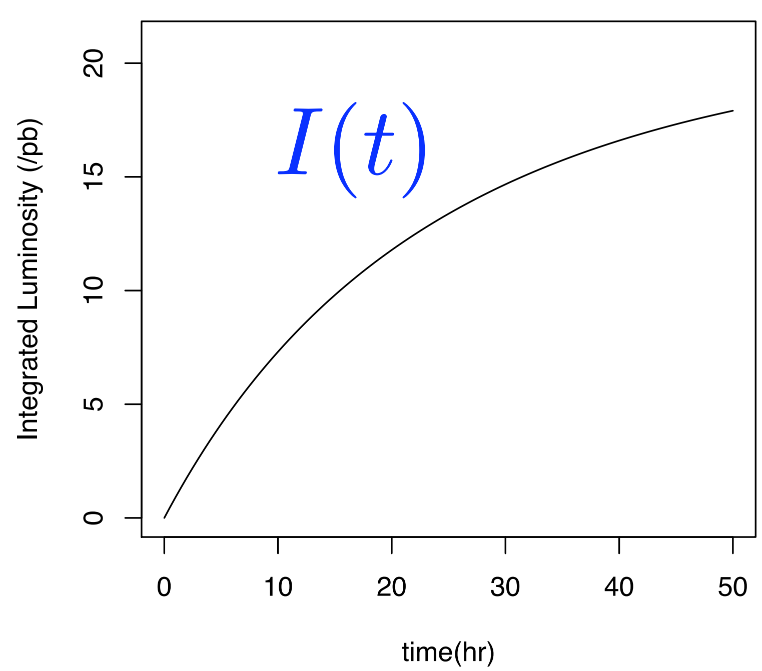
Ultimate Number of Collisions

- Since $\mathcal{R} = \mathcal{L} \cdot \Sigma$ then, #events = $\int \mathcal{L}(t)dt \cdot \Sigma$

- So, our integrated luminosity is

$$I(T) \equiv \int_0^T \mathcal{L}(t)dt = \frac{\mathcal{L}_0 T}{1 + \mathcal{L}_0 T(n\Sigma/BN_0)} = I_0 \cdot \frac{\mathcal{L}_0 T/I_0}{1 + \mathcal{L}_0 T/I_0}$$





asymptotic limit:

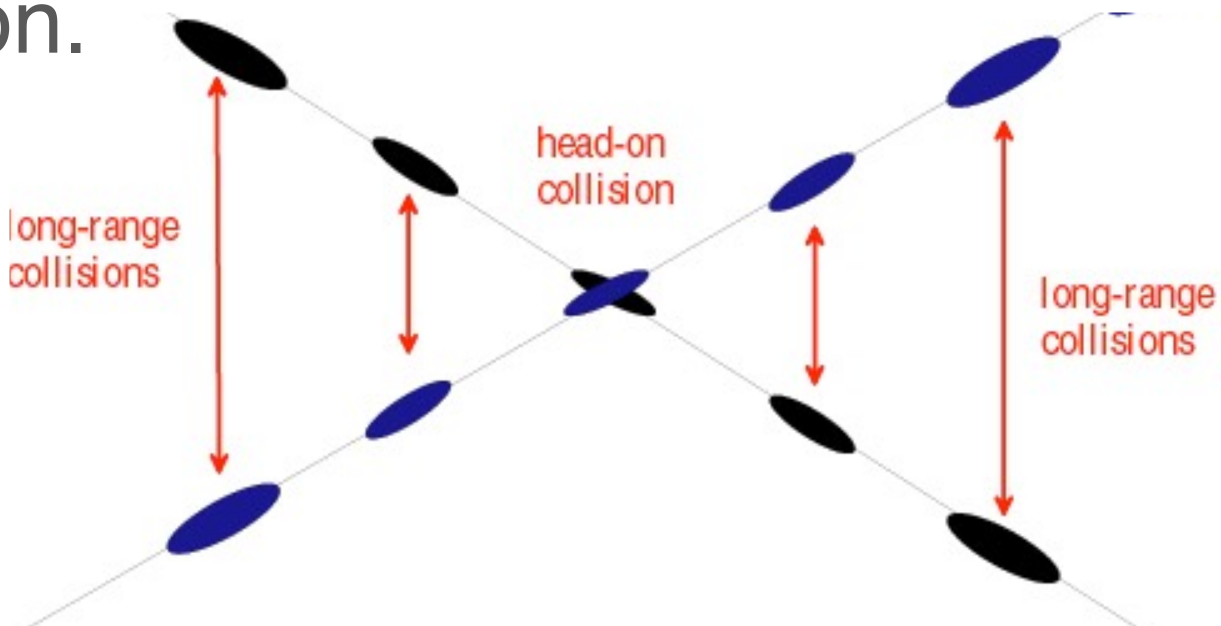
$$I_0 \equiv \frac{BN_0}{n\Sigma}$$

so, ...

$$\mathcal{L} = \frac{f_0 B N^2}{A}$$

Crossing Angle and Effects on Luminosity

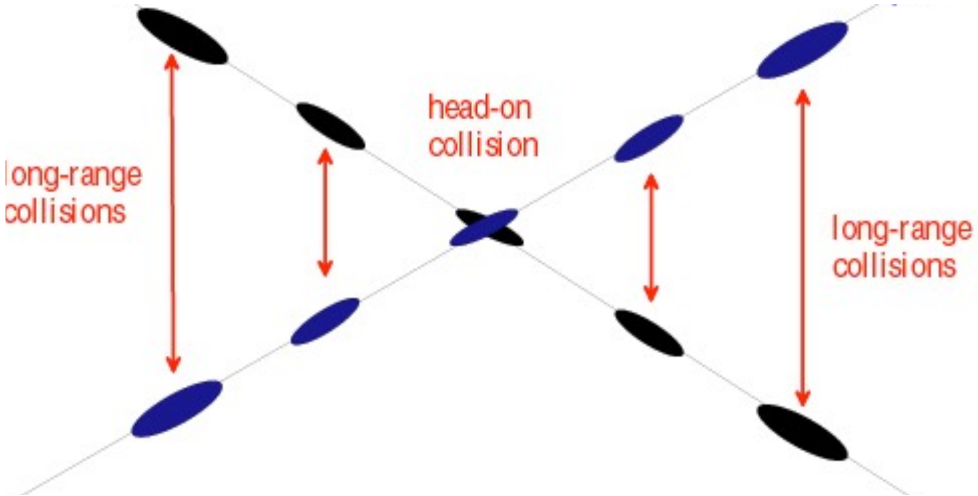
- Large collider detectors require large longitudinal space; collisions every (*bunch spacing*)/2
  - *ex*: 25 ns spacing —> (3e8*25e-9)/2 = 3.75 m
  - if bunches hit "head-on", would have *many* collision points within the detector region.



- Introduce a crossing angle
  - reduction of overlap, hence reduces luminosity
  - luminous region reduced as well

Crossing Angle and Effects on Luminosity

- luminous region: $z_{rms} = \sigma_z/\sqrt{2}$ $\qquad \frac{d\mathcal{L}}{dz} = \mathcal{L}_0 \cdot \frac{1}{\sqrt{2\pi}(\sigma_z/\sqrt{2})} e^{-z^2/(\sigma_z/\sqrt{2})^2}$
- events per bx $\sim \Sigma \cdot t_b \cdot \mathcal{L}$
  - (bunch crossing)
- max events per bx per cm $\sim \Sigma \cdot t_b \cdot \frac{d\mathcal{L}}{dz}\Big|_{max} = \frac{\mathcal{L}_0 t_b \Sigma}{\sqrt{\pi}\sigma_z}$
- luminosity reduction from crossing angle



$$\mathcal{L} = \mathcal{L}_0 \cdot \frac{1}{\sqrt{1 + (\alpha\sigma_z/2\sigma_0)^2}}$$

(length of luminous region reduces by same factor)

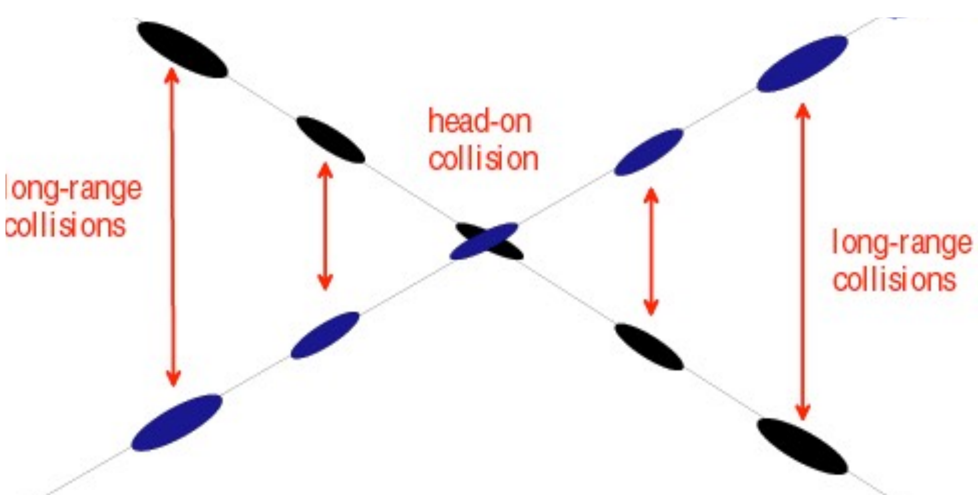
Note: if beams are offset:

$$\mathcal{L} = \mathcal{L}_0 \cdot e^{-(\Delta/2\sigma_0)^2}$$

Crossing Angle and Effects on Luminosity

Numerical examples —

- luminous region: $z_{rms} = \sigma_z/\sqrt{2}$ = 8 cm/√2 = 5.6 cm
- events per bx $\sim \Sigma \cdot t_b \cdot \mathcal{L}$ = (6e-26)(25e-9)(1e34)= 15
- max events per bx per cm: 15/(8 cm)/√π ~ 1/cm
- luminosity reduction from crossing angle ~ -15% (LHC)



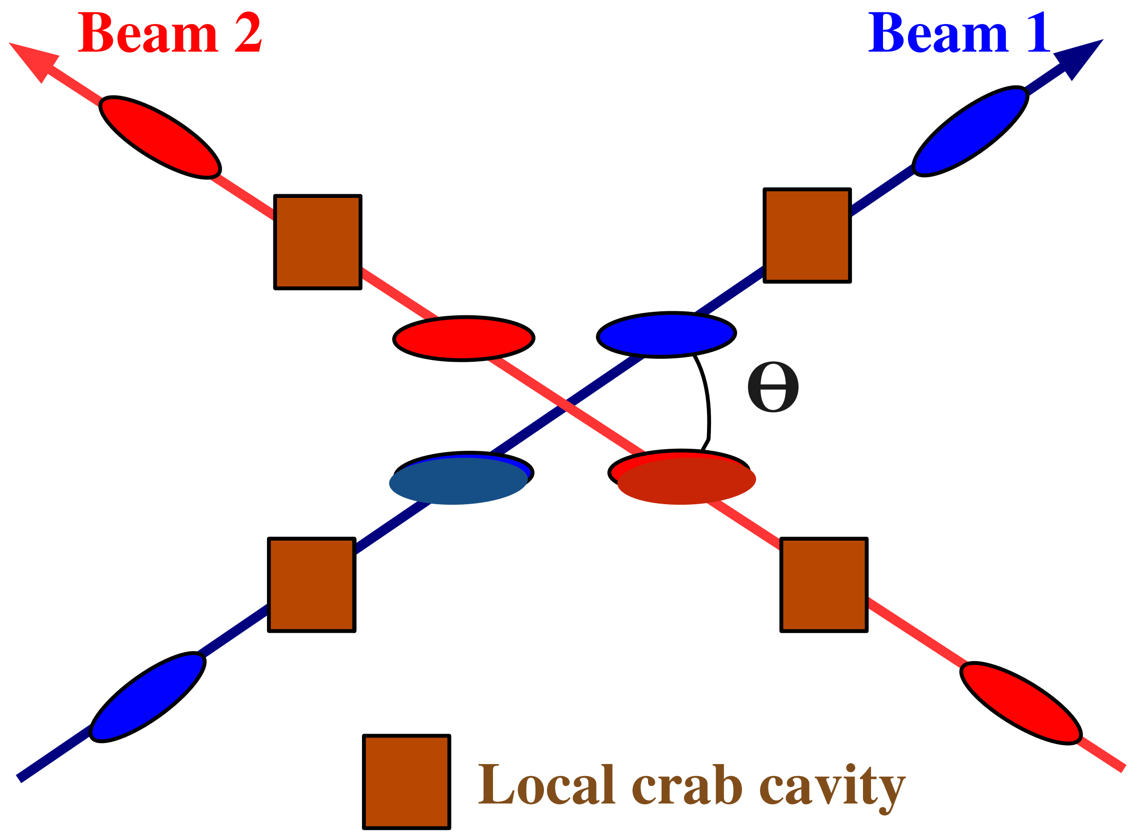
$$\mathcal{L} = \mathcal{L}_0 \cdot \frac{1}{\sqrt{1 + (\alpha\sigma_z/2\sigma_0)^2}}$$

(length of luminous region reduces by same factor)

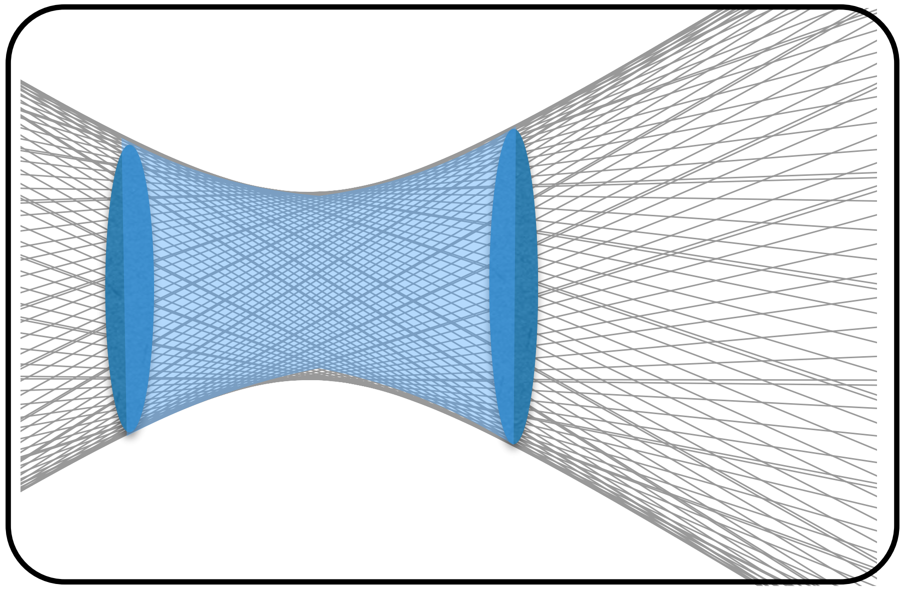
- beams offset by 1 sig: -22% $\mathcal{L} = \mathcal{L}_0 \cdot e^{-(\Delta/2\sigma_0)^2}$

The Use of Crab Cavities

- Future upgrades to LHC will employ “crab cavities”
  - using deflecting-mode RF cavities, give tail/head deflections up/down in transverse direction to create total overlap, thus mitigating the loss of luminosity due to the crossing angle



Hour Glass Factor



$$L = L_0 \cdot \mathcal{H} = L_0 \cdot \frac{1}{\sqrt{\pi}\sigma_z} \int_{-\infty}^{\infty} \frac{e^{-z^2/\sigma_z^2}}{1 + \left(\frac{z}{\beta^*}\right)^2} dz$$

