Basic formulas for electron rings

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K. Oide

katsunobu.oide @ cern.ch

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where momenta are normalized by *design momentum* P_d as:

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where we have introduced the total momentum P, the design arrival time $t_0 = t_0(s)$, and the total velocity $v = cP/\sqrt{m^2c^2 + P^2}$. For a storage ring, usually P_d is chosen constant over the entire ring.

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These variables are functions of s, which is the length along the coordinate line.

Hamiltonian

For the variables defined above, the Hamiltonian is written as

$$H = -(1 + x/\rho)\sqrt{(1 + \delta + \varphi/c)^2 - (p_x - A_x)^2 - (p_y - A_y)^2}$$
$$- (1 + x/\rho)A_z - (xp_y - yp_x)/\tau + \frac{E}{v_d} + \left(\frac{1}{v} + \frac{1}{c}\right) z \frac{\partial \varphi}{\partial s}\,, \qquad (3)$$

where ρ, τ, and v_d are the bending radius, the torsion, and the design velocity, respectively, and $E = c\sqrt{(1 + \delta + \varphi/c)^2 + m^2c^2/P_d^2} + \varphi$ is the (normalized) energy.

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The external fields are expressed by (normalized) electromagnetic potentials (A_x, A_y, A_z, φ), which are functions of $(x, y, z; s)$. In the case of magnets or RF cavities without a solenoid component B_z, only A_z is necessary to express the field. In such cases the Hamiltonian is simplified to

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Transfer Matrix

If a particle does a motion from s_0 to s, a transfer matrix M from s_0 to s is defined by

$$\mathrm{M} = \frac{\partial(x, p_x, y, p_y, z, \delta)}{\partial(x_0, p_{x0}, y_0, p_{y0}, z_0, \delta_0)}\,, \qquad (5)$$

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$${}^t\mathrm{MJM} = \mathrm{J} , \qquad \mathrm{J} \equiv \begin{pmatrix} . & 1 & . & . & ... \\ -1 & . & . & . & ... \\ . & . & . & 1 & ... \\ . & . & -1 & . & ... \\ ... & ... & ... & ... & ... \end{pmatrix} . \tag{6}$$

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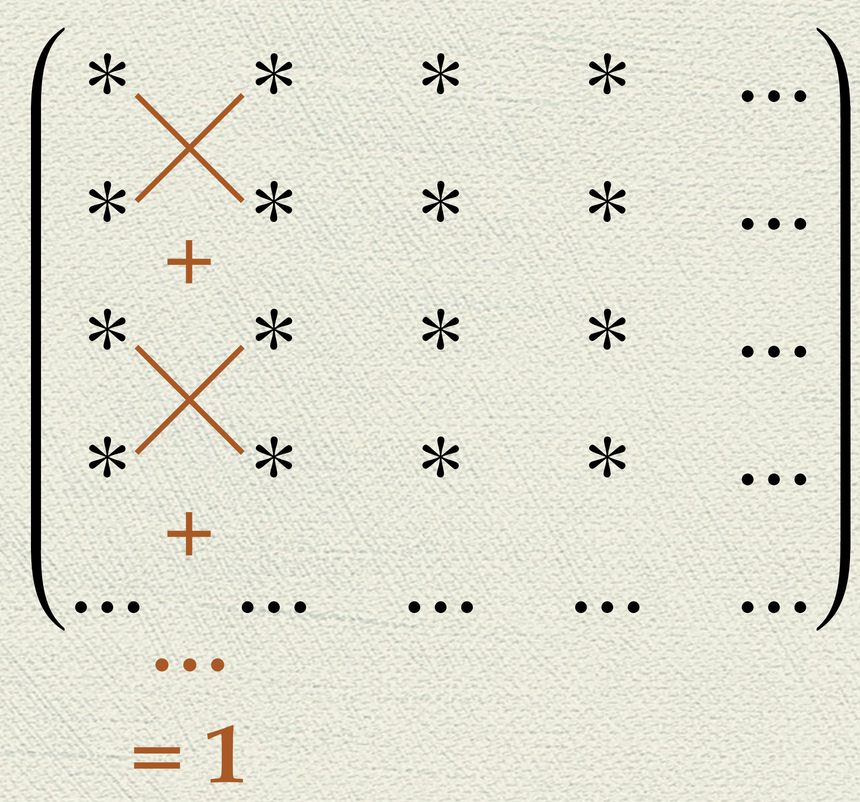
It is known for a symplectic motion that the beam emittance, the spread of the beam in the phase space, is preserved *in each plane*, once the variables are chosen properly.

Symplectic condition

$$\begin{pmatrix} * & * & * & * & \dots \\ * & * & * & * & \dots \\ * & * & * & * & \dots \\ * & * & * & * & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Symplectic condition



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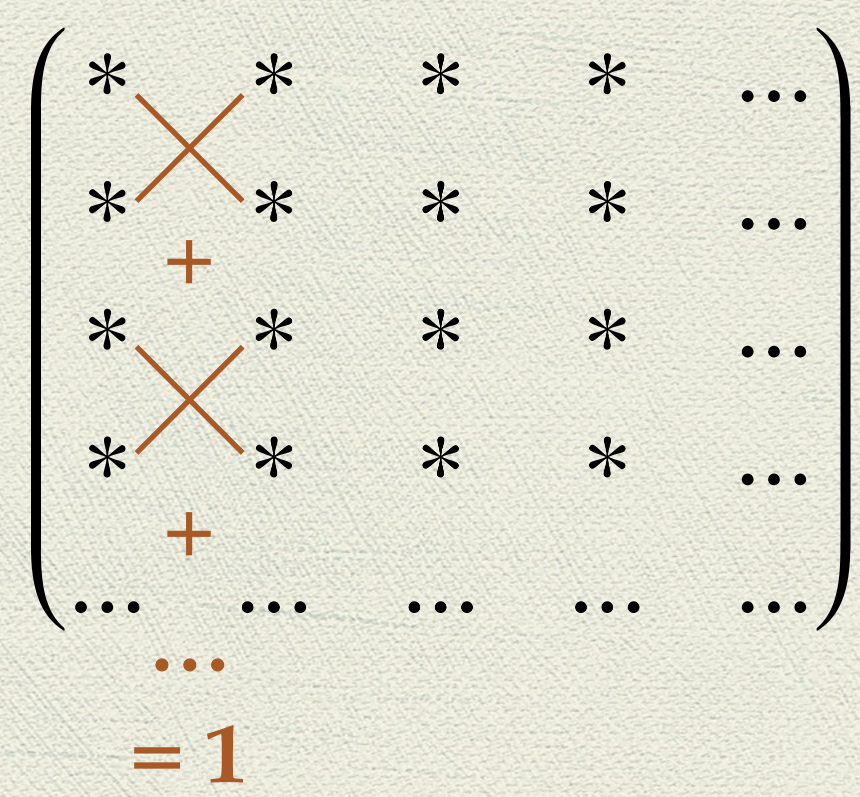

 : 2 by 2 determinant

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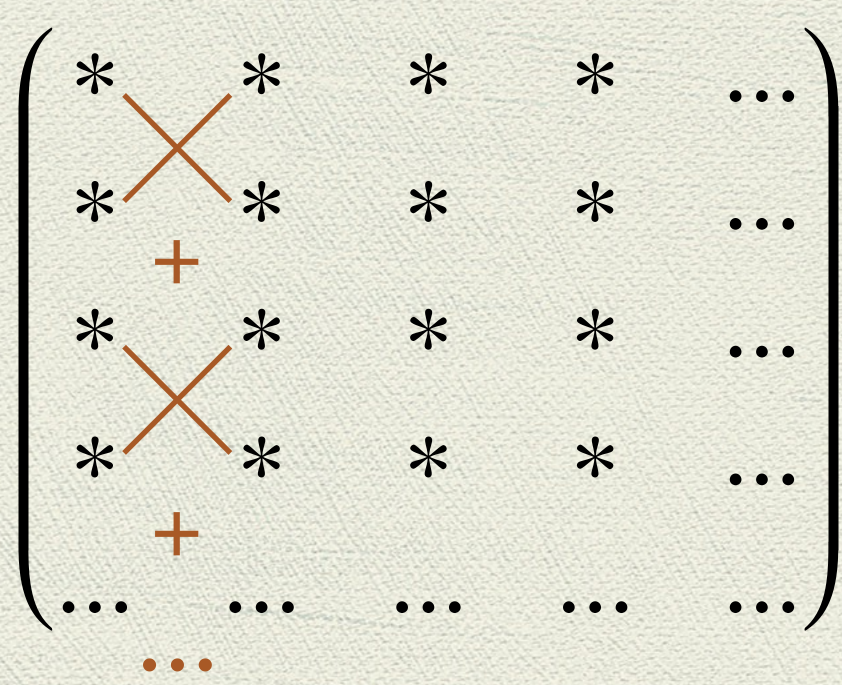
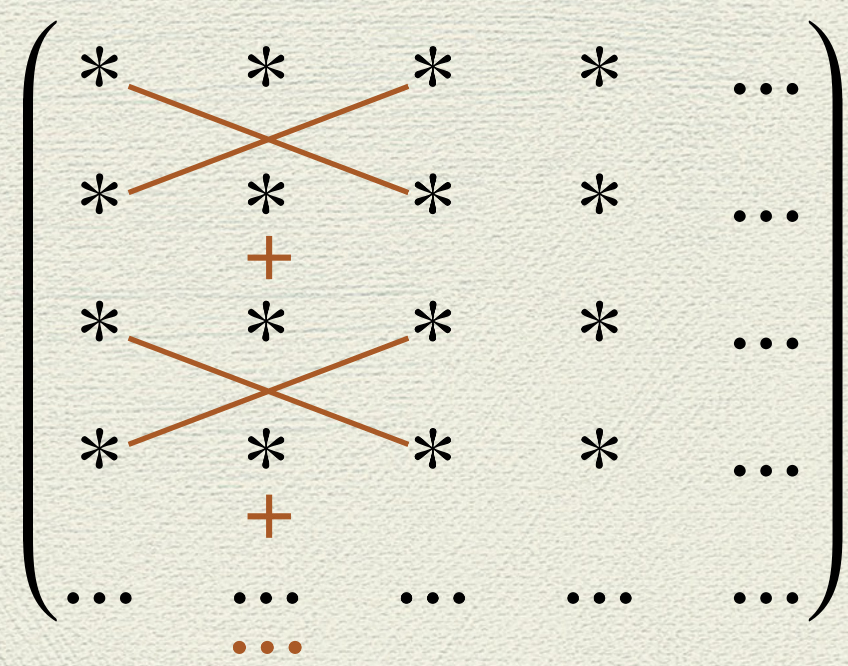