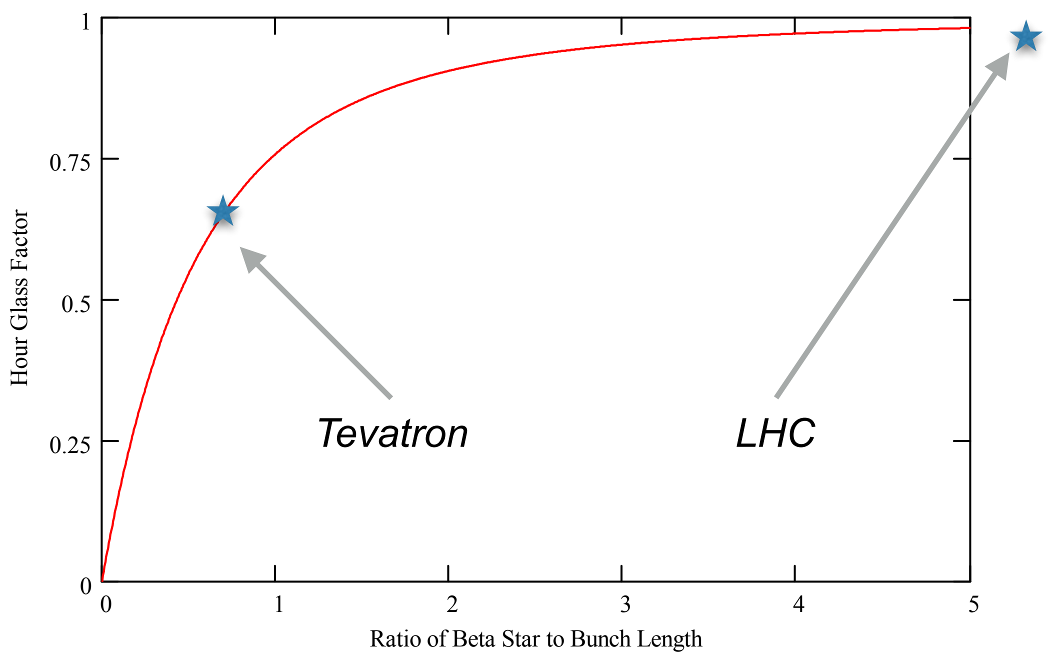
The integral on the right, $\mathcal{H}$, is called the "hour glass factor," and can be evaluated in terms of the Error Function,

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

which can be looked up in standard tables, and is often provided numerically in standard software packages. The luminosity result is,

$$L = L_0 \cdot \mathcal{H}(\beta^*/\sigma_z) = L_0 \cdot \sqrt{\pi} \left(\frac{\beta^*}{\sigma_z}\right) e^{(\beta^*/\sigma_z)^2} \left[1 - \mathrm{erf}(\beta^*/\sigma_z)\right].$$

The hour glass factor as a function of β^*/σ_z is plotted in Figure 2. Note that the luminous region



β^*/σ_z :

LHC: ~ 50 cm / 8 cm ~ 6

Tevatron: ~ 35 cm / 50 cm ~ 0.7

Review of Longitudinal Dynamics

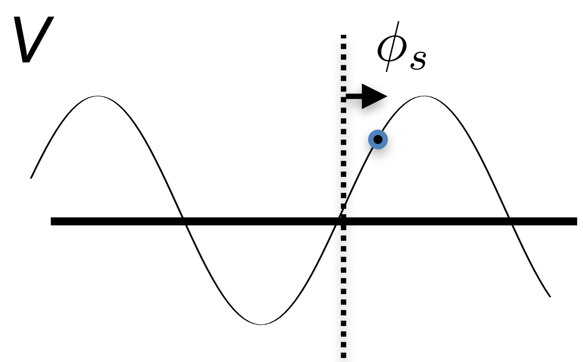
- So, how do we create and manipulate particle bunches to give us the desired time structure? desired bunch length, spacing?
- Particles propagate through a system of accelerating cavities; each cavity has oscillating fields with frequency f_{RF}, and maximum "applied" voltage V. The ideal particle would arrive at the cavity at phase ϕ_s (relative to the "positive zero-crossing" of the RF wave)
- ideal particle acquires an energy gain of

$$\Delta E_s = eV \sin \phi_s$$

Acceleration of Ideal Particle

Wish to accelerate the ideal particle. As this particle exits the RF cavities/stations for the *(n+1)-th* time we would have

$$E_s^{(n+1)} = E_s^{(n)} + eV \sin\phi_s$$



The ideal energy gain per second would be:

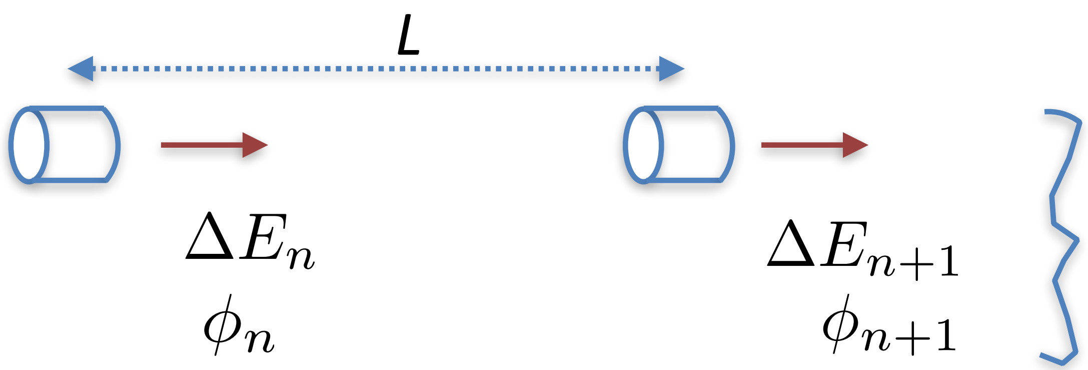
$$dE_s/dt = f_0 eV \sin\phi_s$$

$f_0 = \text{revolution frequency}$

Next, look at (longitudinal) motion of particles near the ideal particle: ϕ = phase w.r.t. RF system

$\Delta E \equiv E - E_s$ = energy difference from the ideal

- Relative to the ideal particle,



$$\phi_{n+1} = \phi_n + \frac{2\pi h \eta}{\beta^2 E} \Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_{n+1} - \sin\phi_s)$$

(*difference equations*)

Notes:

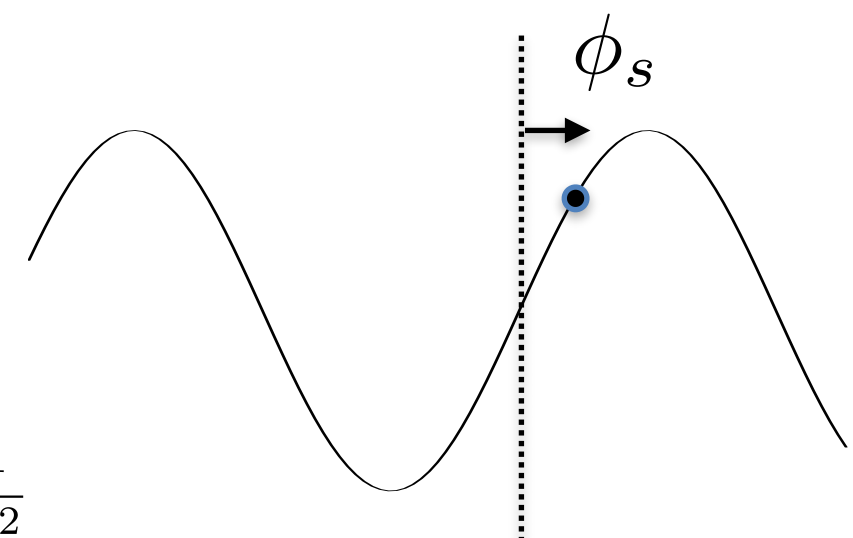
$$h = L/\beta\lambda, \quad \lambda = c/f_{\rm rf} \qquad \text{or,} \qquad h = f_{\rm rf} L/v$$

Desire *h* to be an integer.

If *L* is circumference of a synchrotron then: $h = f_{\rm rf}/f_0$

where f_0 is the revolution frequency,

In this case, *h* is called the "harmonic number"

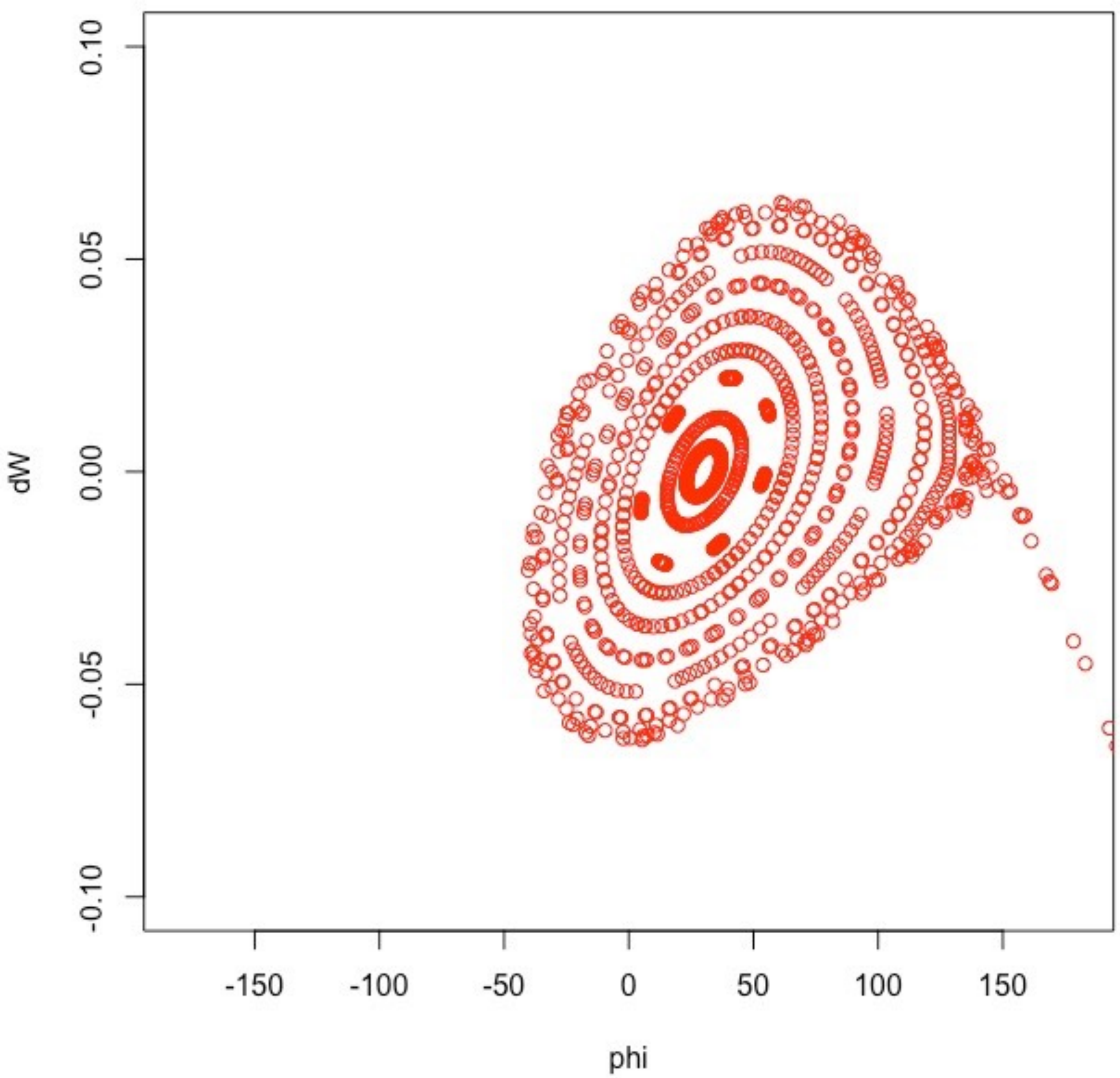


$$E = mc^2 + W; \qquad \Delta E \Leftrightarrow \Delta W$$

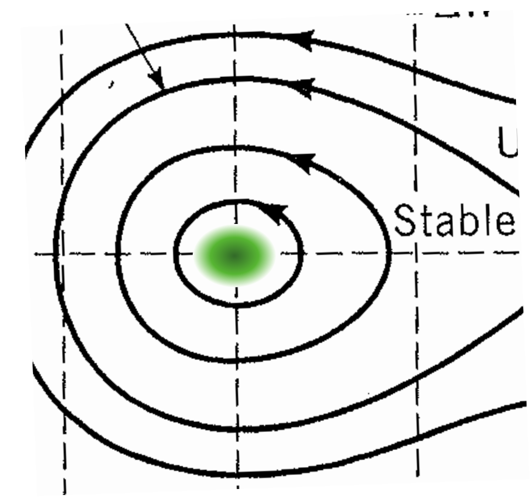
$$\eta \equiv \frac{\Delta C/C}{\Delta p/p} - \frac{1}{\gamma^2}$$

v0_RFtrack.R

```r
# Program to plot longitudinal phase space motion
#  through a system of cavities (just an example...)

Nturns = 100

# Some Parameters
Ws = 1.0 # MeV/u
phis = 30*pi/180  # synchronous phase angle
eV = 0.2 # MeV/u
QonA = 0.25
gamma = (931+Ws)/931
beta = sqrt(1-1/gamma^2)
eta = -1/gamma^2
h = 1/(beta*3e8/80.5e6)
k = 2*pi*h*eta/beta^2*(gamma-1)/gamma/Ws

# initialize the phase space plot
phi    = 0
dW     = 0
plot(phi, dW, xlim=c(-180,180), ylim=c(-0.1,0.1), typ="n")

trk = 1
while (trk < 16) {
# initialize particle positions in phase space
  u0 <- locator(1)
  phi    <- u0$x/180*pi
  dW     <- u0$y
# track the particle...
  i = 1
  while (i < Nturns+1) {
    phi = phi + k*dW
    dW  = dW + QonA*eV*(sin(phi)-sin(phis))
    points(phi*360/2/pi, dW, pch=21,col="red")
    i = i + 1
  }
trk = trk + 1
}

```



Motion Near the Ideal Particle



Linearize the motion, and write in matrix form…

$$
\begin{aligned}
\phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_{n+1} - \sin\phi_s) \\
&= \Delta E_n + eV\cos\phi_s\left[\Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n\right]
\end{aligned}
$$

$$\phi = \phi_s + \Delta\phi$$

small

Thus,

$$
\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= eV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}\;eV\cos\phi_s\right)\Delta E_n
\end{aligned}
$$

$$
\begin{pmatrix}\Delta\phi \\ \Delta E\end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ eV\cos\phi_s & \left(1+\frac{2\pi h\eta}{\beta^2 E}eV\cos\phi_s\right)\end{pmatrix}\begin{pmatrix}\Delta\phi \\ \Delta E\end{pmatrix}_n = \begin{pmatrix} 1 & 0 \\ eV\cos\phi_s & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta\phi \\ \Delta E\end{pmatrix}_n
$$

M = *M*$_c$ · *M*$_d$

"thin" cavity *drift*

Stability and Synchrotron Tune

- Frequency of the small amplitude motion is given by the trace of the matrix:

$$\cos\Delta\psi_s \approx 1 - \frac{1}{2}(\Delta\psi_s)^2 = 1 + \frac{\pi h\eta}{\beta^2 E} eV\cos\phi_s \left[= \frac{1}{2} trM \right]$$

- For small angles we have simple harmonic motion, with "frequency"

$$2\pi\nu_s = \sqrt{-\frac{2\pi h\eta}{\beta^2 E_s} eV\cos\phi_s}$$

- The quantity ν_s is the *synchrotron tune* and is the number of *synchrotron oscillations* which occur per revolution. At 150 GeV in the Tevatron, for example, we had

$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E_s} eV\cos\phi_s} = \sqrt{-\frac{1113\cdot 1/18^2}{2\pi(150\times 10^3)}(1)(-1)} \approx \frac{1}{500}$$

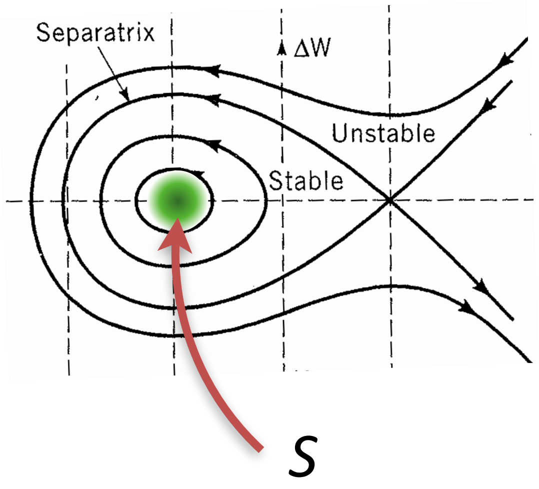
Beam Longitudinal Emittance

Suppose beam is well contained within an ellipse in phase space, and suppose we know either $\Delta\hat{E}$ or $\Delta\hat{\phi}$ (or, $\Delta\hat{t}$) of the distribution (i.e., maximum extent).