= 1



= 0

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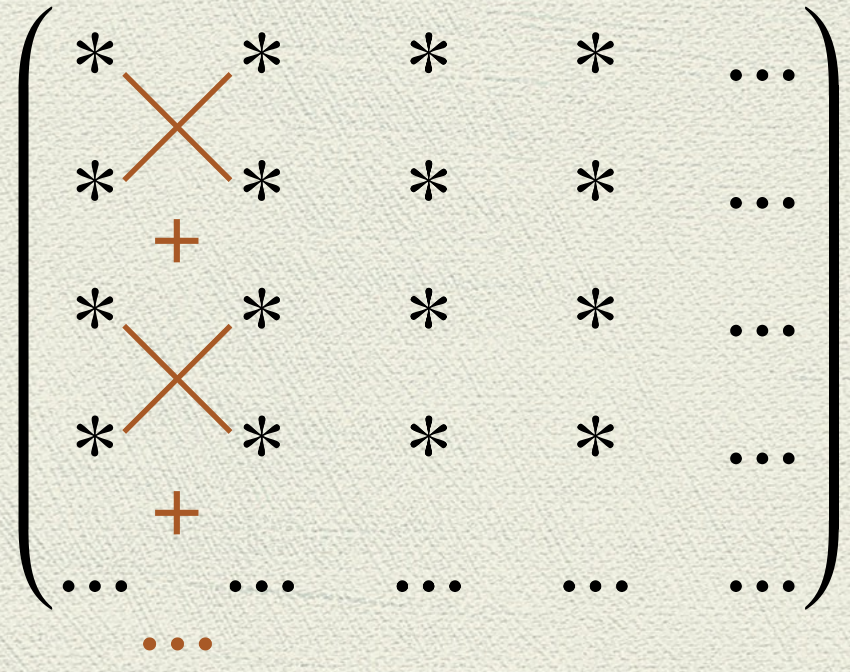
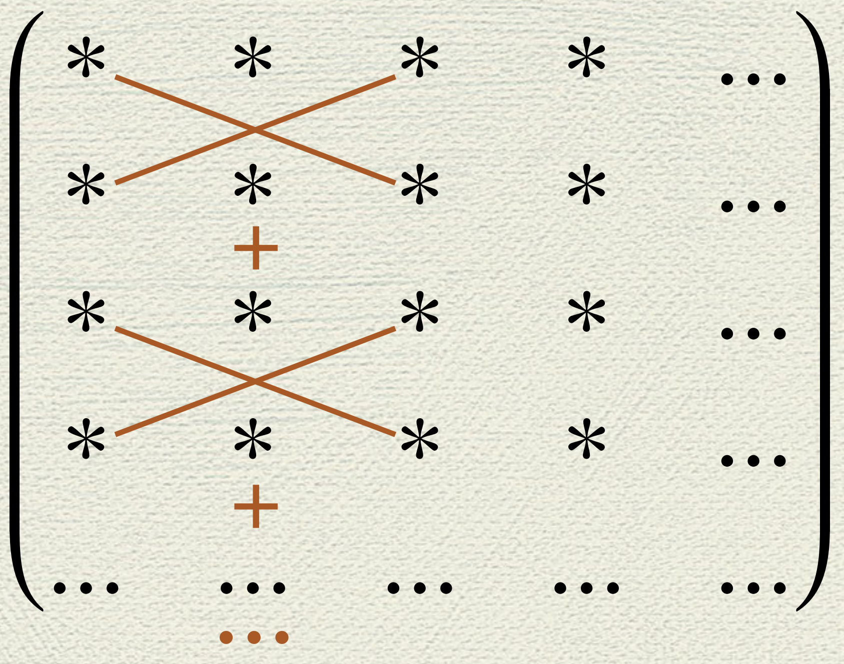


$= 1$



$= 0$

 **: 2 by 2 determinant**

The sum of 2 by 2 determinants between two columns or two rows are:

$$\begin{cases} = 1 \text{ for } (u, p_u)\ . \\ = 0 \text{ for } (u, v), (u, p_v), (p_u, p_v),\ \text{when } u \neq v\ , \end{cases}$$

Drift space

A *drift space* is a field-free region in the beam line. Its Hamiltonian is

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{E}{v_d}\,. \tag{7}$$

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The solution of the motion for a drift space with a length L is

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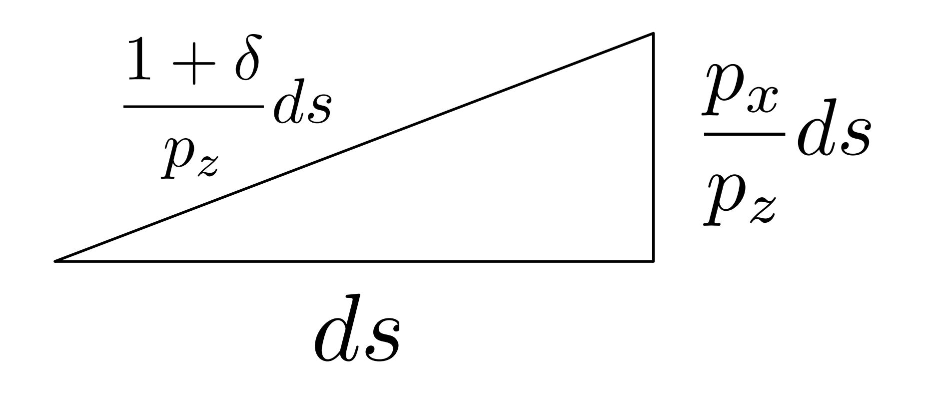
$$p_z \equiv \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}\,. \tag{9}$$



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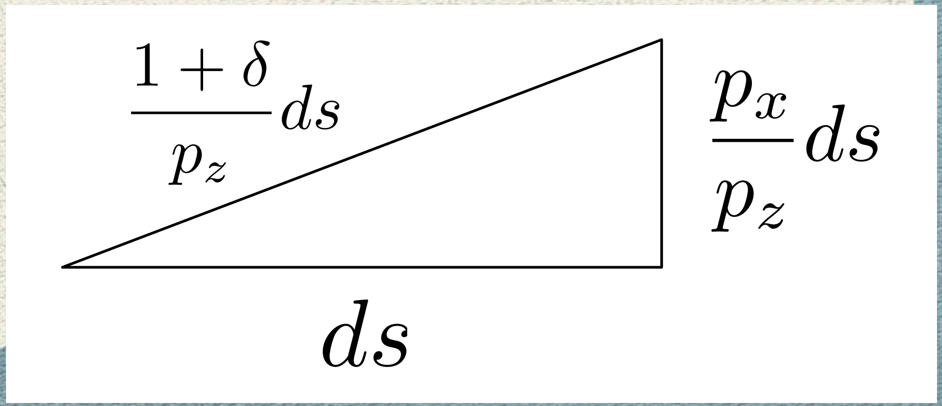
Note that the motion in a drift space is *nonlinear* in the momenta due to the $1/p_z$ dependence.



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For an on-axis particle $(x_0, p_{x0}, y_0, p_{y0}) = (0,0,0,0)$, the transfer matrix of a drift space becomes

$$\mathrm{M} = \begin{pmatrix} 1 & \frac{L}{1+\delta} & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & \frac{L}{1+\delta} & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & \frac{v - v_d}{v_d} L \\ . & . & . & . & . & 1 \end{pmatrix} , \tag{10}$$

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since, for instance,

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Thus the "length" of a drift has a $1/(1+\delta)$ dependence on the momentum offset δ. This is the source of *natural chromaticity*.

Chromaticity

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{E}{v_d} .$$

If we approximate the Hamiltonian of a drift space up to the second order of p_x and p_y:

$$H \approx -(1+\delta)\left(1 - \frac{p_x^2}{2(1+\delta)^2} - \frac{p_y^2}{2(1+\delta)^2}\right) + \frac{E}{v_d} , \tag{11}$$

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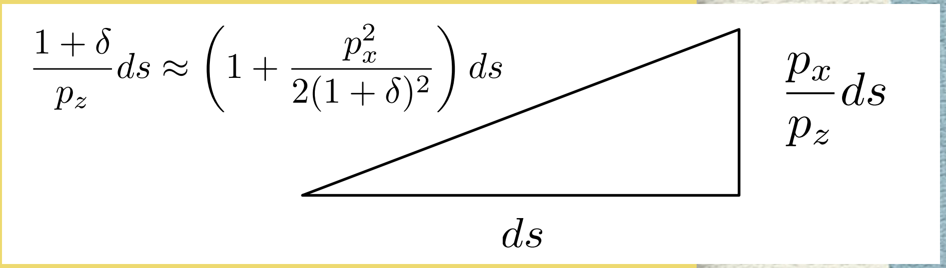
$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{p_x^2}{2(1+\delta)^2} - \frac{p_y^2}{2(1+\delta)^2} + \frac{v - v_d}{v_d} \,. \qquad (12)$$



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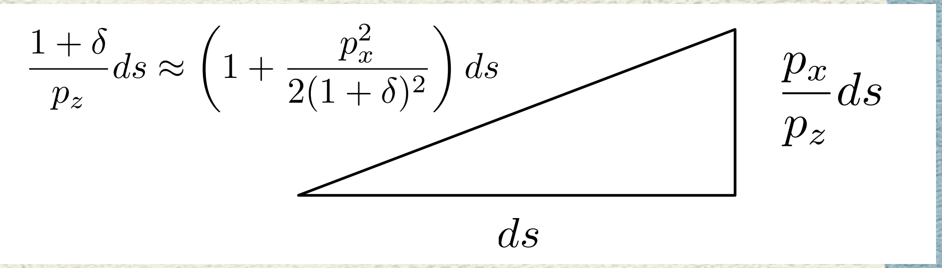
It means that the change of the path length should be proportional to the transverse actions (another canonical variable, square of the amplitudes in the normalized phase space, to be introduced later):



$$\Delta z = 2\pi\xi_x J_x + 2\pi\xi_y J_y$$

with coefficients ξ_x, ξ_y.

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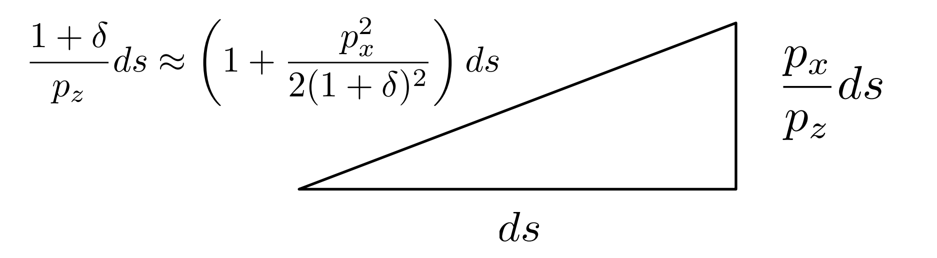
It means that the change of the path length should be proportional to the transverse actions (another canonical variable, square of the amplitudes in the normalized phase space, to be introduced later):



$$\Delta z = 2\pi\xi_x J_x + 2\pi\xi_y J_y$$

with coefficients ξ_x, ξ_y.

Then since

$$2\pi\xi_x = \frac{\partial \Delta z}{\partial J_x} = \frac{\partial}{\partial J_x}\frac{\partial H}{\partial \delta} = \frac{\partial}{\partial \delta}\frac{\partial H}{\partial J_x} = \frac{\partial \mu_x}{\partial \delta} \, , \tag{14}$$

we have realized that the coefficients $\xi_{x,y}$ correspond to the momentum derivatives of *tunes* $\frac{1}{2\pi}\frac{\partial \mu_{x,y}}{\partial \delta}$.

Chromaticity (2)

For an off-momentum particle having $d\delta$, a small section of a drift space ds has an effective length $ds/(1+d\delta) \approx -ds d\delta$. Thus the entire transfer matrix of a stable ring changes to

$$\mathrm{M}' = \begin{pmatrix} \cos\mu' + \alpha' \sin\mu' & \beta' \sin\mu' \\ -\dfrac{1+\alpha'^2}{\beta'} \sin\mu' & \cos\mu' - \alpha' \sin\mu' \end{pmatrix} =$$

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$$= \begin{pmatrix} \cos\mu + \left(\alpha + \frac{1+\alpha^2}{\beta}dsd\delta\right)\sin\mu & \beta\sin\mu - (\cos\mu - \alpha\sin\mu)dsd\delta \\ \frac{1+\alpha^2}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}. \qquad (15)$$

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Note that $\xi_{x,y}$ is always negative.

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$$\mathbf{B} = (0, B, 0), \;\; B = \frac{P_d}{e\rho} \tag{19}$$

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$$\begin{aligned} A_z &= -\frac{1}{2}\left(1 + \frac{x}{\rho}\right), \\ \frac{eB_y}{P_d} &= -\frac{1}{1 + x/\rho}\frac{\partial}{\partial x}\left[(1 + x/\rho)A_z\right] \\ &= \frac{1}{\rho} \end{aligned}$$

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\frac{dy}{ds} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{p_z}\left(1 + \frac{x}{\rho}\right), & \frac{dp_y}{ds} &= -\frac{\partial H}{\partial y} = 0, \\
\frac{dz}{ds} &= \frac{\partial H}{\partial \delta} = -\frac{1+\delta}{p_z}\left(1 + \frac{x}{\rho}\right) + \frac{v}{v_d}, & \frac{d\delta}{ds} &= -\frac{\partial H}{\partial z} = 0 \ .
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The motion in a uniform magnetic field is helical motion along the field line, which is in *y*-direction in this case. First we can derive

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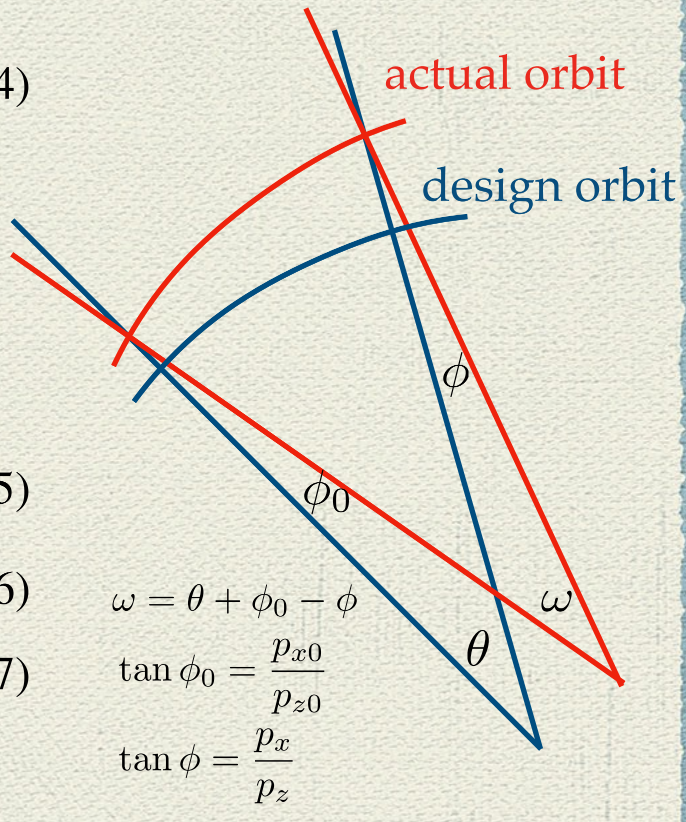

Dipole magnet (3)

The transfer matrix for a flat dipole can be obtained by differentiating above. For an on-axis particle $(x_0, p_{x0}, y_0, p_{y0}) = (0, 0, 0, 0)$ with a small bending angle $\theta \ll 1$, it is written as

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$$\mathrm{M} = \begin{pmatrix} 1 & \dfrac{L}{1+\delta} & \cdot & \cdot & \cdot & \dfrac{L\theta}{2(1+\delta)^2} \\ \cdot & 1 & \cdot & \cdot & \cdot & \theta \\ \cdot & \cdot & 1 & \dfrac{L}{1+\delta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -\theta & -\dfrac{L\theta(1+2\delta)}{2(1+\delta)^2} & \cdot & \cdot & 1 & \dfrac{v-v_d}{v_d}L \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + O(\theta)^2 , \tag{29}$$

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where $L \equiv \rho\theta$. If we compare this with the transfer matrix for a drift, the differences are the non-zero $M_{16}, M_{26}, M_{51}, M_{52}$ components, which are related to *dispersions*, and will be discussed later.