So, area of ellipse (the *longitudinal emittance*) is: $\pi\ \Delta\hat{E}\Delta\hat{\phi}$

or, in *E*-*t* coordinates, $$S \equiv \pi\ \Delta\hat{E}\Delta\hat{t} = \pi\ \Delta\hat{E}\frac{\Delta\hat{\phi}}{2\pi f_{\rm rf}}$$

$$S = \frac{1}{2f_{\rm rf}}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}\quad\cos\phi_S}\ \ \Delta\hat{\phi}^2$$

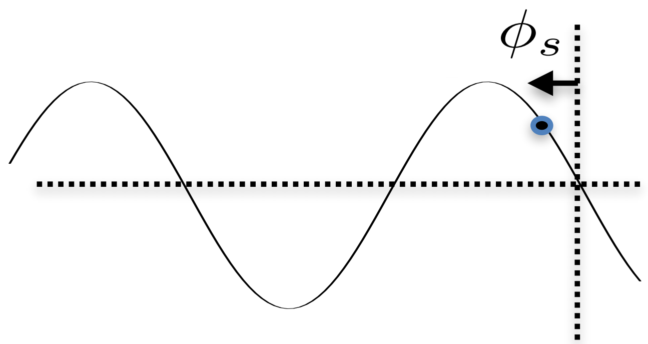


or,

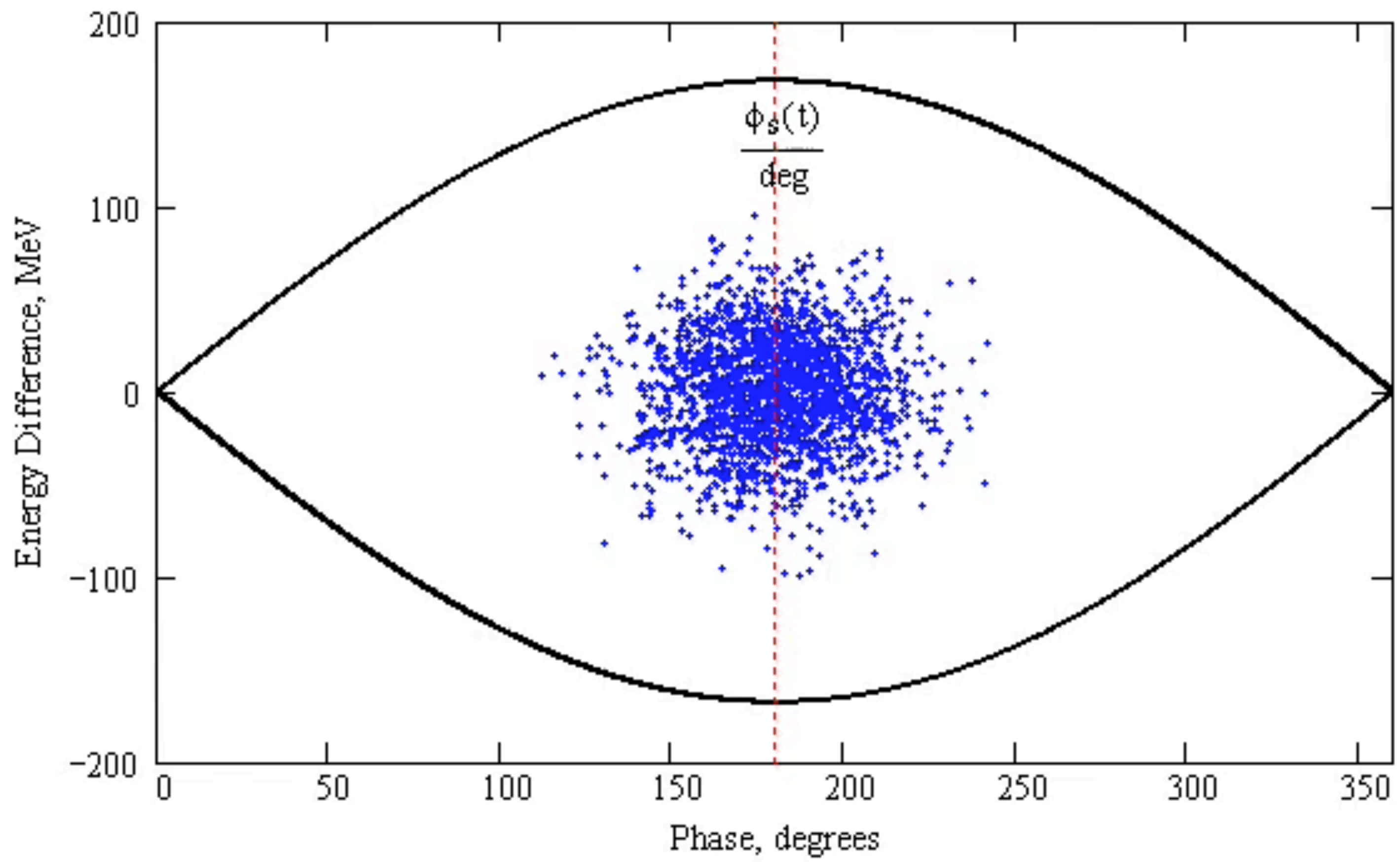
$$S = 2\pi^2 f_{\rm rf}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}\quad\cos\phi_S}\ \ \Delta\hat{t}^2$$

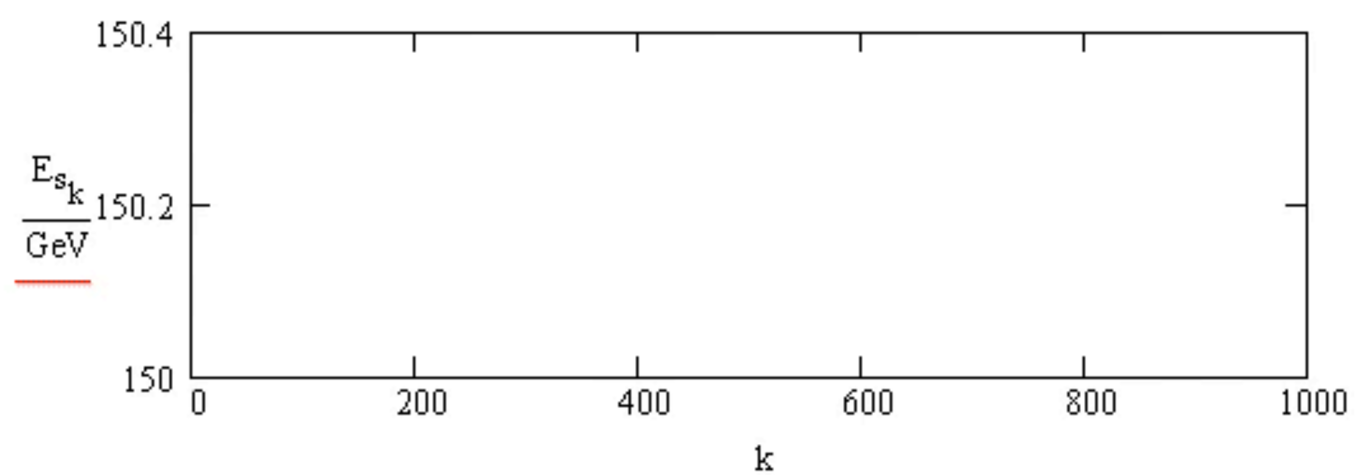
units: “eV-sec”

Acceleration

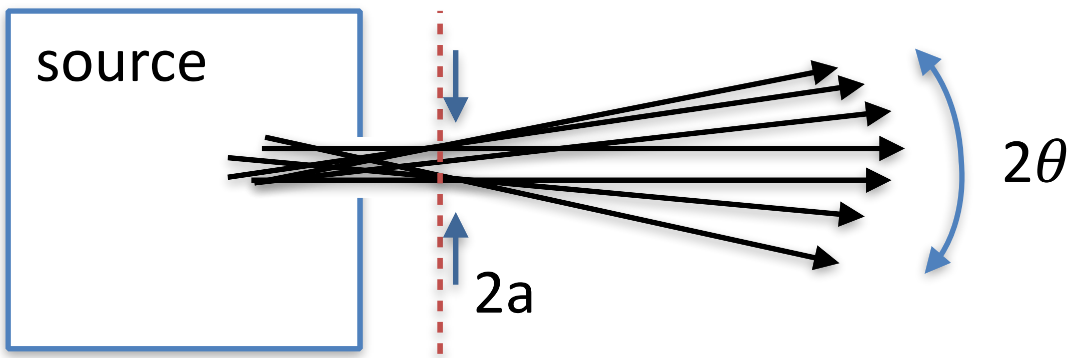


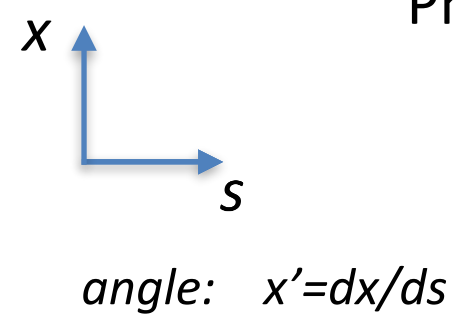
- Stable regions shrink as begin to accelerate
- If beam phase space area is too large (or if DC beam exists), can lose particles in the process



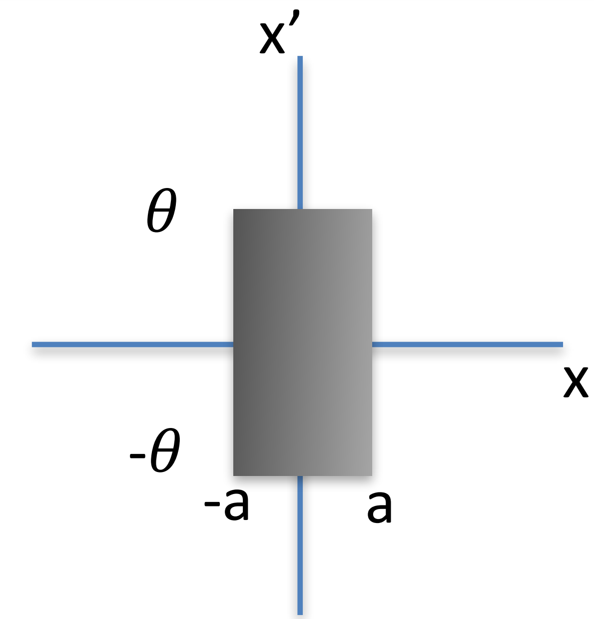


Review of Transverse Motion



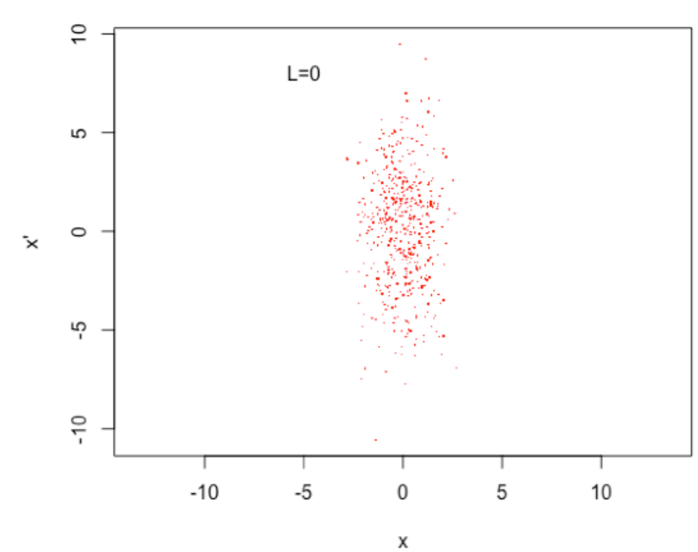


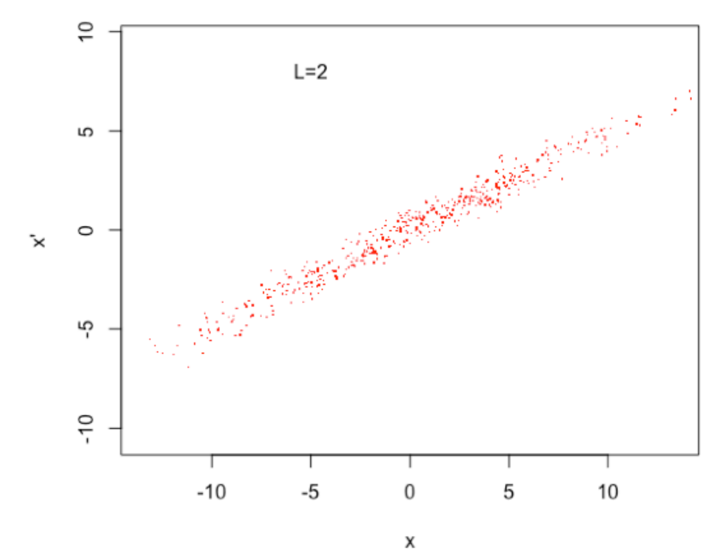
Phase Space:



- Allow to "drift" a distance L, then $x = x_0 + Lx'_0$ for each particle; the particles with largest $x' = x'_0$ will quickly drift to larger x values, and the distribution will "shear" —

Shape, orientation of distribution in "phase space" will change, but effective "area" of distribution will remain constant

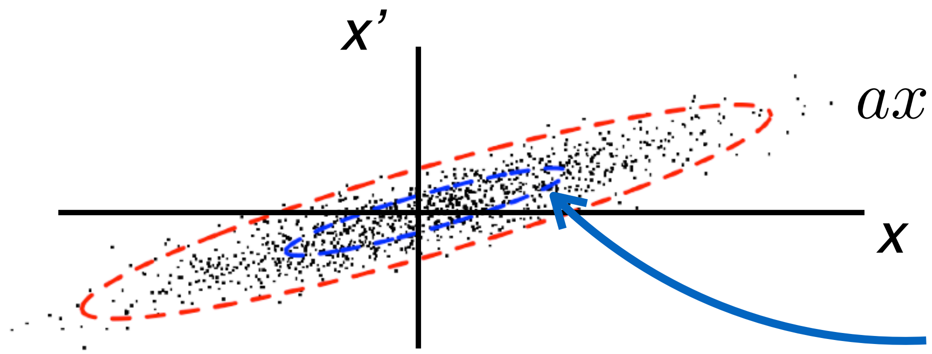




Emittance in Terms of Moments

- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to the coefficients by:

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$



$$ax^2 + bxx' + cx'^2 = 1$$

Area (blue curve): ϵ

- Define scaled quantities from our distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi}$$

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi}$$

$$\gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

Can show:

$$\epsilon = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

the "rms emittance"

α, β, γ collectively are called the ***Courant-Snyder*** parameters, or ***Twiss*** parameters

We find that the equation of the **blue** curve above is:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon/\pi$$

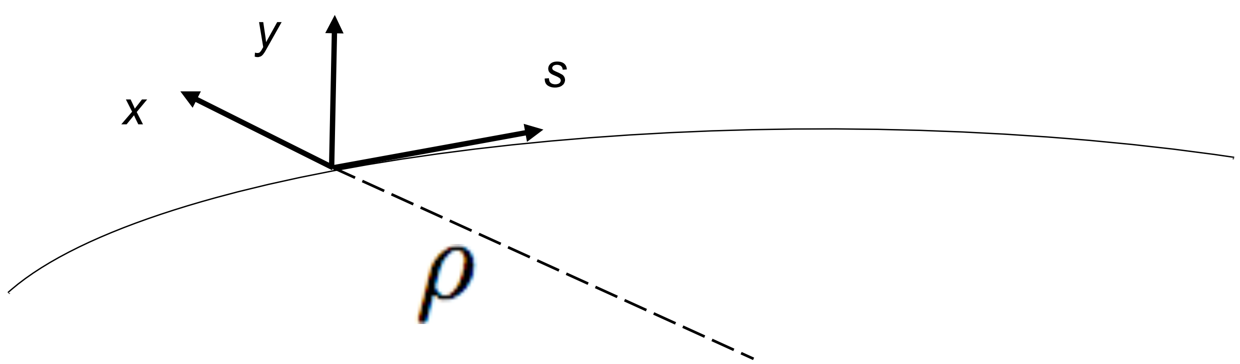
Note:

$$\beta\gamma - \alpha^2 = 1$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

Transverse Equations of Motion

- Lorentz Force: $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$
- Magnetic Rigidity
  - particle of unit charge, q = e: $$B\rho \equiv \frac{p}{q} = \frac{p}{e} = \frac{10}{2.9979} \cdot (pc)_{GeV} \ [\mathrm{T} \cdot \mathrm{m}]$$
- Reference Trajectory
  - Local Coordinate System



Linear Restoring Forces and Hill's Equation

- Wish to look at motion "near" the ideal trajectory of the accelerator system
- Assume linear guide fields —