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Except for these terms, the transfer matrix of a flat, small-bending dipole is just equal to that of the drift space with the length L, including the chromatic behavior.

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A $2n$-pole multipole magnet has a magnetic field

$$B_x + iB_y = \frac{B_n}{(n-1)!}(x+iy)^{n-1} \ . \tag{30}$$

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There is no analytic solution of the motion with the Hamiltonian above, due to the $\sqrt{}$ term. The simples way of approximations is to concentrate the field into a *thin lens*. In this case the resulting Hamiltonian with the thin field at $s = 0$ is expressed as

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$$k_{n-1} \equiv \int K_{n-1} ds \approx K_{n-1} L \,. \tag{33}$$

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Then the solution is

$$p_x + ip_y = p_{x0} + ip_{y0} - \frac{k_{n-1}}{(n-1)!}(x_0 + iy_0)^{n-1} \,, \tag{34}$$

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Thin multipole magnets (2)

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In the case of a quadrupole, M is obtained by setting $n = 2$ in Eq. (36):

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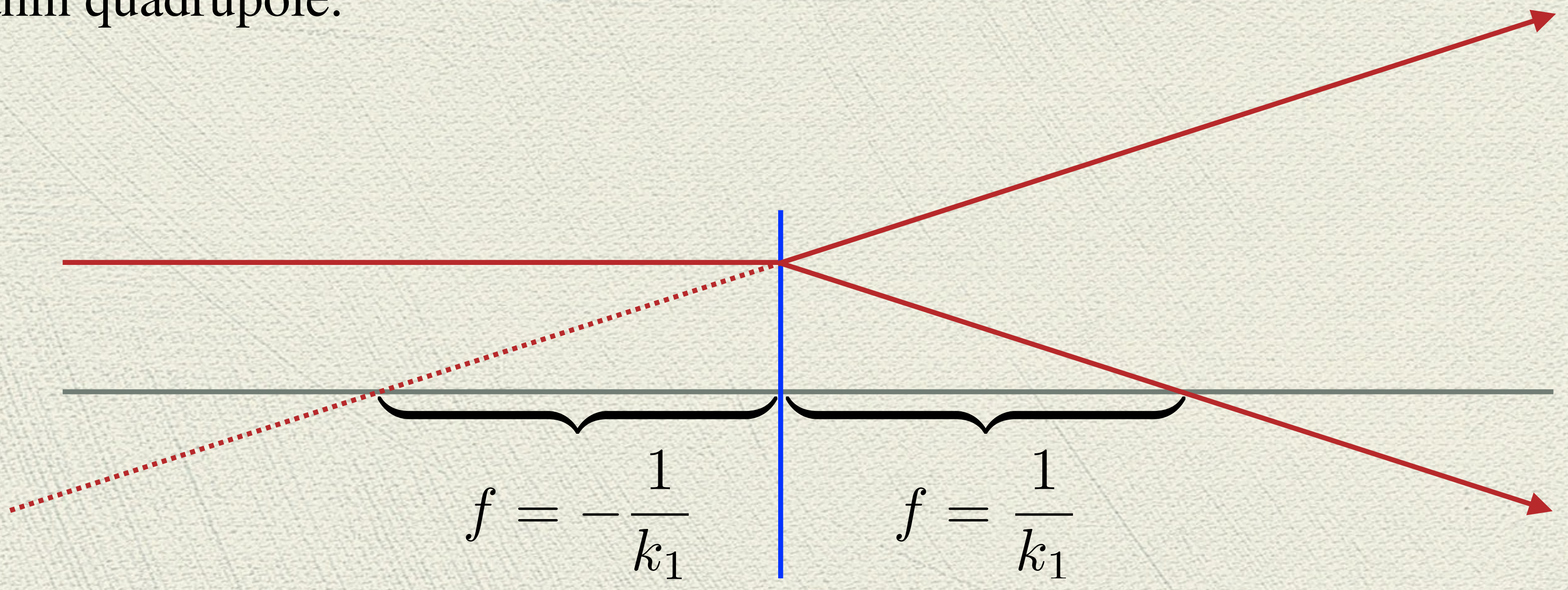

RF cavity

A thin accelerating RF cavity can be expressed by a Hamiltonian, consisting of only A_z:

$$H = \frac{eV_c}{cP_d\omega_{\mathrm{RF}}}\cos\left(-\omega_{\mathrm{RF}}\frac{z}{v} + \phi_{\mathrm{RF}}\right)\delta(s) \tag{38}$$

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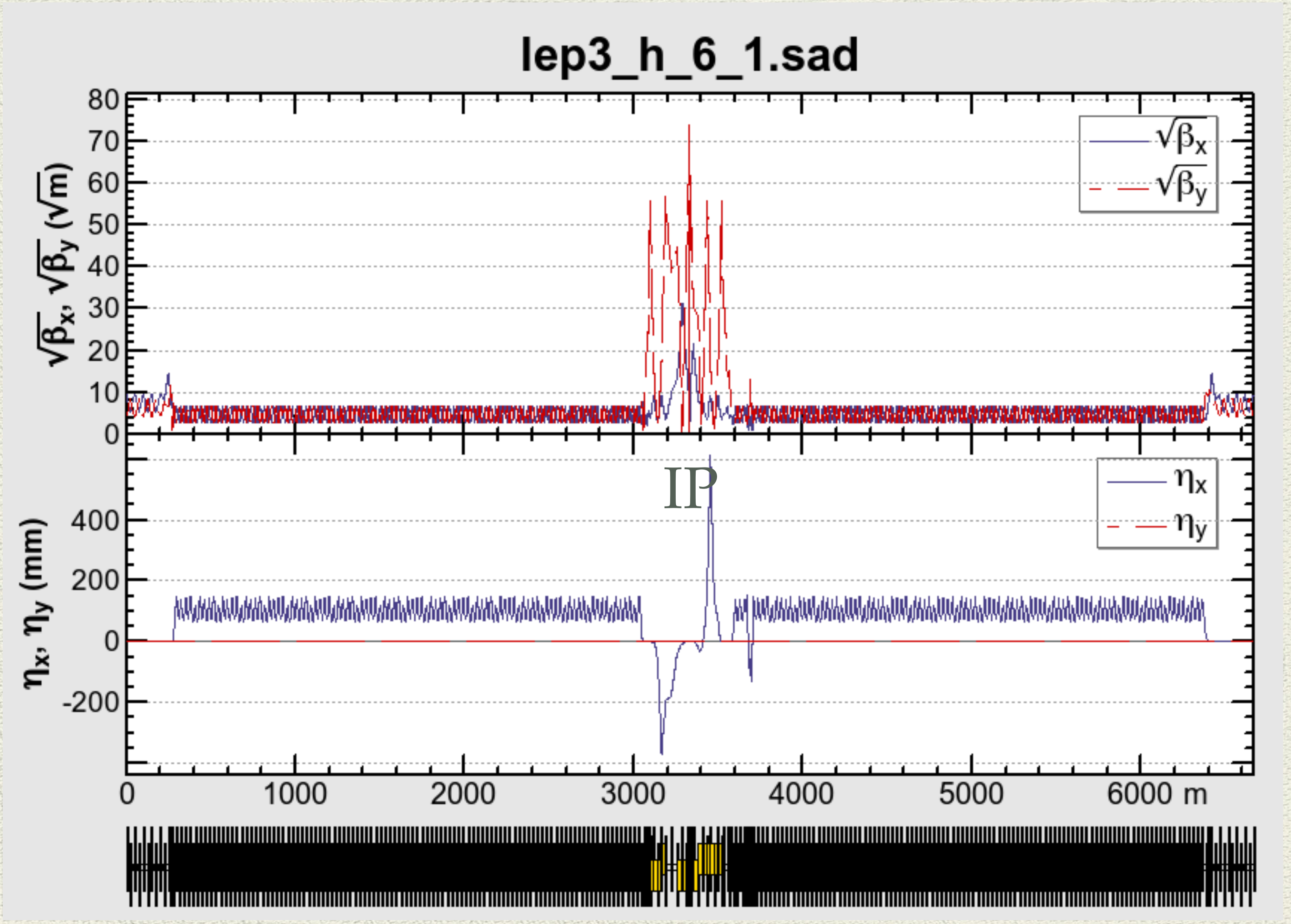
Since the RF cavity is expressed by Hamiltonian, the emittance is preserved in each plane.

FODO cell

A "typical" collider consists of

FODO cells, interaction region, dispersion suppressors, and RF.



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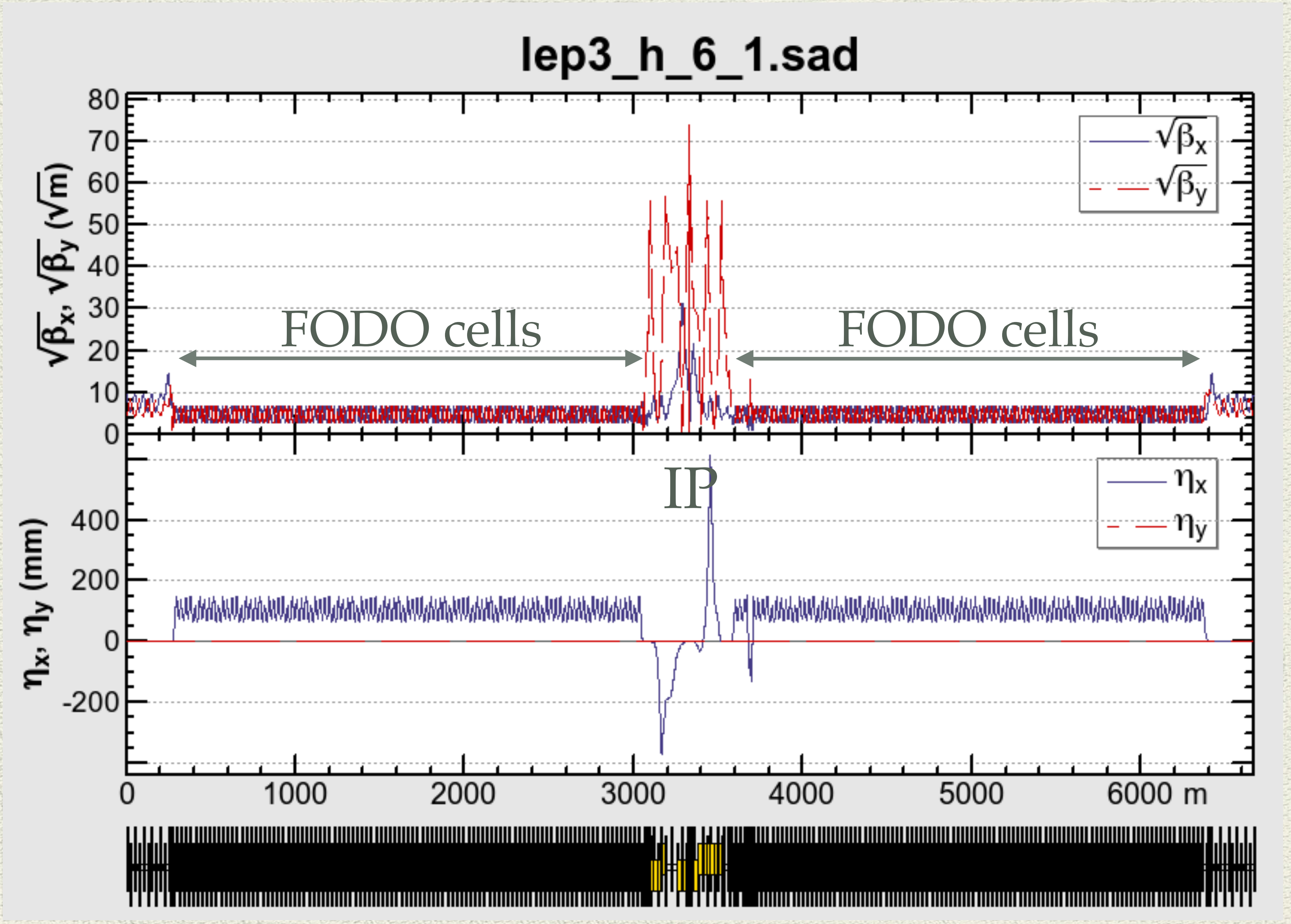

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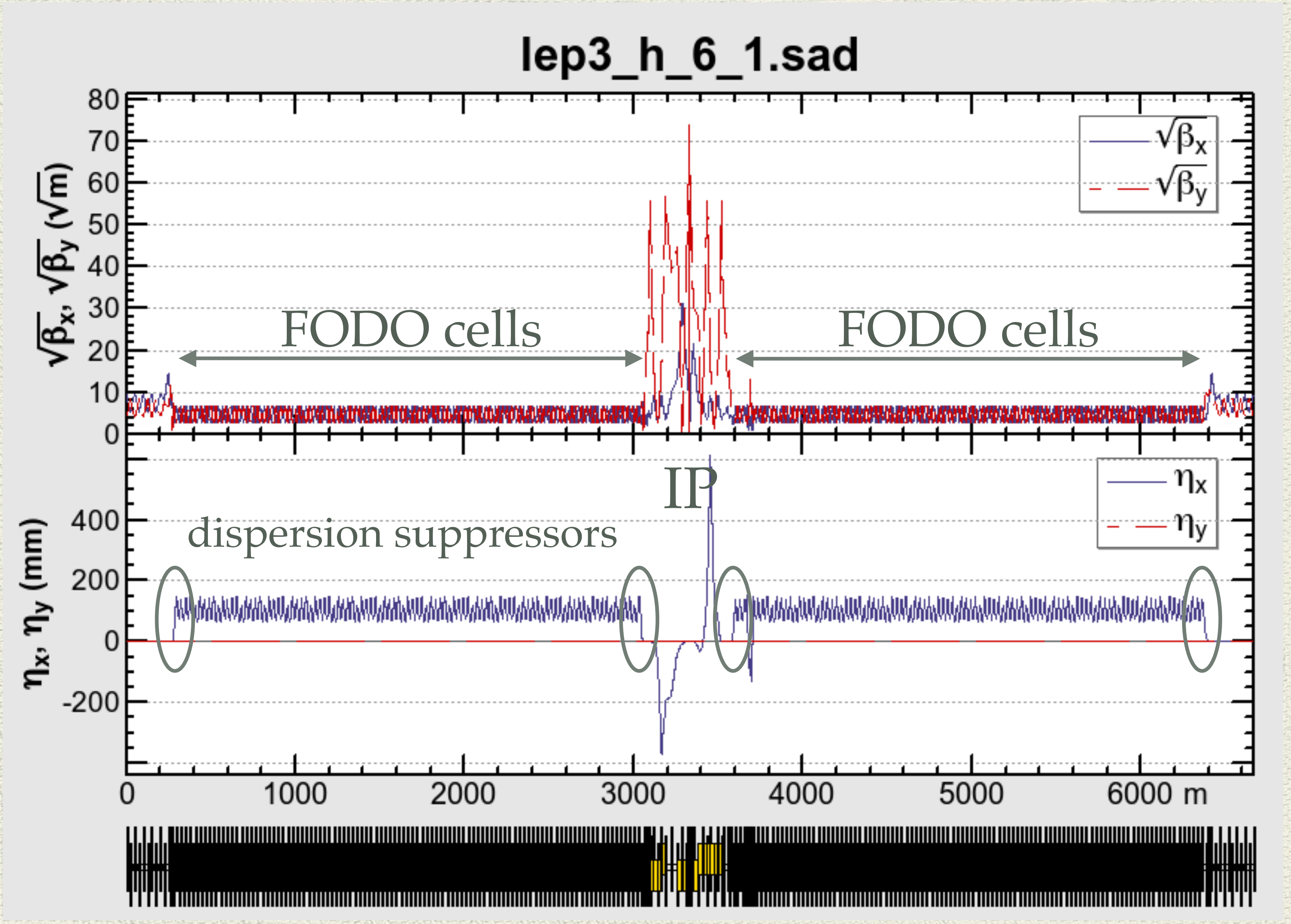