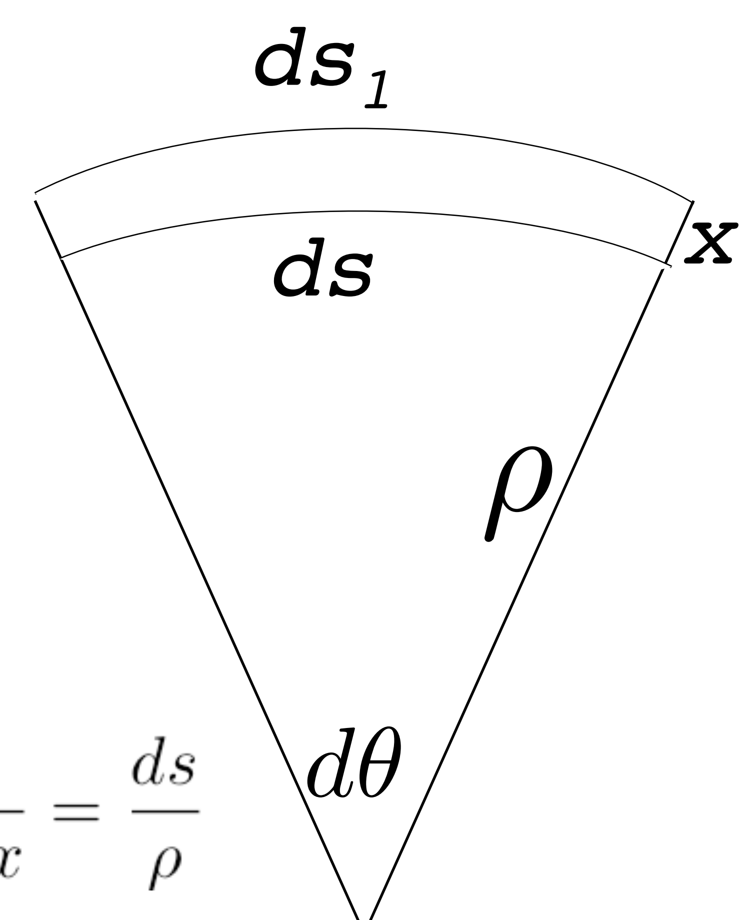
$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$B' \equiv \frac{\partial B_y}{\partial x}$$

- Then, to lowest order,

$$x' = \frac{dx}{ds}$$

$$x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right)x = 0$$
$$y'' - \left(\frac{B'}{B\rho}\right)y = 0$$



$$\frac{ds_1}{\rho + x} = \frac{ds}{\rho}$$

General Form —

Hill's Equation: $$x'' + K(s)x = 0$$

Piecewise Method -- Matrix Formalism

- Hill's Equation: $x'' + K(s)x = 0$
- Solutions for K = *constant* from $s = 0$ to $s = L$

(matrix form)

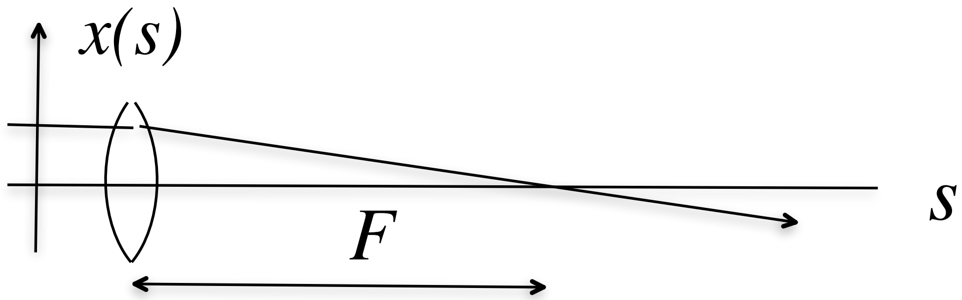
- K = 0: $\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

- K > 0: $\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

- K < 0: $\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

“Thin Lens” Quadrupole

- If quadrupole magnet is short enough, particle’s offset through the quad does not change by much, but the slope of the trajectory does -- acts like a “thin lens” in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

  - (similarly, for defocusing quadrupole)

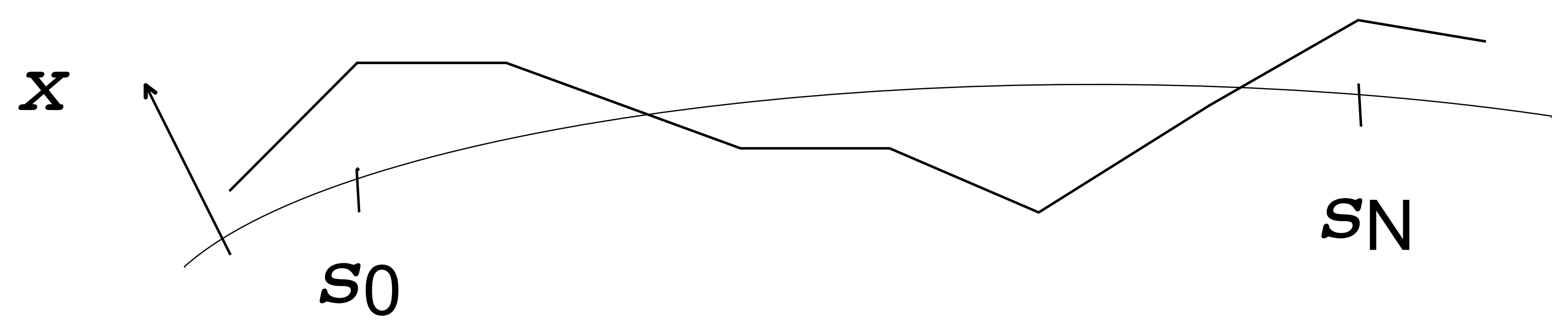
- Valid approximation, if *F* >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



TRANSPORT of Beam Moments

- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad\qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

- So, since $$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$
- where $$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$
- then, $$K = M \; K_0 \; M^T$$
- If know matrices *M*, then can “transport” beam parameters from one point to any point downstream, which determines beam distribution along the way.

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \longrightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Conservation of Emittance

- Note that from $$\Sigma = M \Sigma_0 M^T$$

$$\Sigma = \epsilon \cdot K \qquad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\det \Sigma = \det M \ \det \Sigma_0 \ \det M^T = \det \Sigma_0$$

- and

note: det$M$ = 1

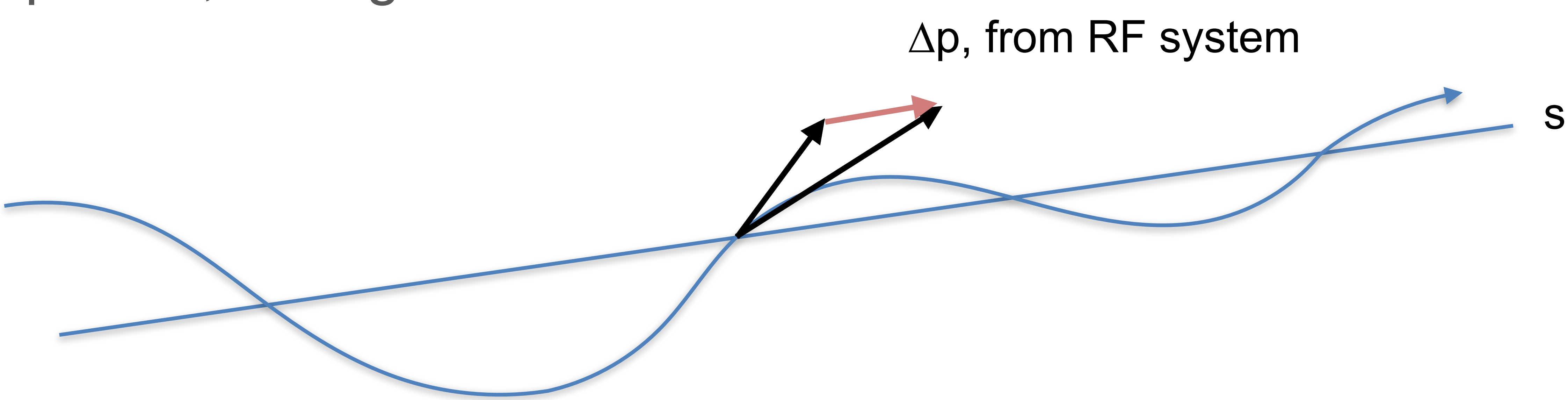
$$\det \Sigma = \epsilon^2 \det K = \epsilon^2 (\beta\gamma - \alpha^2) = \epsilon^2$$

$$\beta\gamma - \alpha^2 = 1$$

- Thus, the emittance is conserved upon transport through the system

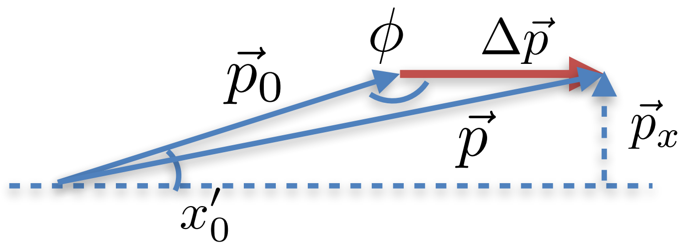
Adiabatic Damping from Acceleration

- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.

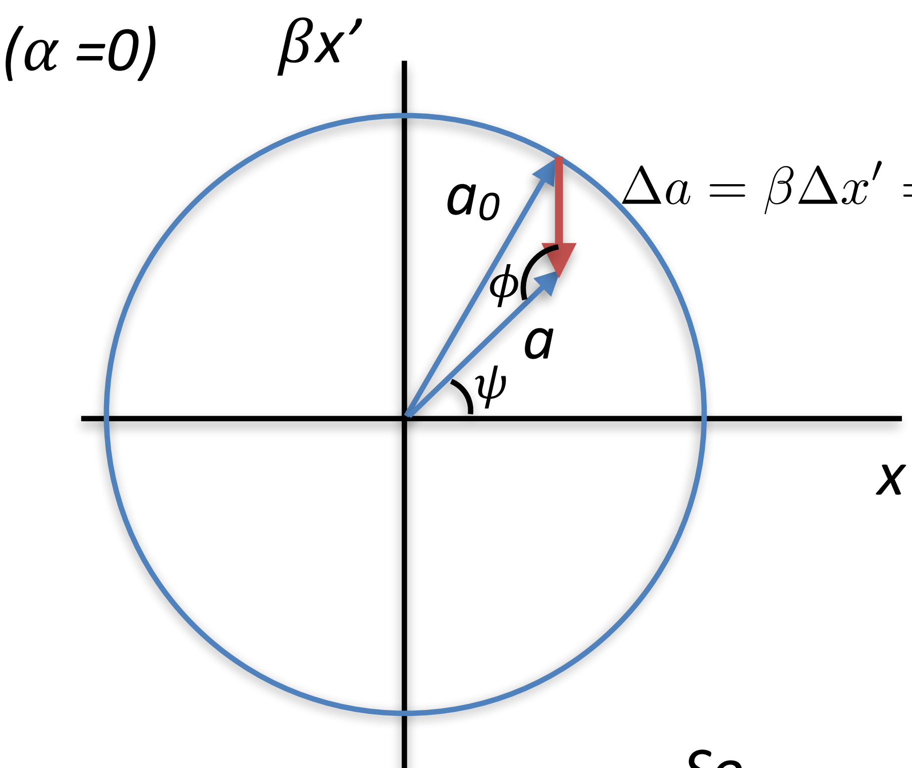


- The coordinates x-x' are not canonical conjugates, but x-p_x are; thus, from classical mechanics, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration



$$x' = \frac{p_x}{p} = \frac{p_x}{\sqrt{p_0^2 + \Delta p^2 - 2\Delta p\, p_0 \cos\phi}} = \frac{p_x}{p_0}\left(1 - \frac{\Delta p}{p_0} + \ldots\right) \approx x_0'\left(1 - \frac{\Delta p}{p_0}\right)$$



$$\Delta a = \beta \Delta x' = -\beta x_0' \frac{\Delta p}{p}$$

$$\begin{aligned}
a^2 &= a_0^2 + \Delta a^2 - 2\Delta a\, a_0 \cos\phi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\, a_0 \sin\psi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\, \beta\, x_0' \\
&= a_0^2 + (-\beta x_0' \frac{\Delta p}{p_0})^2 + 2(-\beta x_0' \frac{\Delta p}{p_0})\beta x_0' \\
&= a_0^2 + (\beta x_0')^2 (\frac{\Delta p}{p_0})^2 - 2(\beta x_0')^2 \frac{\Delta p}{p_0}
\end{aligned}$$

$$\Longrightarrow \quad \langle a^2 \rangle = \langle a_0^2 \rangle - 2\langle (\beta x_0')^2 \rangle \frac{\Delta p}{p_0}$$

Note:
$$2\langle (\beta x')^2 \rangle = 2\langle x^2 \rangle = \langle a^2 \rangle$$

So,

$$\Delta \langle a^2 \rangle = -\langle a^2 \rangle \frac{\Delta p}{p} \qquad \frac{\Delta\epsilon}{\epsilon} = -\frac{\Delta p}{p}$$

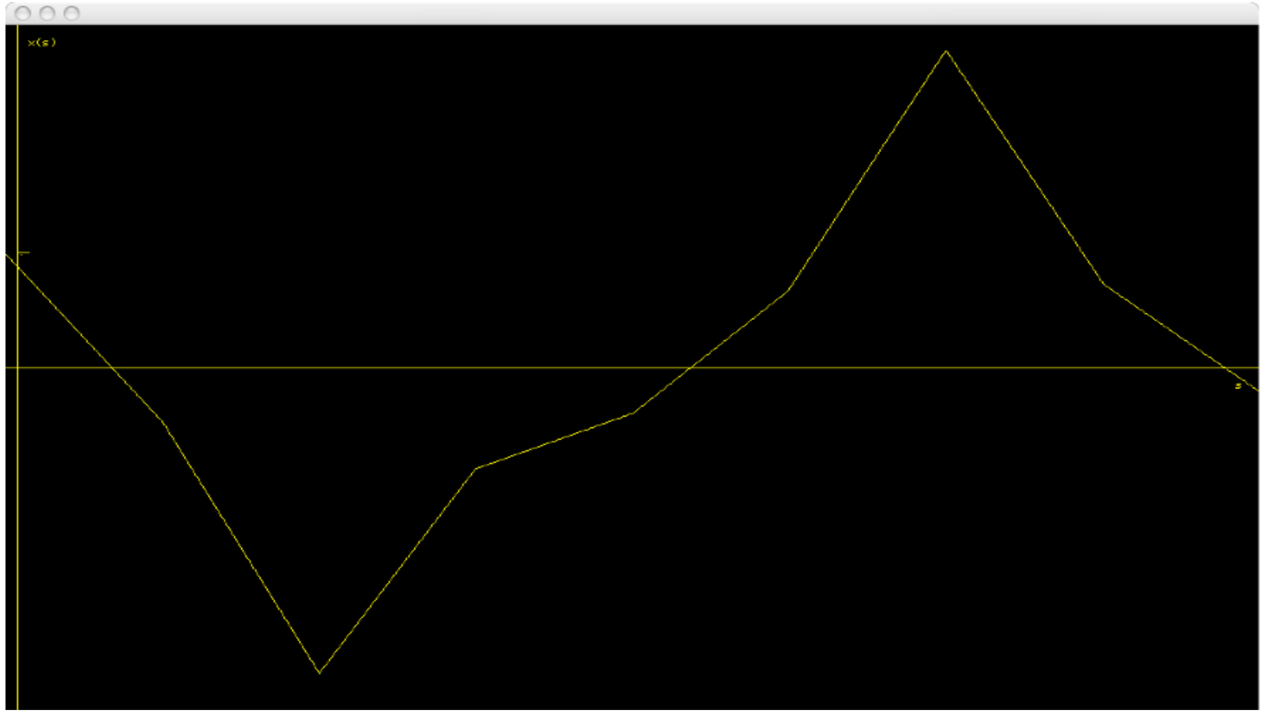
$$\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}$$