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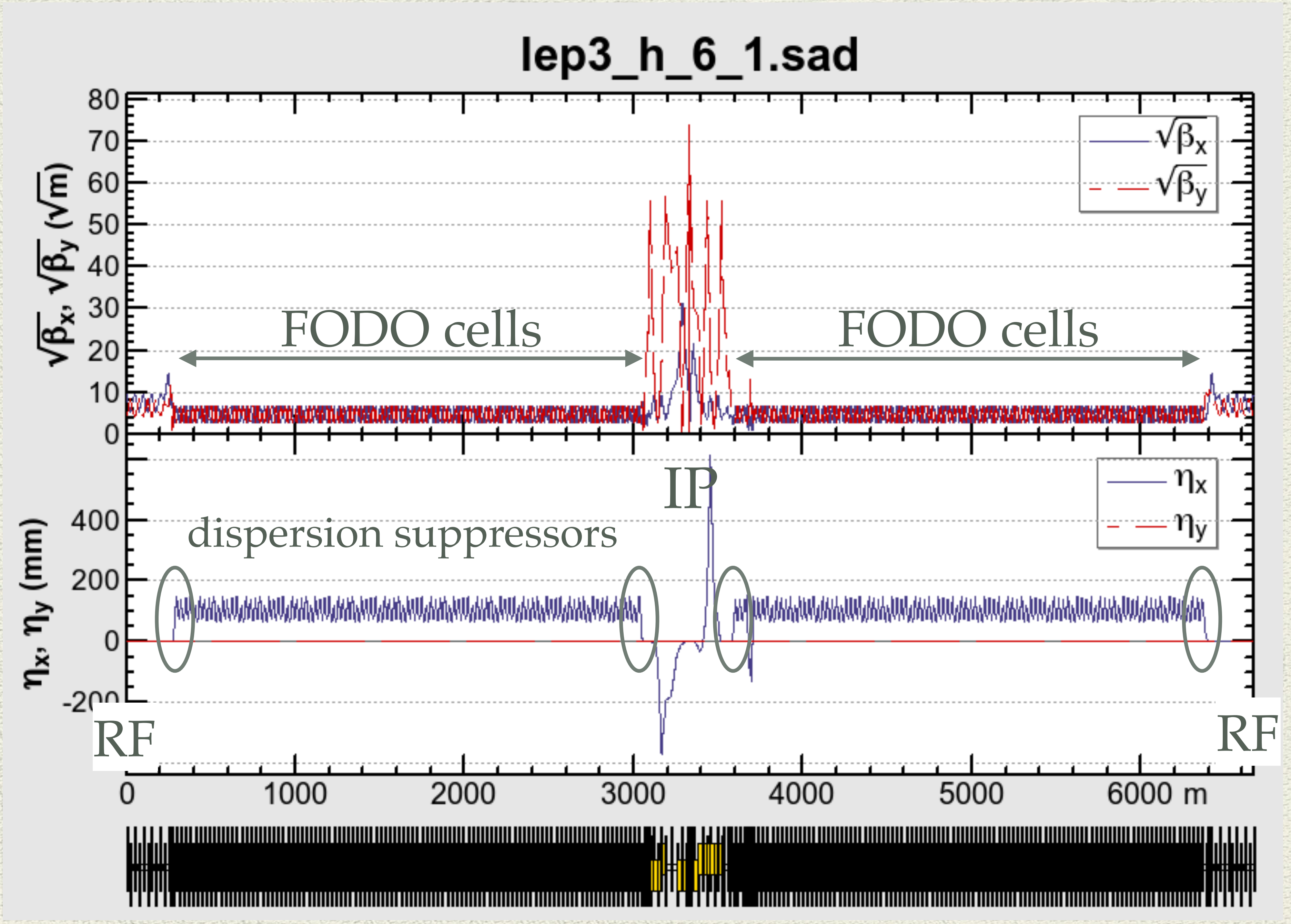

Various arc lattices

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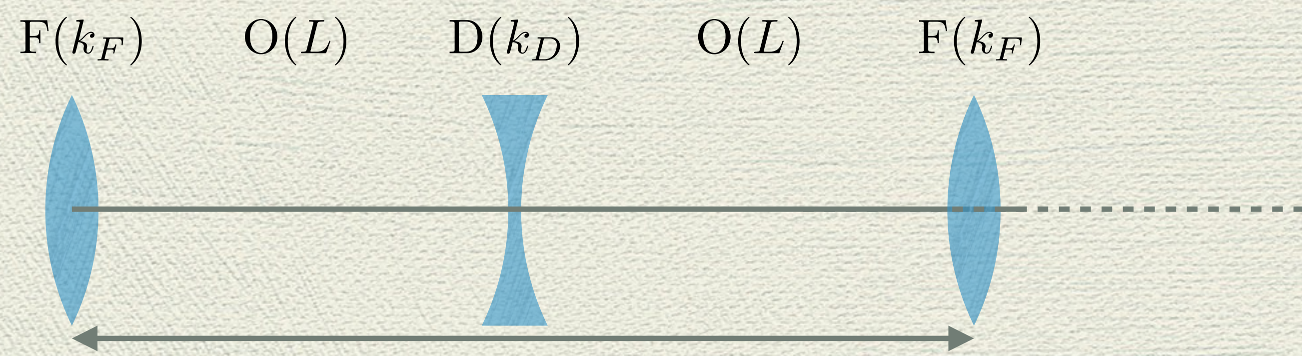

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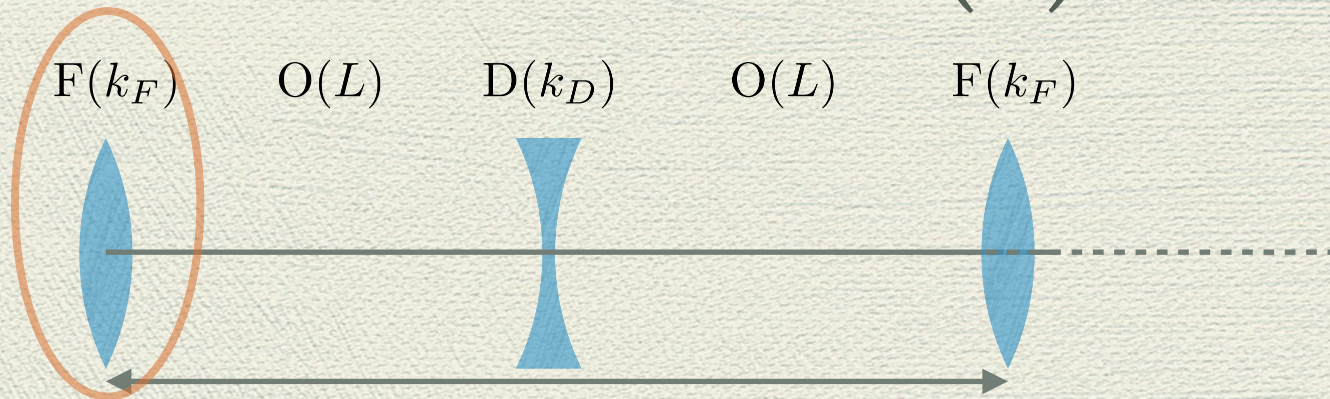

Let us consider the on-axis, on-momentum transfer matrix M of a FODO cell, between the midpoints of QF. For the time being, we consider the 2 by 2 part for x and y. The $z\delta$ parts will be discussed later.

$$M_{\substack{x\\y}} = \cdots \begin{pmatrix} 1 & 0 \\ \mp k_F/2 & 1 \end{pmatrix} \tag{42}$$

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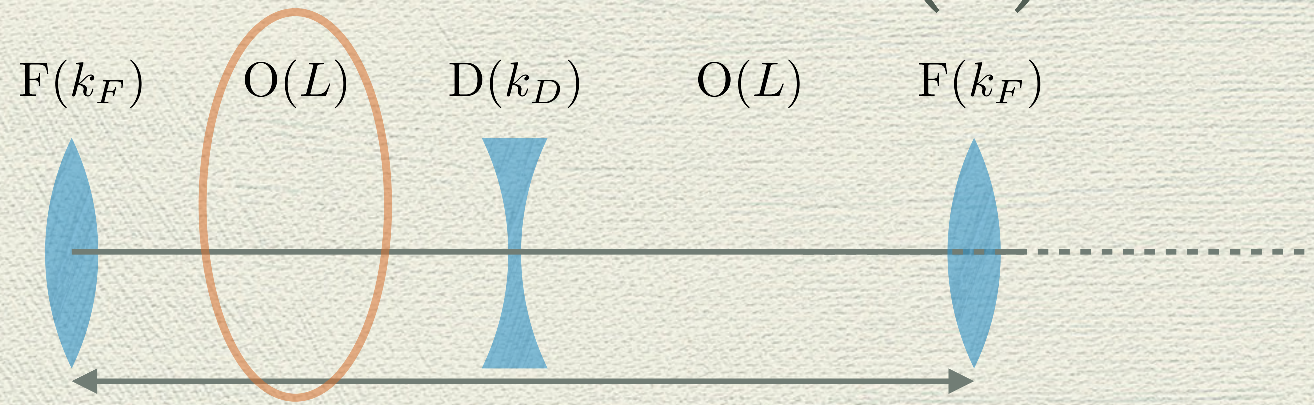

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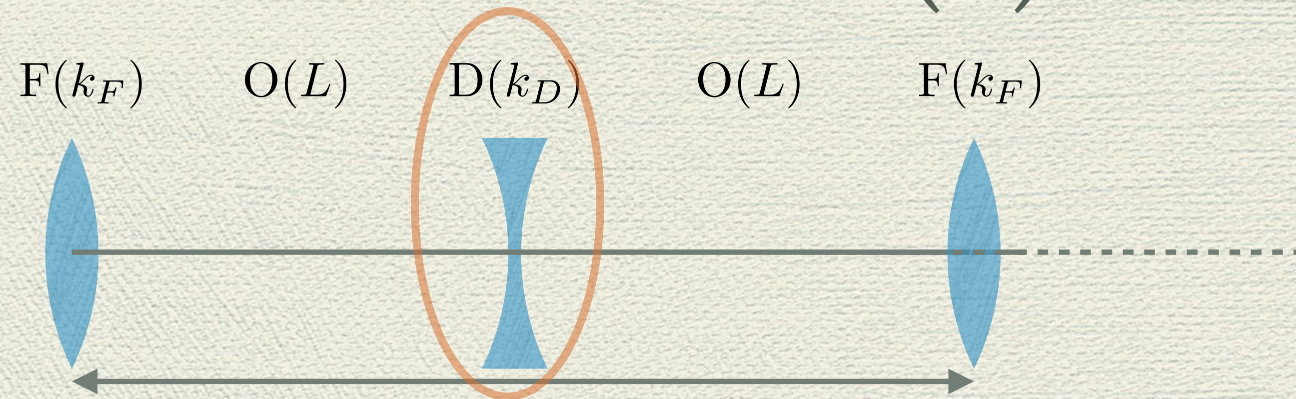

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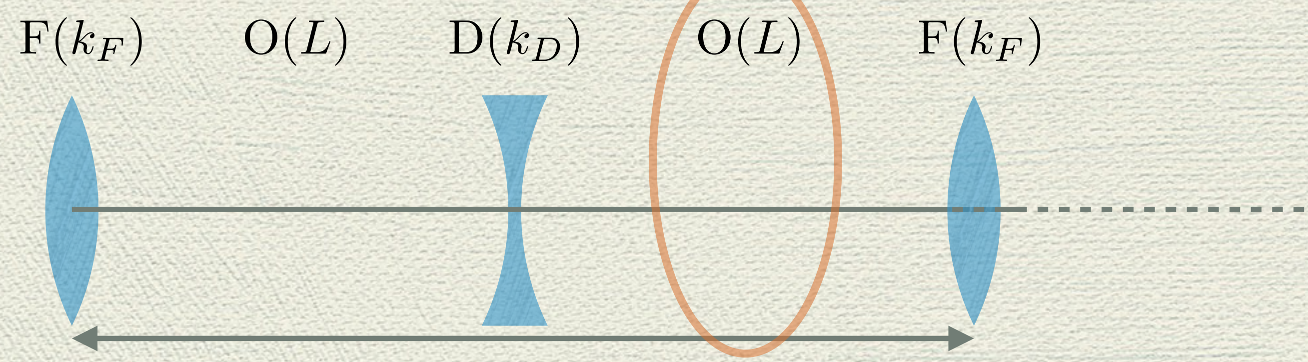

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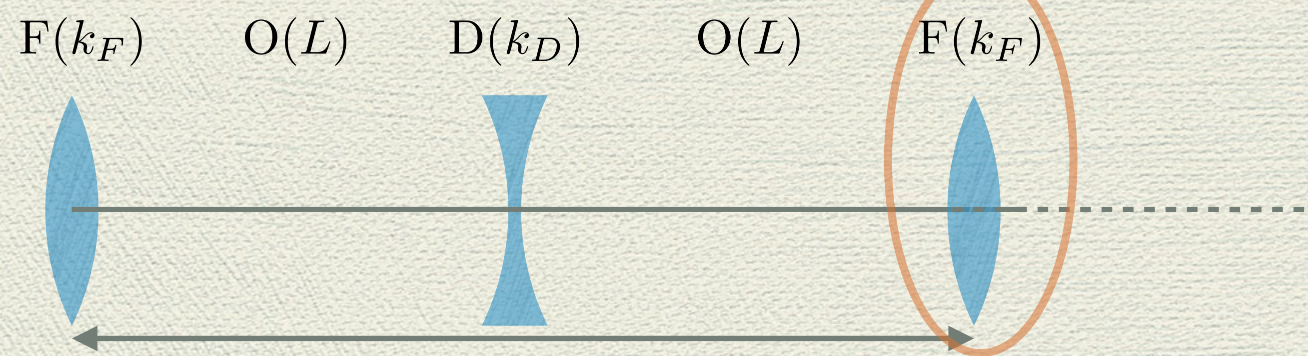

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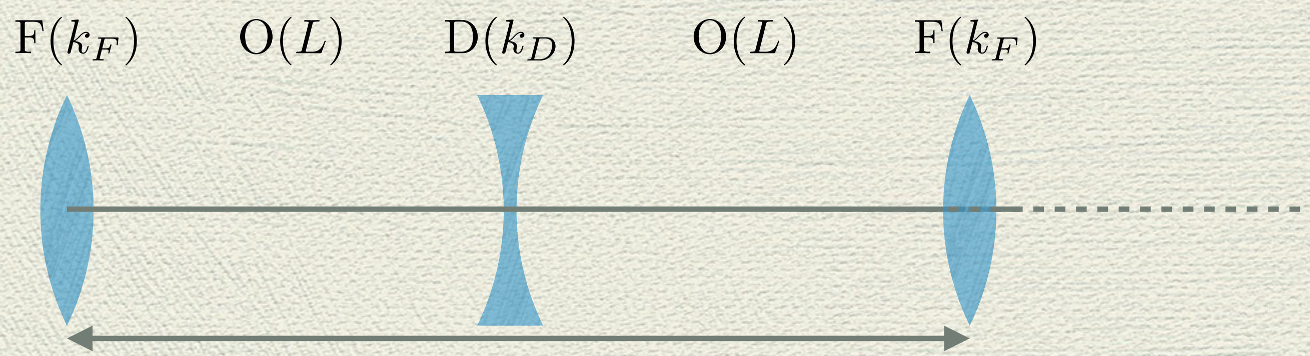

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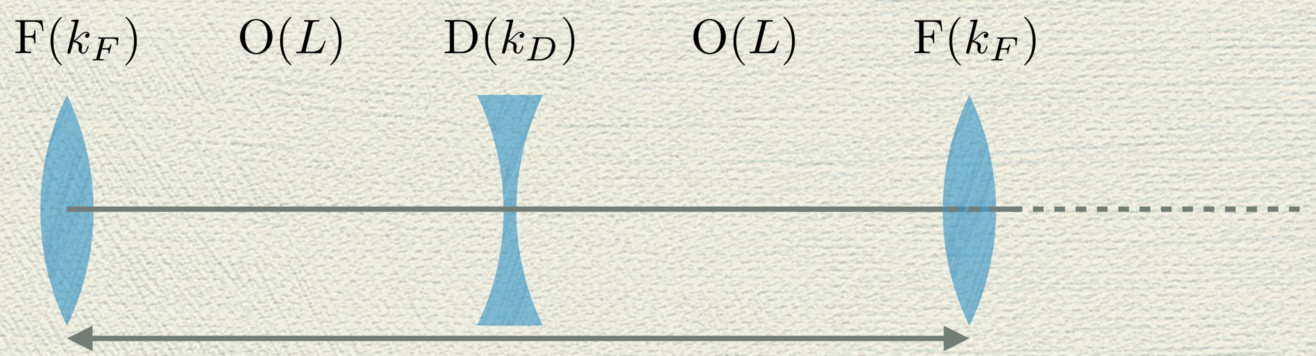

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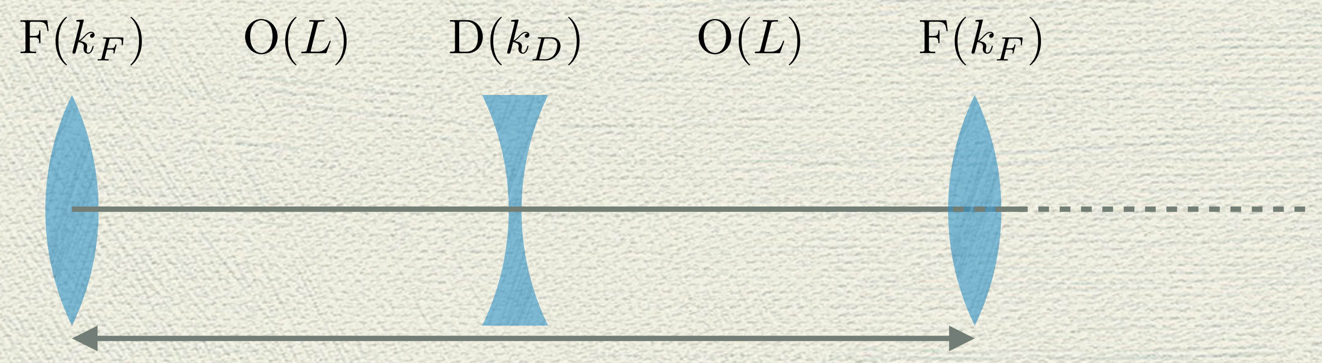

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We have used the thin-lens approximation for the quadrupoles, and introduced dimensionless parameters $c_{F,D} \equiv k_{F,D}L$. We assume $c_{F,D} > 0$.