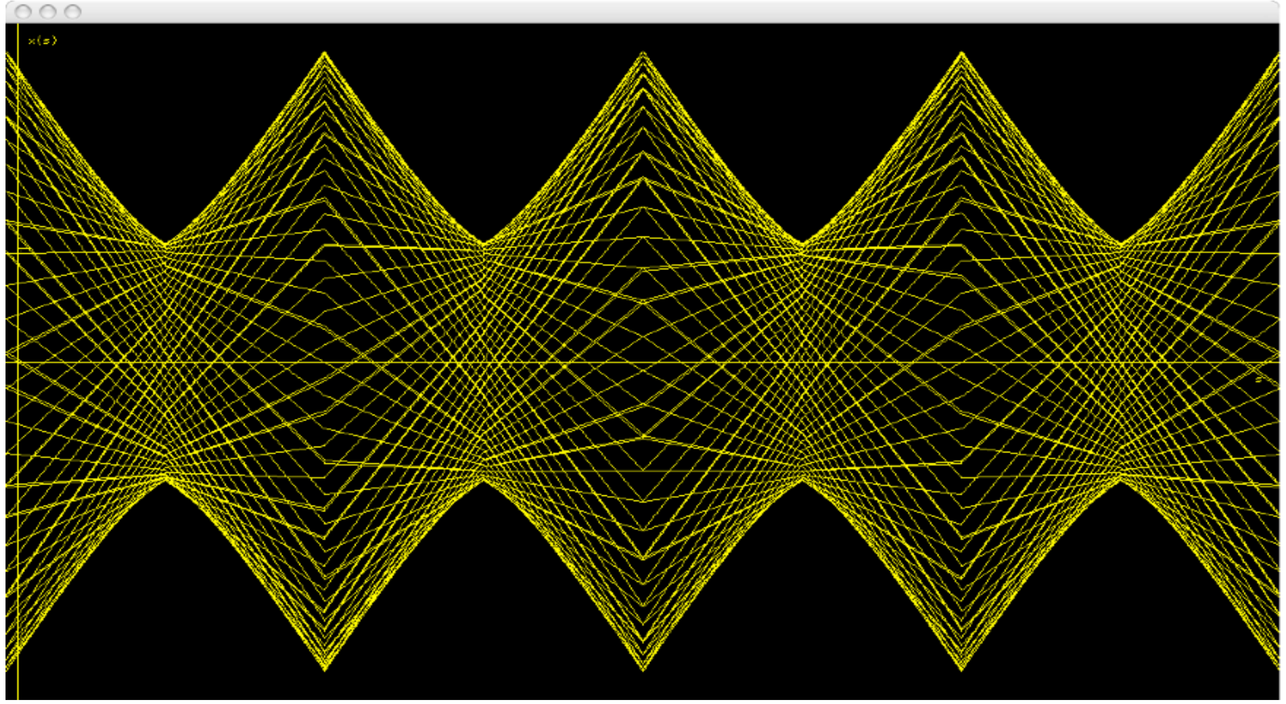
Define “normalized emittance”: $\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{Lorentz}$ should be preserved

The Notion of an Amplitude Function…

- Track single particle(s) through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

Hill's Equation — Analytical Solution

- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, s.
- So, assume solution is sinusoidal, with a phase which advances as a function of location s; also assume amplitude is modulated by a function which also depends upon s:

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- Then, plug into Hill's Equation ...

Analytical Solution (cont'd)

$$x'' + K(s)x = 0$$

$$\begin{aligned} x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\ x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\ x'' &= \ldots \end{aligned}$$

Plugging into Hill's Equation, and collecting terms...

$$\begin{aligned} x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\ &\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0 \end{aligned}$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously for *sum* = 0.

Analytical Solution (cont'd)

First,

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of `const`, then β would just scale accordingly; thus, valid to choose `const` = 1.

Next,

From

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\beta'' + 4K\beta = const.$$

is the general equation of motion for the amplitude function, β.
(in regions where K is either zero or constant)

(see back-up slides)

The Transport Matrix

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$$

- We can always write: $x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$

- Solve for *a* and *b* in terms of initial conditions and write in matrix form; we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

So, can write any of our transport matrices in terms of values of C-S parameters at the two end points, and the phase advance between them.

$\Delta\psi$ is the phase advance from point s_0 to point *s* in the beam line

Evolution of the Phase Advance

- If know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase *and* the Courant-Snyder parameters along a beam line or segment of the synchrotron from one point to another

remember, $K = M\ K_0\ M^T$

Betatron Tune

- Since $x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$ and $\psi' = 1/\beta$,

  then the total change in phase around the circumference is given by

  $$\psi_{tot} \equiv 2\pi\nu = \oint \frac{ds}{\beta}$$ or, will see: $$2\pi\nu = \cos^{-1}(\frac{1}{2}traceM_0)$$

  The tune, ν, is the number of transverse "*betatron oscillations*" per revolution. For the PS, the tune is ~8; for the Tevatron, the tune was about ~ 20; for the LHC, on the order of ~60.

- Note: since betatron tune ~ 20-60, and synchrotron tune ~ 0.002, it *is* (relatively) safe to consider the longitudinal motion independently from the transverse motion

- "circular" accelerators --> resonances; choose *tunes* carefully!

Periodic Transport Matrix

- Suppose M_0 represents a repeat period
  - FODO cell, say — or, could be the *whole ring* !

$$M_0 = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix}$$

ψ is the phase advance through the periodic system; if represents the whole ring, then ψ = $2\pi\nu$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\psi \equiv I\cos\psi + J\sin\psi$$

Notes: $J^2 = -I$ $\quad M_0 = e^{J\psi}$ $\quad J = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = K \cdot S$

Dispersion

$$B\rho = \frac{p}{q} \qquad \theta = \frac{qB \cdot \ell}{p}$$

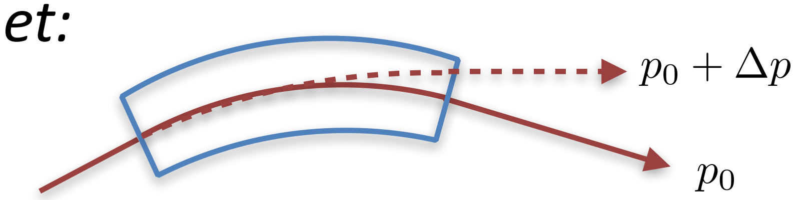
The bend angle (and/or focusing strength) depends upon momentum

Similar to index of refraction depending upon frequency

dipole steering “error” due to a different momentum —> “dispersion”

focusing “error” due to a different momentum —> “chromatic aberration"

dipole magnet:



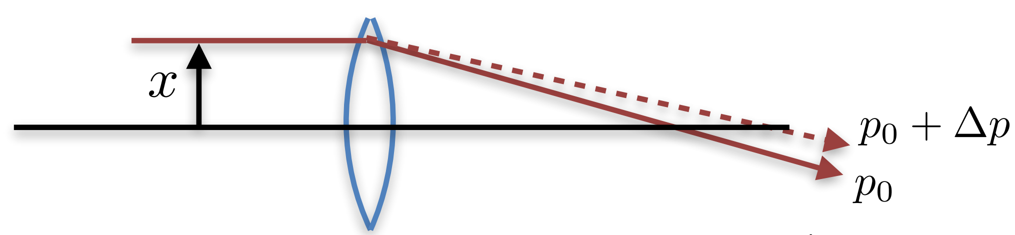
$$\frac{\Delta\theta}{\theta_0} = -\frac{\Delta p}{p}$$

[i.e., in “opposite” direction of bend]

at exit, to lowest order, $\Delta x' = \theta_0 \frac{\Delta p}{p}$

$$\Delta x \approx \frac{1}{2}\ell\theta_0 \frac{\Delta p}{p}$$

likewise, for quadrupole:



$$f = f_0(1 + \frac{\Delta p}{p})$$

Trajectory differences due to momentum differences referred to as “dispersion”

and, $$D(s, \Delta p/p) \approx D(s) \equiv \frac{\Delta x(s)}{\Delta p/p}$$ *“dispersion function”*

Dispersion

$$D'' + K\ D = \frac{1}{\rho_0}$$

In terms of matrices…

K = 0 :

$$D'' = \frac{1}{\rho},\quad D' = \frac{s}{\rho} + D_0'$$

$$D = D_0 + D_0' s + \frac{1}{2}\frac{s^2}{\rho}$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & s & \frac{1}{2}s^2/\rho \\ 0 & 1 & s/\rho \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

same 2x2 as before

1/ρ = 0 :

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

in the limit of short, or "thin" elements, a bending magnet primarily changes the slope of the dispersion function by an amount equal to the bend angle of the magnet

otherwise, the *D* transports roughly like a betatron oscillation

So, can use matrix methods (3x3 now; and 2x2 in "vertical" plane) to solve for:

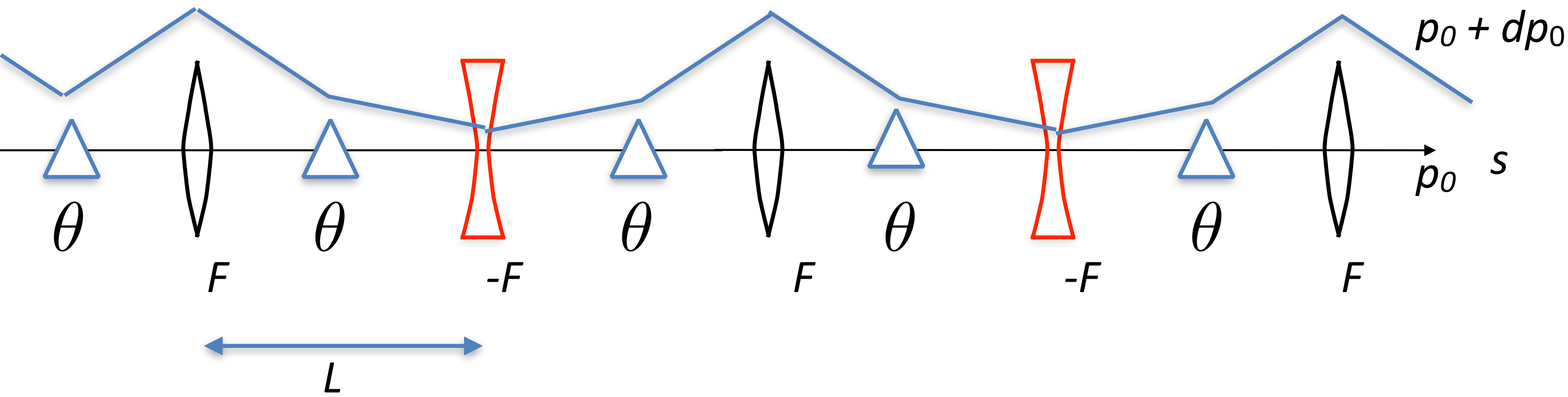
$$\beta_x,\quad \alpha_x,\quad \psi_x$$

$$\beta_y,\quad \alpha_y,\quad \psi_y$$

$$D_x,\quad D_x'$$

(& D_y, D_y' , if also have vertical bending)

Ex: String of FODO Cells with Bending



$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

$$D_{max,min} = \frac{L\theta}{2\sin^2(\mu/2)}\left(1 \pm \frac{1}{2}\sin(\mu/2)\right)$$

Values of dispersion function are typically ~ few meters

Beam Size Including Dispersion

- Total excursion due to “off momentum” plus betatron oscillation:

$$x = x_\beta + D\ \delta \qquad\qquad \delta \equiv \Delta p/p$$

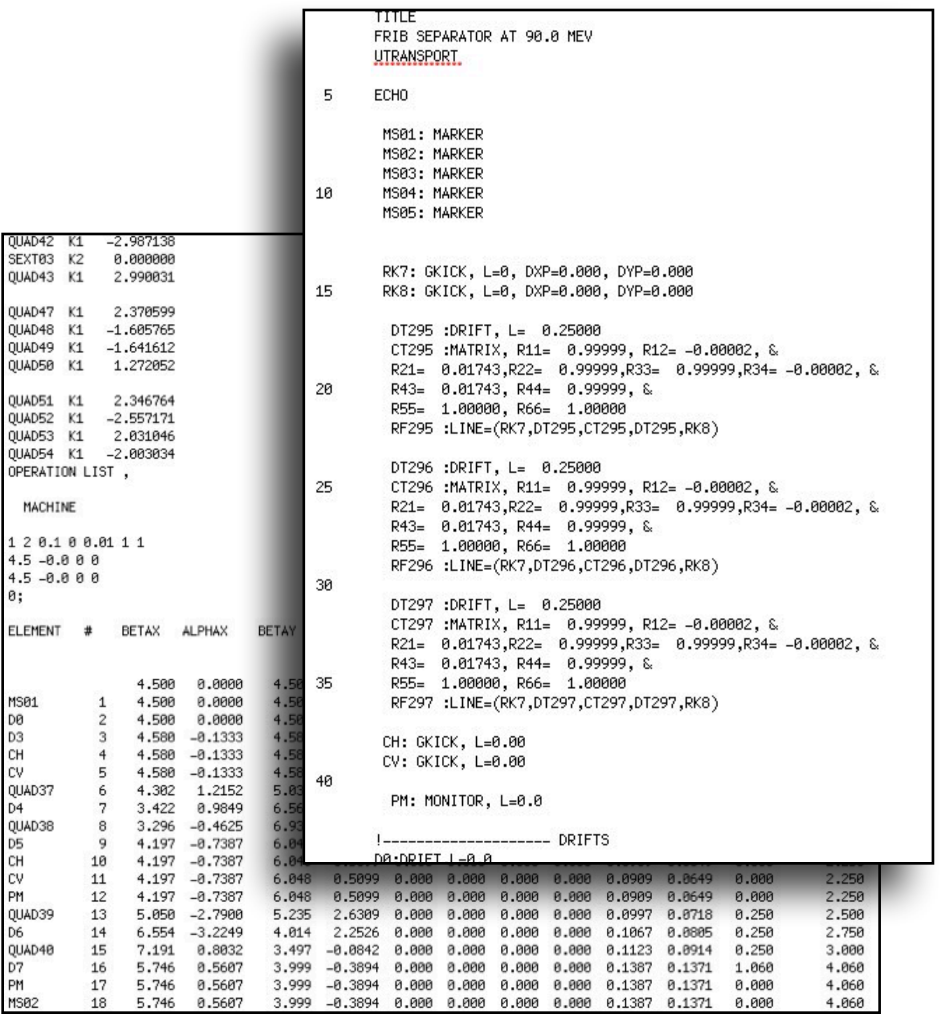
$$x^2 = x_\beta^2 + 2x_\beta D\delta + D^2\delta^2$$

- Assuming no correlation between x_β and particle’s momentum: $\langle x^2\rangle = \langle x_\beta^2\rangle + D^2\langle\delta^2\rangle$

$$\langle x^2\rangle = \epsilon_N\beta(s)/(\pi\gamma v/c) + D(s)^2\langle(\Delta p/p)^2\rangle$$

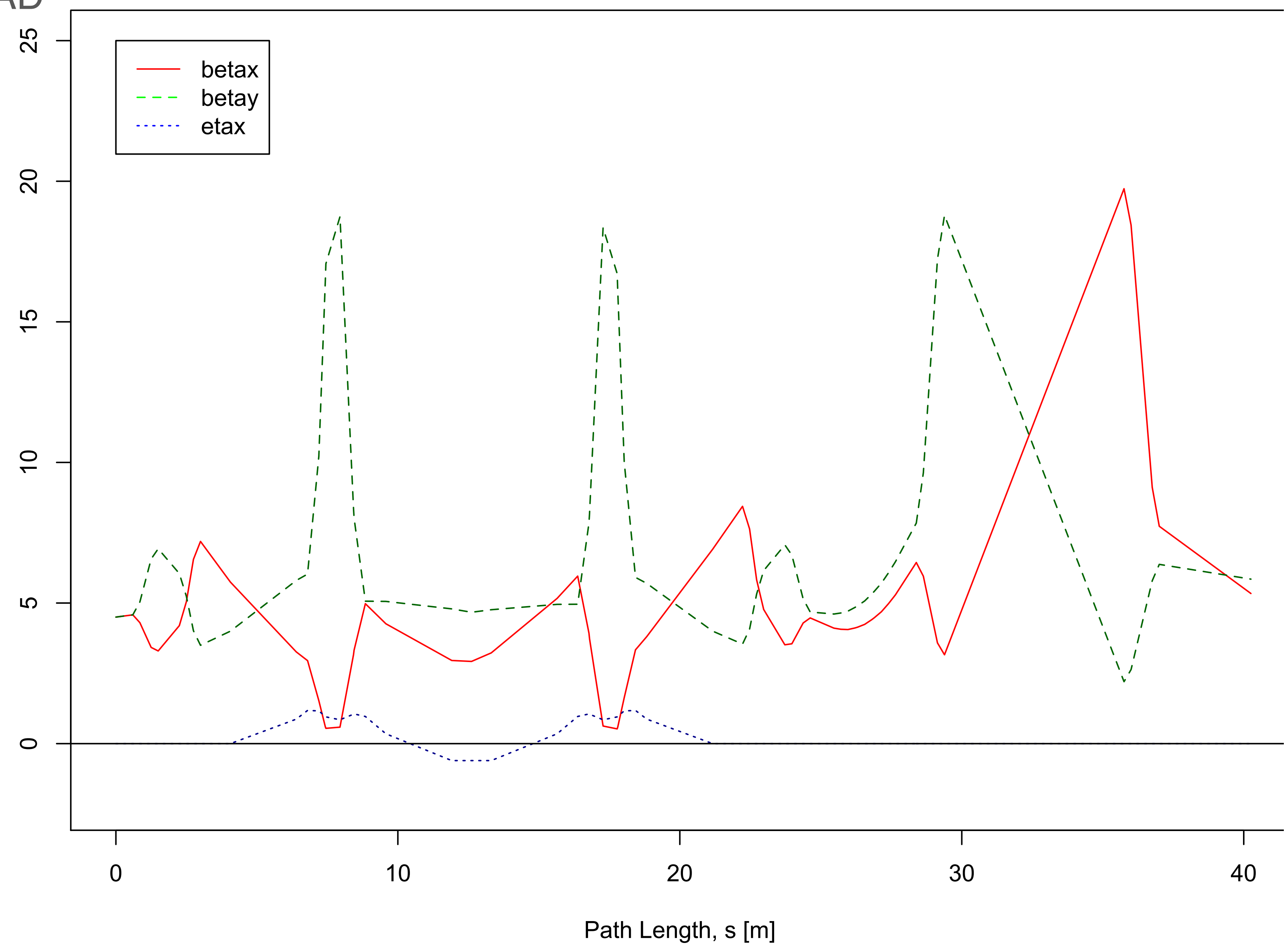
Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD,BMAD
  - TRACE, TRACE3D, COSY
  - SYNCH, CHEF, many more ...



```
QUAD42  K1   -2.987138
SEXT03  K2    0.000000
QUAD43  K1    2.990031

QUAD47  K1    2.370599
QUAD48  K1   -1.605765
QUAD49  K1   -1.641612
QUAD50  K1    1.272052

QUAD51  K1    2.346764
QUAD52  K1   -2.557171
QUAD53  K1    2.031046
QUAD54  K1   -2.003034
OPERATION LIST ,

  MACHINE

1 2 0.1 0 0.01 1 1
4.5 -0.0 0 0
4.5 -0.0 0 0
0;

ELEMENT    #    BETAX   ALPHAX    BETAY

                4.500   0.0000    4.50
MS01       1    4.500   0.0000    4.50
D0         2    4.500   0.0000    4.50
D3         3    4.580  -0.1333    4.58
CH         4    4.580  -0.1333    4.58
CV         5    4.580  -0.1333    4.58
QUAD37     6    4.302   1.2152    5.03
D4         7    3.422   0.9849    6.56
QUAD38     8    3.296  -0.4625    6.93
D5         9    4.197  -0.7387    6.04
CH        10    4.197  -0.7387    6.04
CV        11    4.197  -0.7387    6.048   0.5099  0.000  0.000  0.000  0.000  0.0909  0.0649  0.000   2.250
PM        12    4.197  -0.7387    6.048   0.5099  0.000  0.000  0.000  0.000  0.0909  0.0649  0.000   2.250
QUAD39    13    5.050  -2.7900    5.235   2.6309  0.000  0.000  0.000  0.000  0.0997  0.0718  0.250   2.500
D6        14    6.554  -3.2249    4.014   2.2526  0.000  0.000  0.000  0.000  0.1067  0.0805  0.250   2.750
QUAD40    15    7.191   0.8032    3.497  -0.0842  0.000  0.000  0.000  0.000  0.1123  0.0914  0.250   3.000
D7        16    5.746   0.5607    3.999  -0.3894  0.000  0.000  0.000  0.000  0.1387  0.1371  1.060   4.060
PM        17    5.746   0.5607    3.999  -0.3894  0.000  0.000  0.000  0.000  0.1387  0.1371  0.000   4.060
MS02      18    5.746   0.5607    3.999  -0.3894  0.000  0.000  0.000  0.000  0.1387  0.1371  0.000   4.060
```

```
TITLE
FRIB SEPARATOR AT 90.0 MEV
UTRANSPORT

ECHO

 MS01: MARKER
 MS02: MARKER
 MS03: MARKER
 MS04: MARKER
 MS05: MARKER


 RK7: GKICK, L=0, DXP=0.000, DYP=0.000
 RK8: GKICK, L=0, DXP=0.000, DYP=0.000

  DT295 :DRIFT, L=  0.25000
  CT295 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF295 :LINE=(RK7,DT295,CT295,DT295,RK8)

  DT296 :DRIFT, L=  0.25000
  CT296 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF296 :LINE=(RK7,DT296,CT296,DT296,RK8)

  DT297 :DRIFT, L=  0.25000
  CT297 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF297 :LINE=(RK7,DT297,CT297,DT297,RK8)

 CH: GKICK, L=0.00
 CV: GKICK, L=0.00

  PM: MONITOR, L=0.0

!-------------------- DRIFTS
D0:DRIFT L=0.0
```



Basic Optics of Colliders

- Arcs and FODO cells
- Long straight sections
  - Utility Straight Sections
  - Interaction Regions
- Dispersion suppressors

FODO Cells (arcs)

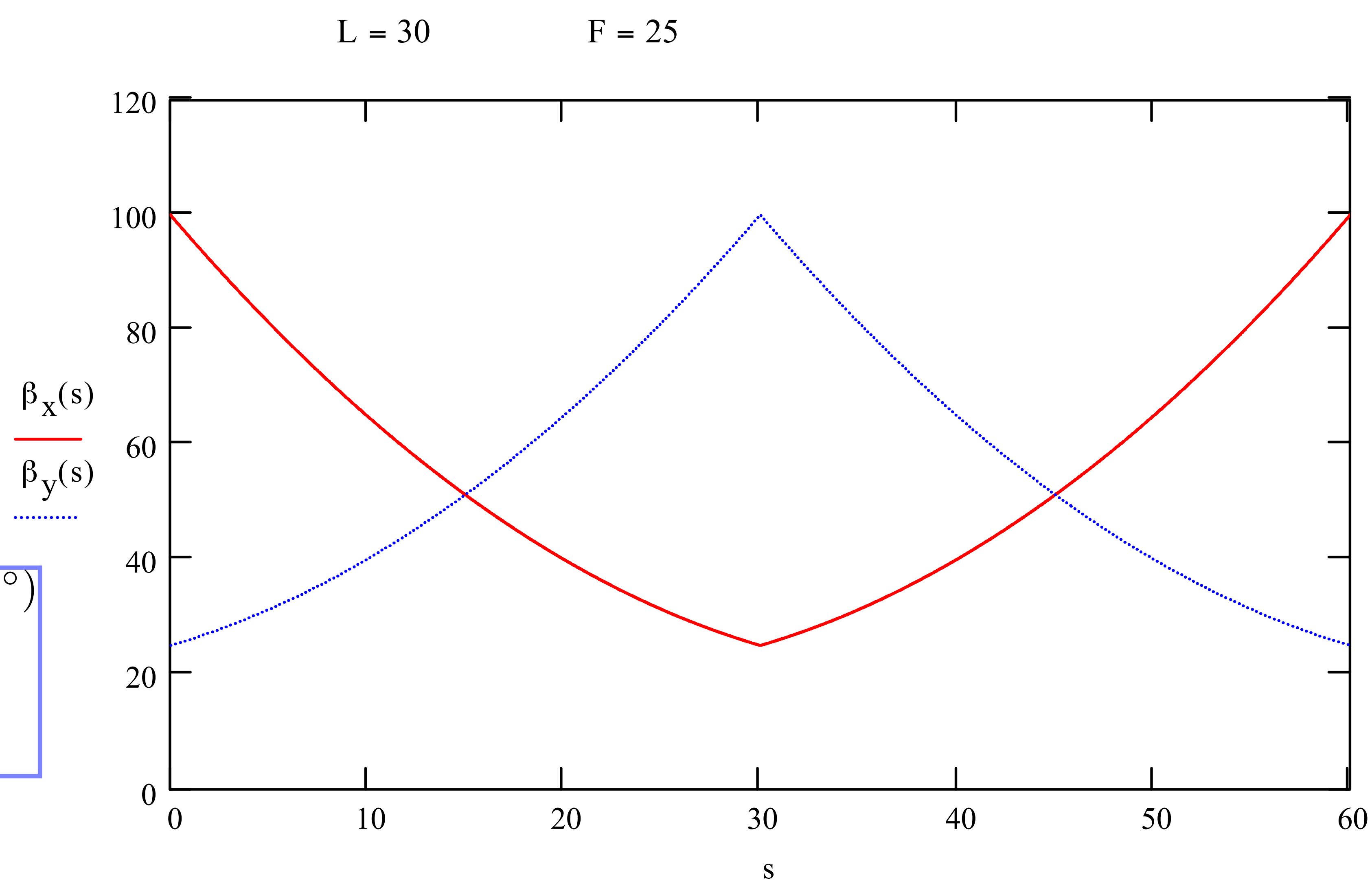
$$\Delta\beta' = \mp 2\beta/F$$ through a thin quad

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$ between quadrupoles

$$\beta_{max,min} = 2F\sqrt{\frac{1 \pm L/2F}{1 \mp L/2F}}$$

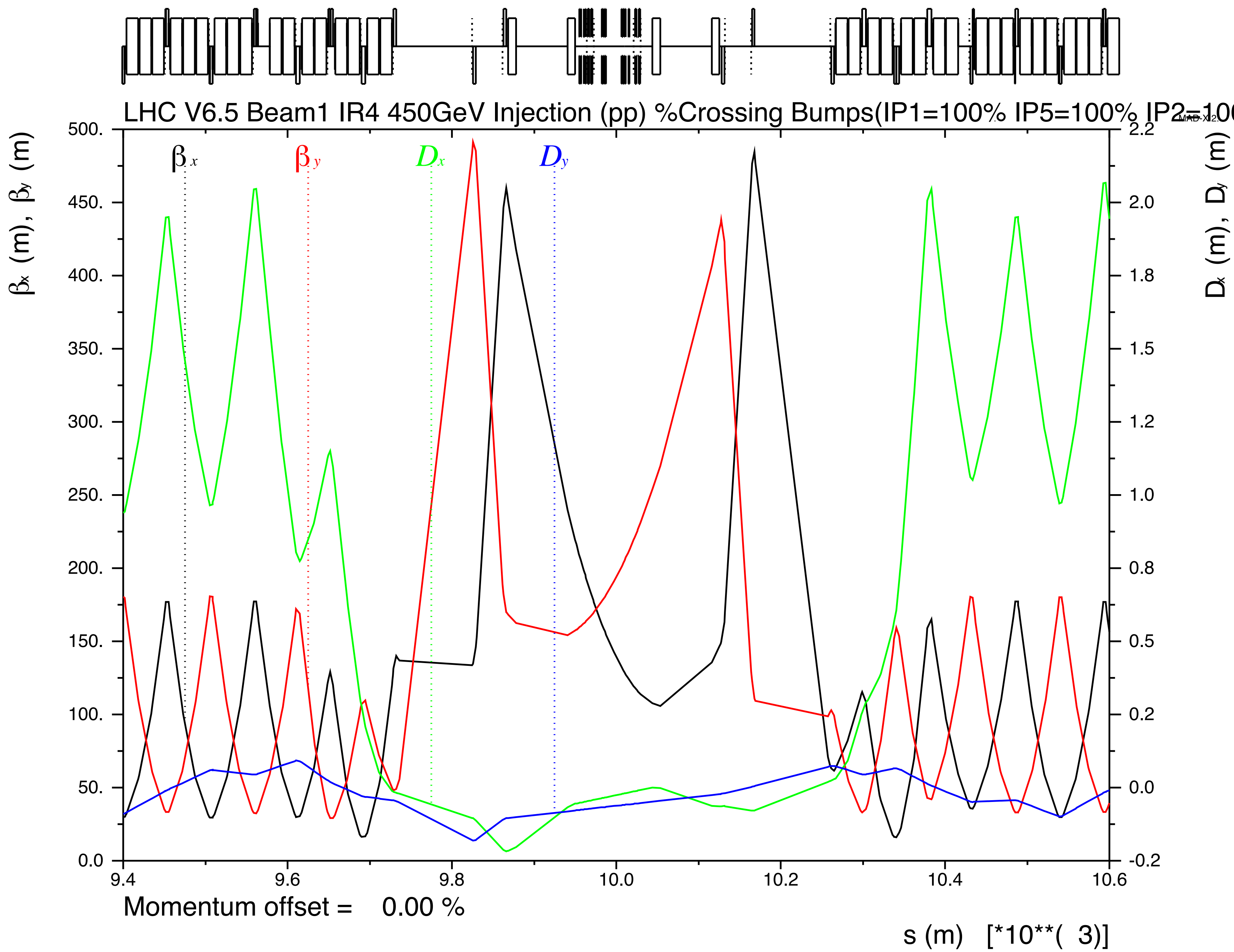
Ex: Tevatron Cell

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25\ \mathrm{m})\sqrt{1.6/0.4} = 100\ \mathrm{m}$$
$$\beta_{min} = 2(25\ \mathrm{m})\sqrt{0.4/1.6} = 25\ \mathrm{m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$



Long Straight Section

- a "matched insertion" that propagates the amplitude functions from their FODO values, through the new region, and reproduces the FODO values on the other side
- Here, we see an LHC section used for beam scraping



Interaction Region

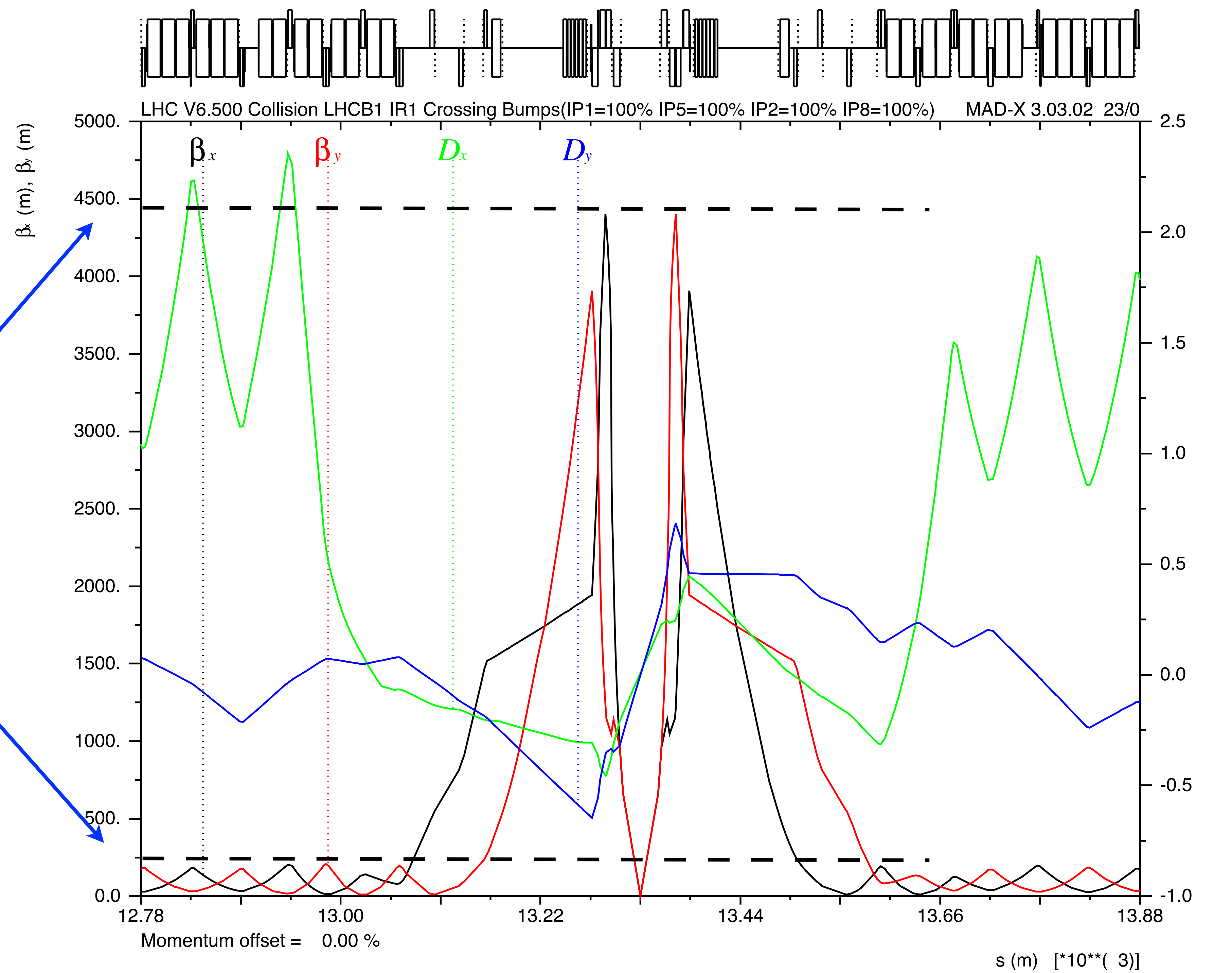
LHC high luminosity IR

- Final Focus

Note scales!

Triplets

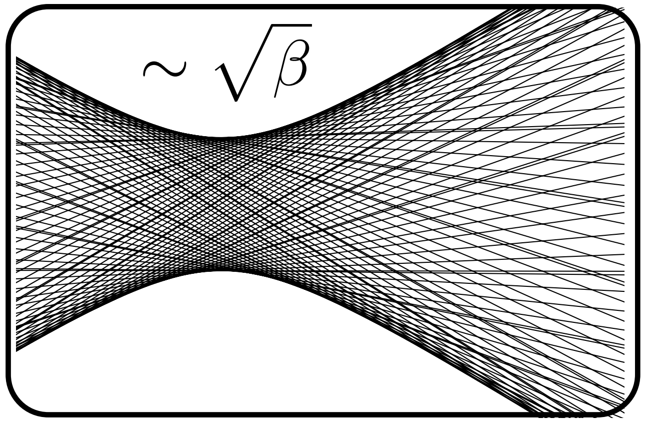
FODO



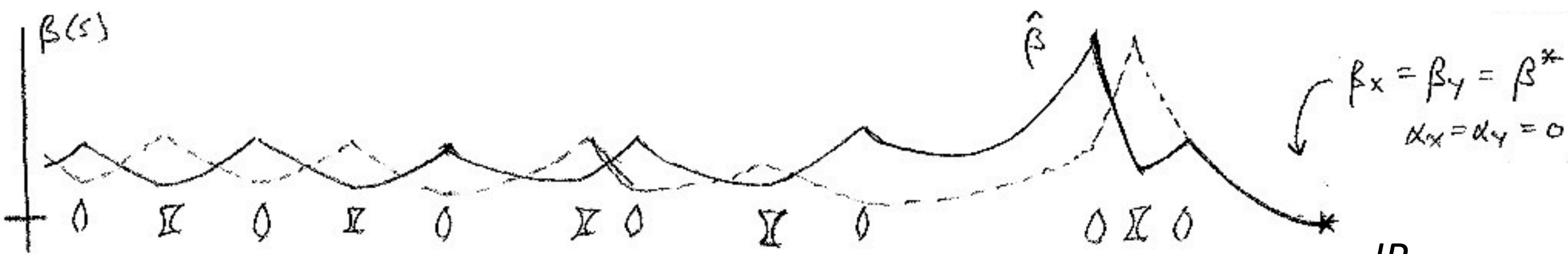
Low-Beta Optics

- In drift, amplitude function is a parabola:

$$\beta(s) = \beta^* \left[1 + (s/\beta^*)^2\right]$$



- Very small beam at IP requires very large beam in the final focus triplet:



IP

transport	adjust/tune	final focus
telescope-like:	"eyepiece"	"objective"

$\leftarrow L^* \rightarrow$

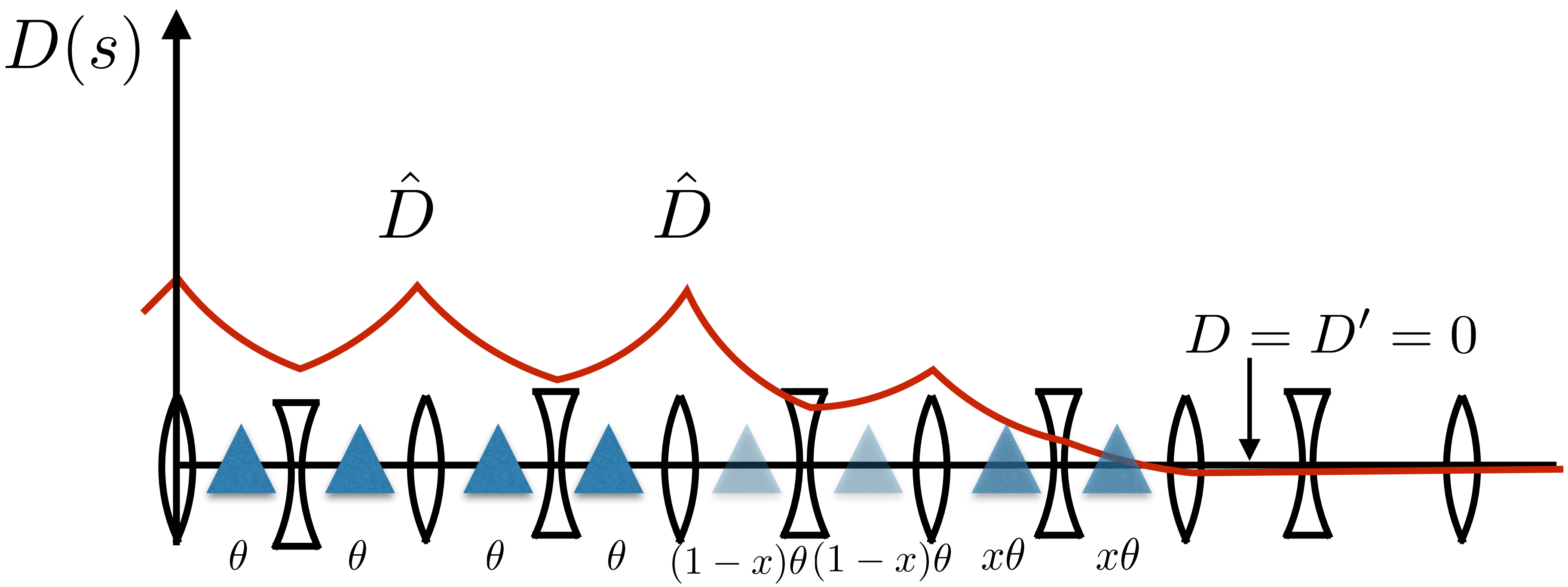
$$\hat{\beta} \approx \frac{L^{*2}}{\beta^*}$$

Dispersion Suppressor

used in rings or large bending segments to bring the periodic dispersion of the FODO cells to zero

$$\hat{D} = \frac{L\theta}{\sin^2\frac{\mu}{2}}\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)$$

μ = phase advance per FODO cell



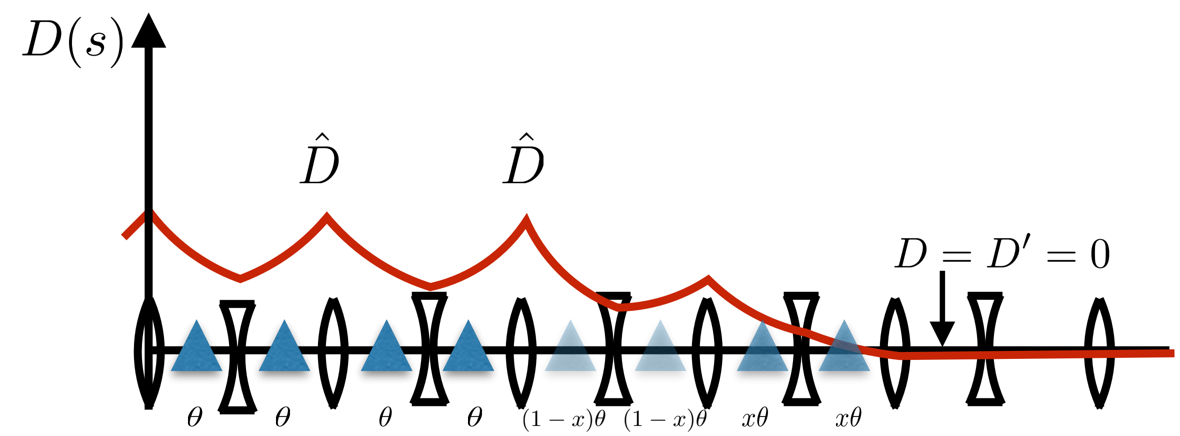
What value of *x* will bring *D*, *D'* both to zero?

Let periodic dispersion be: $\vec{D}_0 = \begin{pmatrix} D_0 \\ D_0' \end{pmatrix}$ and $M_0 = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$

Then, using 3x3 matrix approach:

$$\begin{pmatrix} \vec{D}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \vec{D}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} M_0 & (I - M_0)\vec{D}_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \vec{D}_0 \\ 1 \end{pmatrix}$$

Dispersion Suppressor



So, we transport through a cell with bending reduced by factor (1-*x*) followed by a cell with bending reduced by factor *x*; since *D* ~ *θ*, then the matrix elements scale accordingly and hence…

final dispersion = 0

$$\begin{pmatrix} \vec{0} \\ 1 \end{pmatrix} = \begin{pmatrix} M_0 & x(I - M_0)\vec{D}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_0 & (1-x)(I - M_0)\vec{D}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{D}_0 \\ 1 \end{pmatrix}$$

initial periodic values

which, upon simplification, yields $\vec{0} = \left[(1-2x)I + x(M_0^{-1} + M_0)\right] M_0 \vec{D}_0$

Noting that $M_0^{-1} + M_0 = trace(M_0)I = (2\cos\mu)I$

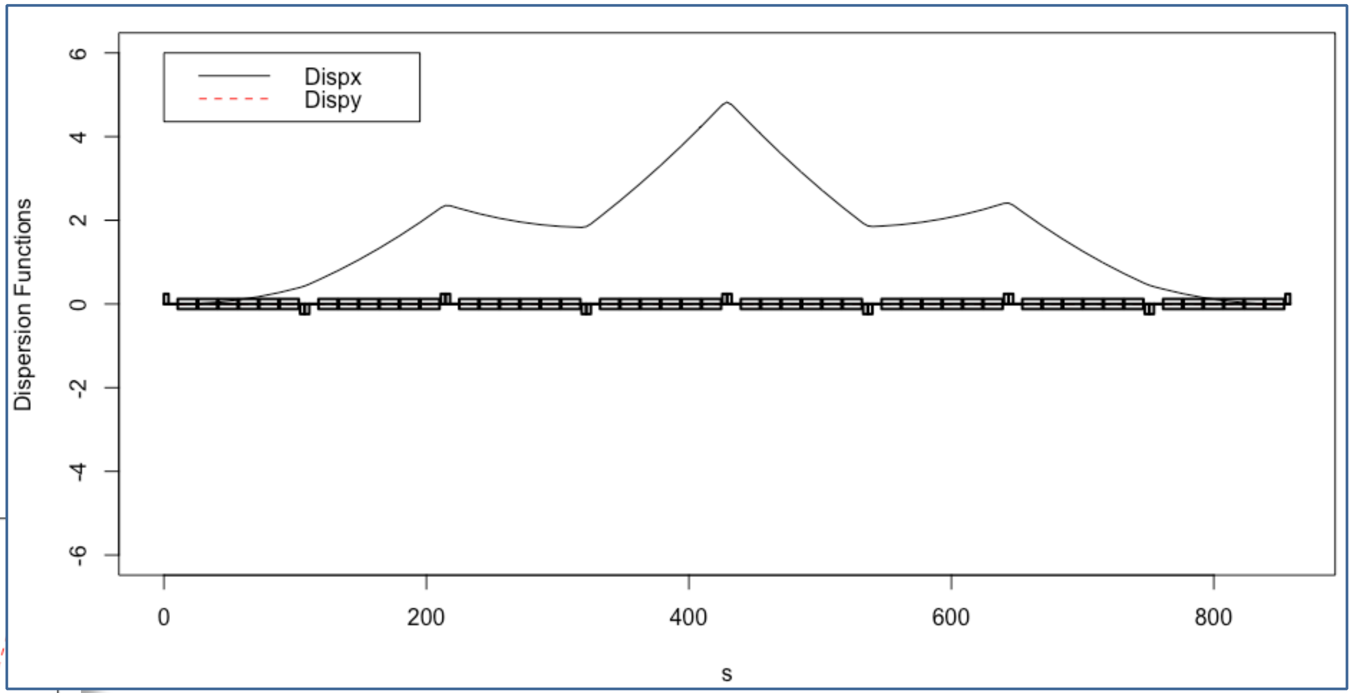
then our equation is satisfied if

$$x = \frac{1}{2(1-\cos\mu)}$$

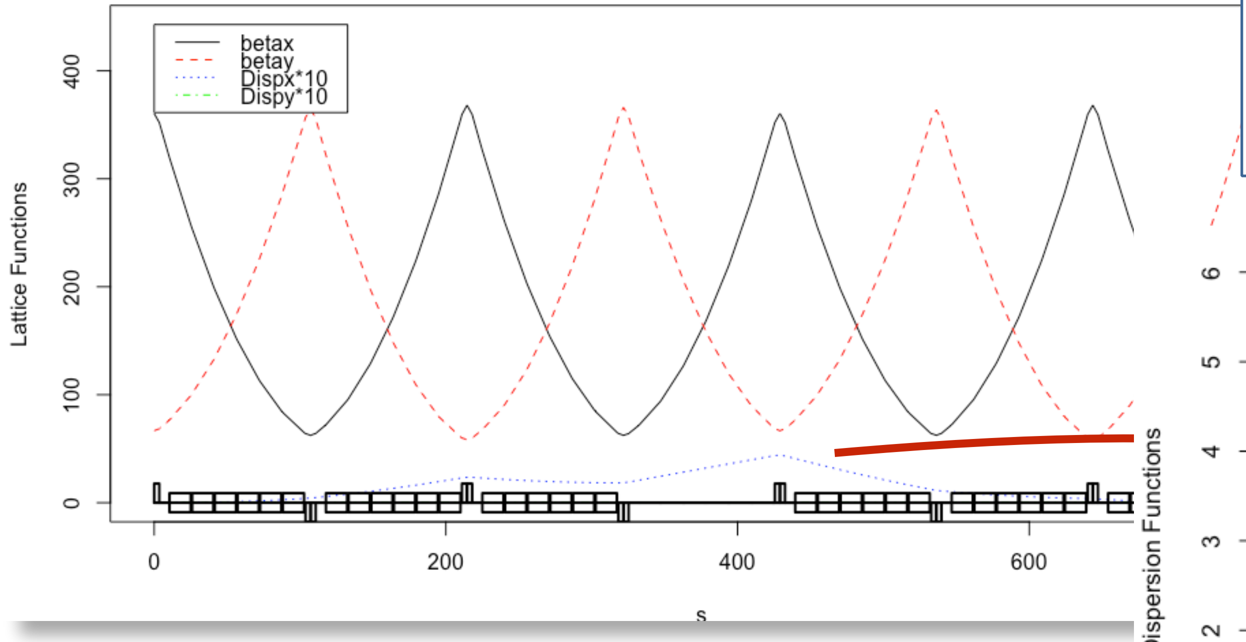
μ	x	1-x
60°	1	0
90°	1/2	1/2

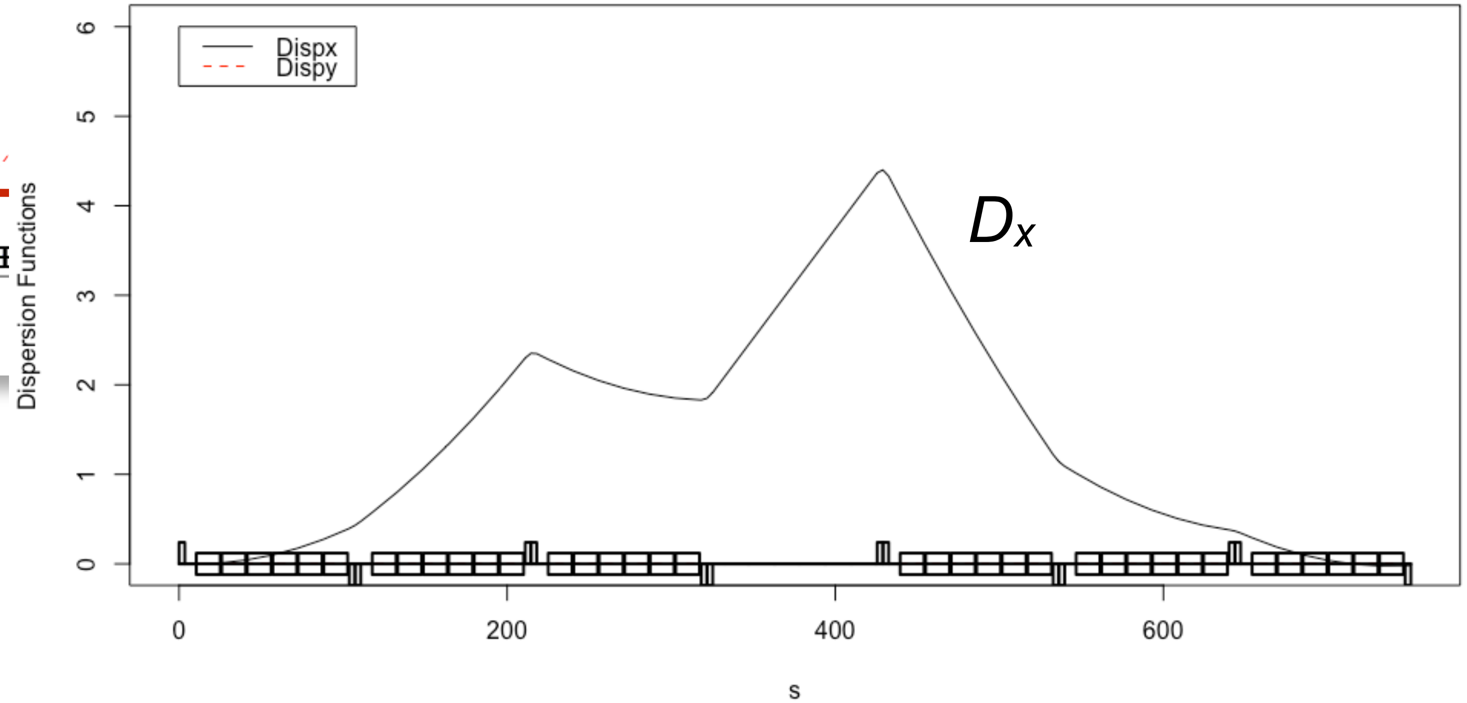
FODO Achromatic Bend Sections

for μ = 90°/cell, use 4 cells;
back-to-back dispersion suppressors —



here, μ = 90°, but
with empty half-cell in middle:





Corrections and Adjustments

- Correction/adjustment systems required for fine control
  - correct for misalignment, construction errors, drift, etc.
  - adjust operational conditions, tune up
- Use smaller magnetic elements for “fine tuning” of accelerator
  - dipole steering magnets for orbit/trajectory adjustment
  - quadrupole correctors for tune adjustment
  - sextupole magnets for chromaticity adjustment

- Typically, place correctors and instrumentation near the major quadrupole magnets -- “corrector package”
  - control steering, tunes, chromaticity, etc.
  - monitor beam position (in particular), intensity, losses, etc.

Steering (dipole) Errors

- dipole field error: $B_y = B_0 \longrightarrow B_y = B_0 + \Delta B$
  - manufacturing; powering; control setting, …

$$\Delta x' = -\frac{\Delta B \ell}{B\rho}$$

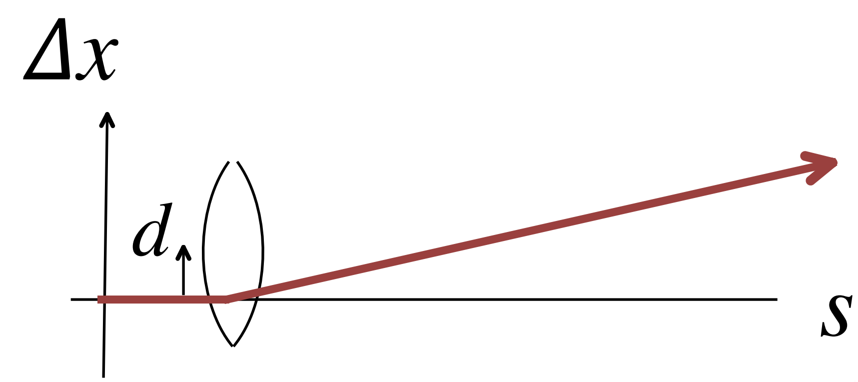
- dipole field "roll" (about the longitudinal axis)

$$B_y = B_0, \quad B_x = 0 \longrightarrow \begin{array}{rcl} B_y & = & B_0 \cos\phi \approx B_0 \\ B_x & = & B_0 \sin\phi \approx \phi B_0 \end{array}$$

$$\Delta y' = \phi \frac{B_0 \ell}{B\rho} = \phi\, \theta_0$$

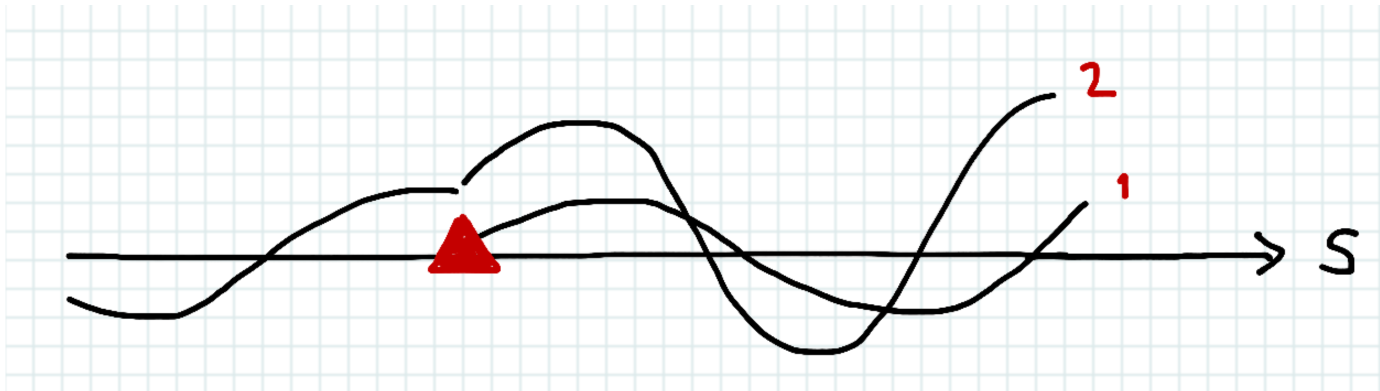
- Quadrupole misalignment:

$$\Delta x' = \frac{d}{F}$$



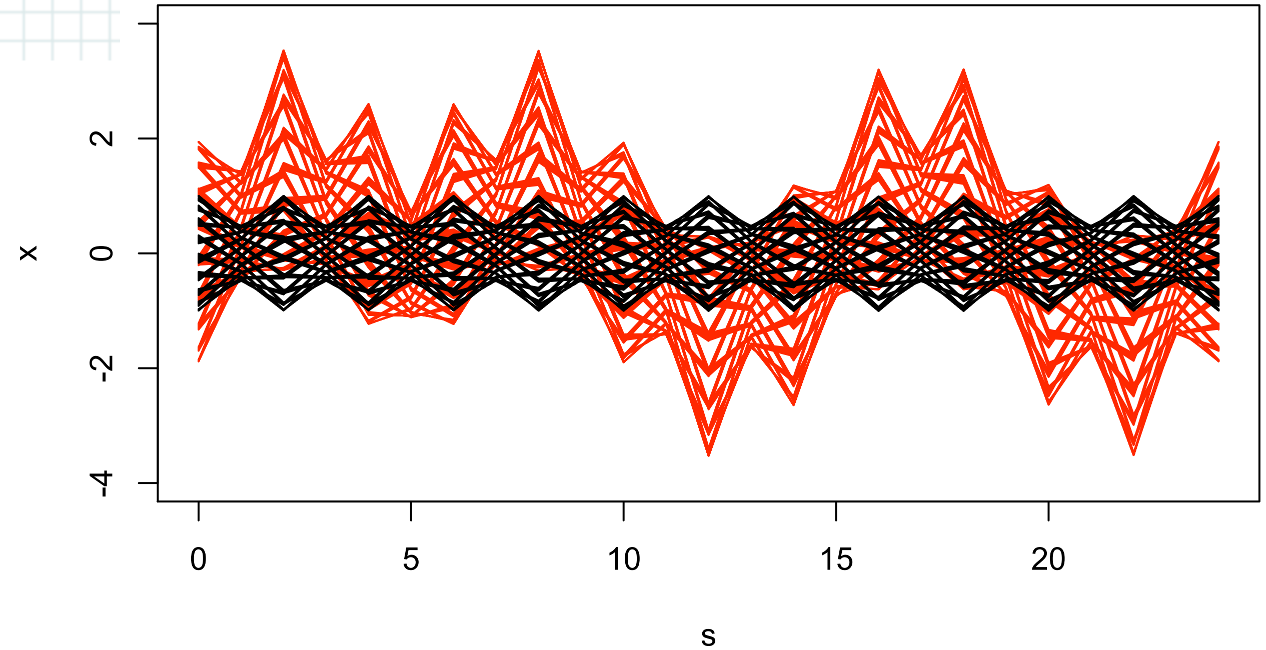
Steering (dipole) Errors

- Closed orbit distortions in a circular accelerator
  - These are not “one-time” kicks; they affect the particle motion every revolution



see ClosedOrbit.R

black = nominal
red = w/ error field



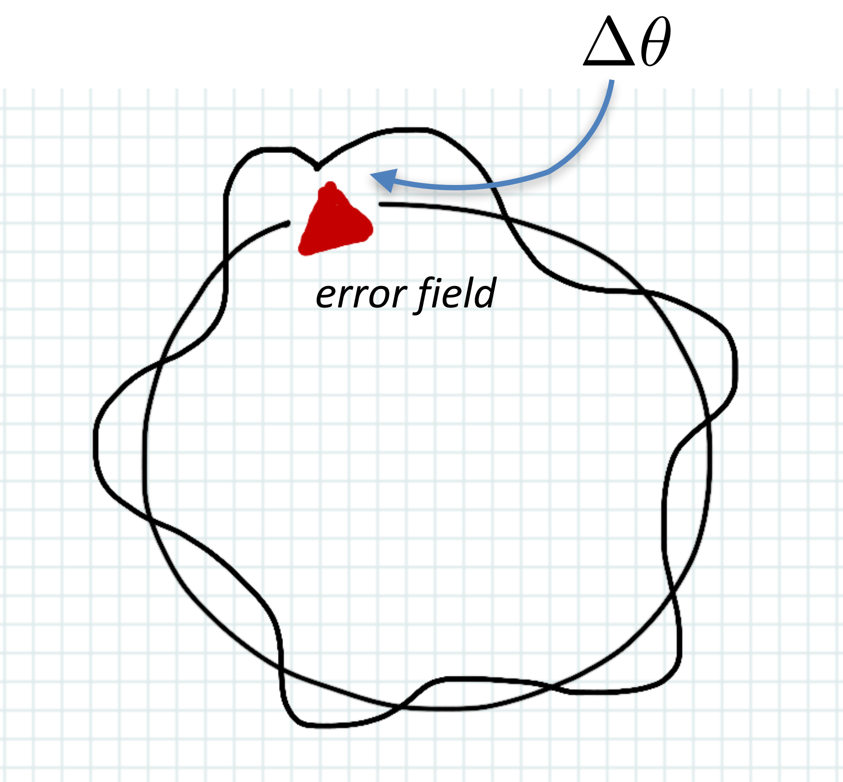
The trajectory of each particle will be altered by the angle $\Delta\theta$ every time it passes through the error field

The Closed Orbit

- Want to find the one trajectory which, upon passing through the error field, will come back upon itself
  - this is the "closed" trajectory, or ***closed orbit***

$$M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$



- When find x_0, x'_0, can find x,x' downstream:

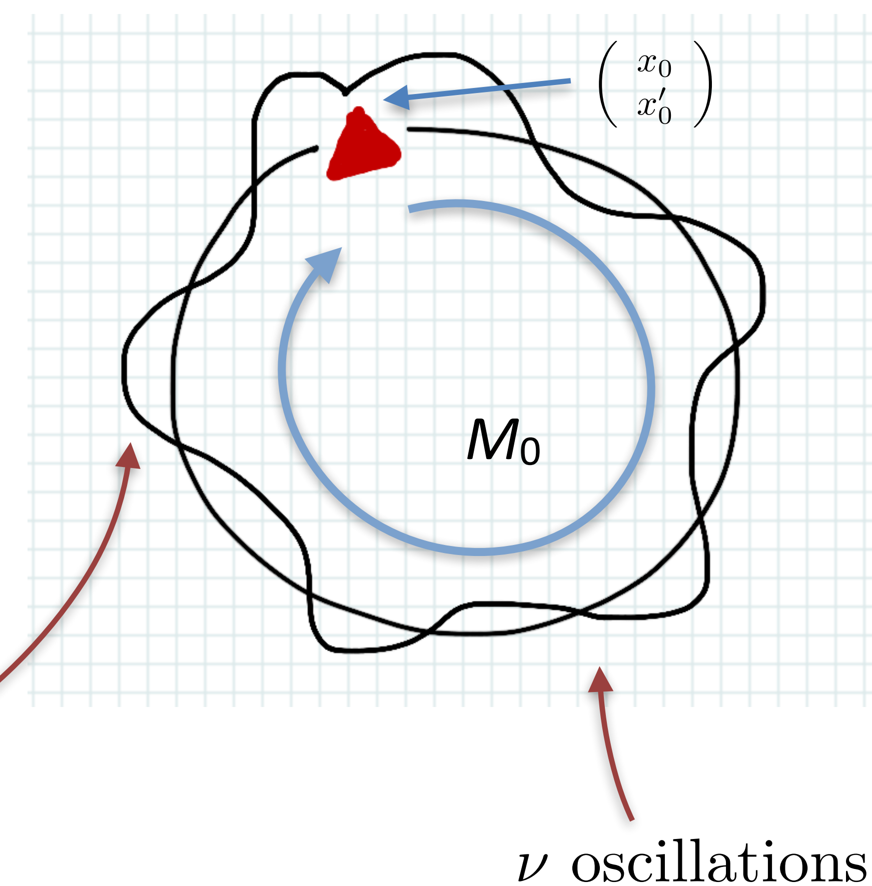
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Closed Orbit Distortion from Single Error

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0 \cos\pi\nu \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}} \sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}} \cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2} (\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\Delta x(s) = \frac{\Delta\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu} \cos\left[|\psi(s) - \psi_0| - \pi\nu\right]$$



If have a collection of errors about the accelerator, then at any one point:

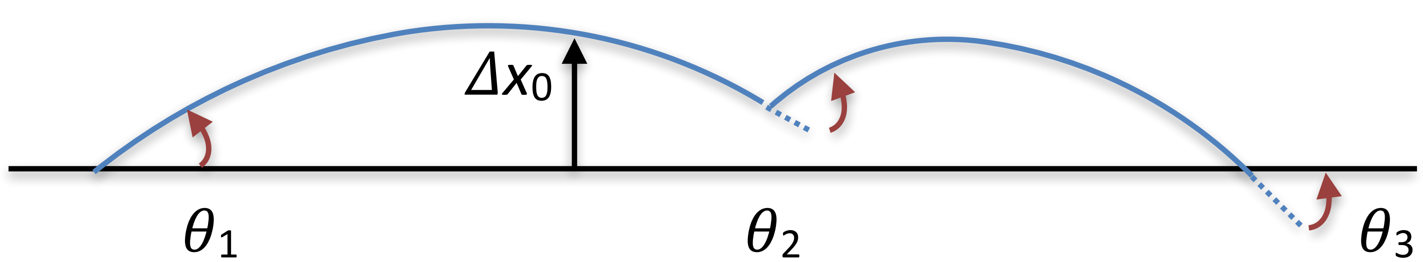
$$\Delta x(s) = \sum_i \frac{\Delta\theta_i\sqrt{\beta_i\beta(s)}}{2\sin\pi\nu} \cos\left[|\psi(s) - \psi_i| - \pi\nu\right]$$

as ν —> **integer**, distortions enhanced

a resonance!

Trajectory/Orbit Correction

- The final closed orbit of the synchrotron will be determined by the superposition of the distortions created by many error fields and correction fields
- To make a local adjustment or correction of the position of the beam in a beam line or synchrotron, three correctors are required (in general):



The trajectory before θ_1 and after θ_3 is left undisturbed

$$\theta_1 = \frac{\Delta x_0}{\sqrt{\beta_0\beta_1}\sin\psi_{10}}$$

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\,\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}$$