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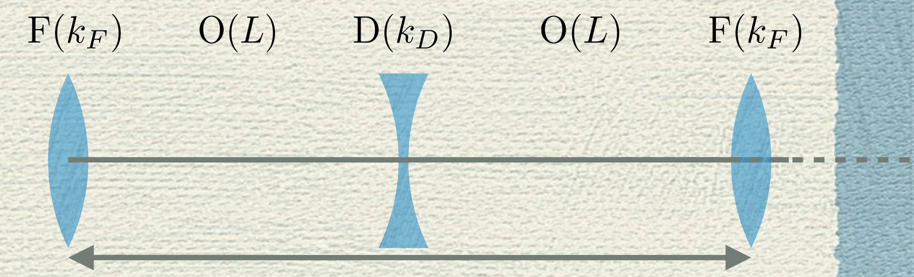

If the beam optics is periodic, the transfer matrix above must be equal to

$$\mathrm{M}_{x,y} = \begin{pmatrix} \cos\mu_{x,y} + \alpha_{x,y}\sin\mu_{x,y} & \beta_{x,y}\sin\mu_{x,y} \\ \dfrac{1+\alpha_{x,y}^2}{\beta_{x,y}}\sin\mu_{x,y} & \cos\mu_{x,y} - \alpha_{x,y}\sin\mu_{x,y} \end{pmatrix}. \tag{45}$$

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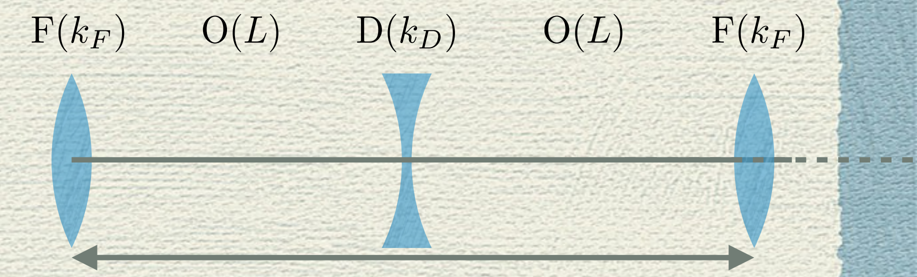

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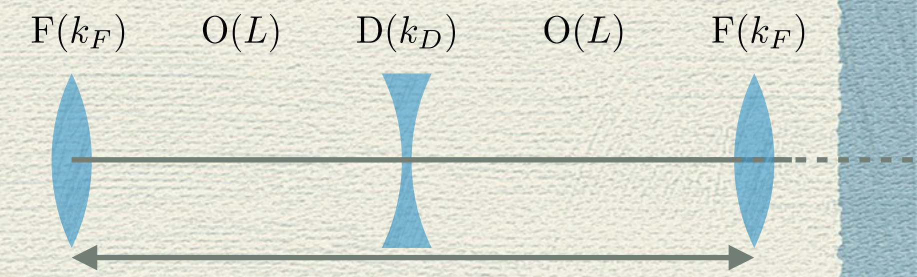

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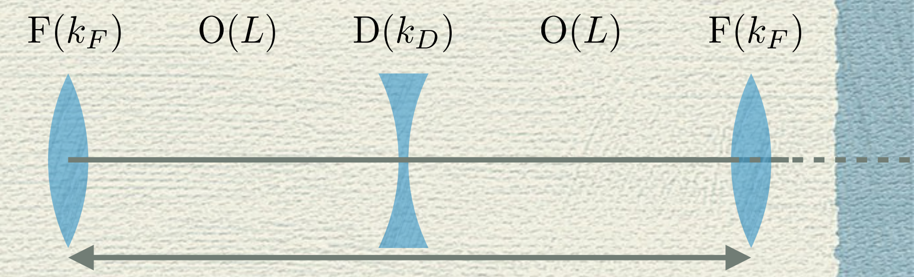

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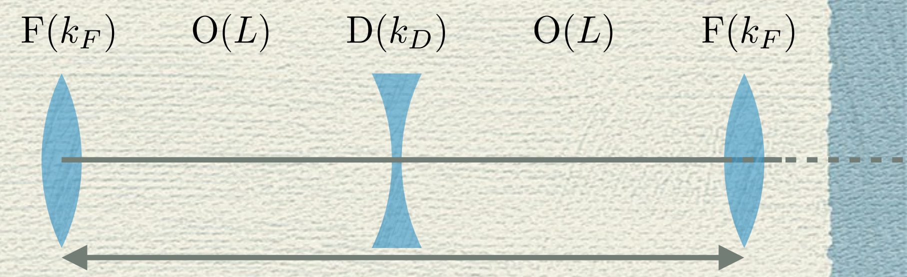

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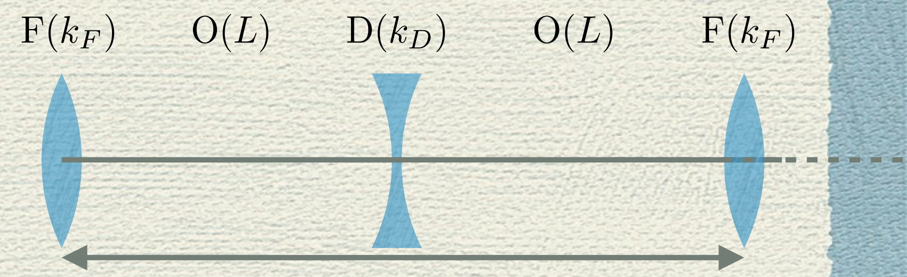
FODO cell (3)



$$\mathrm{M}_{\substack{x\\y}} = \begin{pmatrix} 1 \pm (c_D - c_F) - \dfrac{c_D c_F}{2} & L(2 \pm c_D) \\ \dfrac{\pm(c_D - c_F) - c_F c_D + c_F^2(1 \pm c_D/2)}{L} & 1 \pm (c_D - c_F) - \dfrac{c_F c_D}{2} \end{pmatrix}$$

If the beam optics is periodic, the transfer matrix above must be equal to

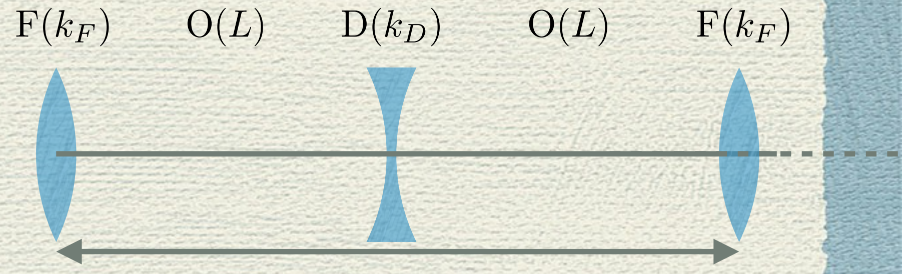
$$\mathrm{M}_{x,y} = \begin{pmatrix} \cos\mu_{x,y} + \alpha_{x,y}\sin\mu_{x,y} & \beta_{x,y}\sin\mu_{x,y} \\ \dfrac{1+\alpha_{x,y}^2}{\beta_{x,y}}\sin\mu_{x,y} & \cos\mu_{x,y} - \alpha_{x,y}\sin\mu_{x,y} \end{pmatrix}. \tag{45}$$

By comparing two expressions, we obtain the values of the Twiss parameters at the center of QF as

$$\alpha_{x,y} = 0\,, \tag{46}$$

$$\cos\mu_{x,y} = 1 + (c_{D,F} - c_{F,D}) - \frac{c_F c_D}{2}\,, \tag{47}$$

$$\beta_{x,y} = \frac{2 \pm c_D}{\sin\mu_{x,y}} L\,. \tag{48}$$



FODO cell (3)

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Then once the phase advances $\mu_{x,y}$ are given, parameters $c_{F,D}$ are determined as

$$c_F = \frac{1}{4}\left(\cos\mu_y - \cos\mu_x + \sqrt{32 - 16(\cos\mu_x + \cos\mu_y) + (\cos\mu_x - \cos\mu_y)^2}\right) \tag{49}$$

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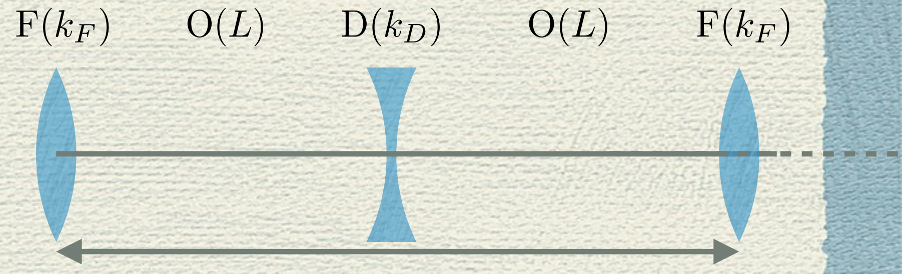

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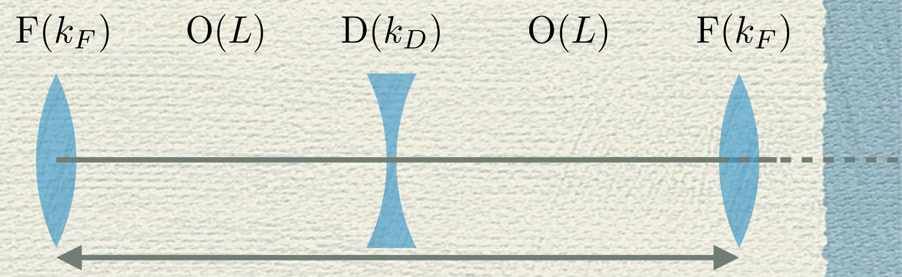
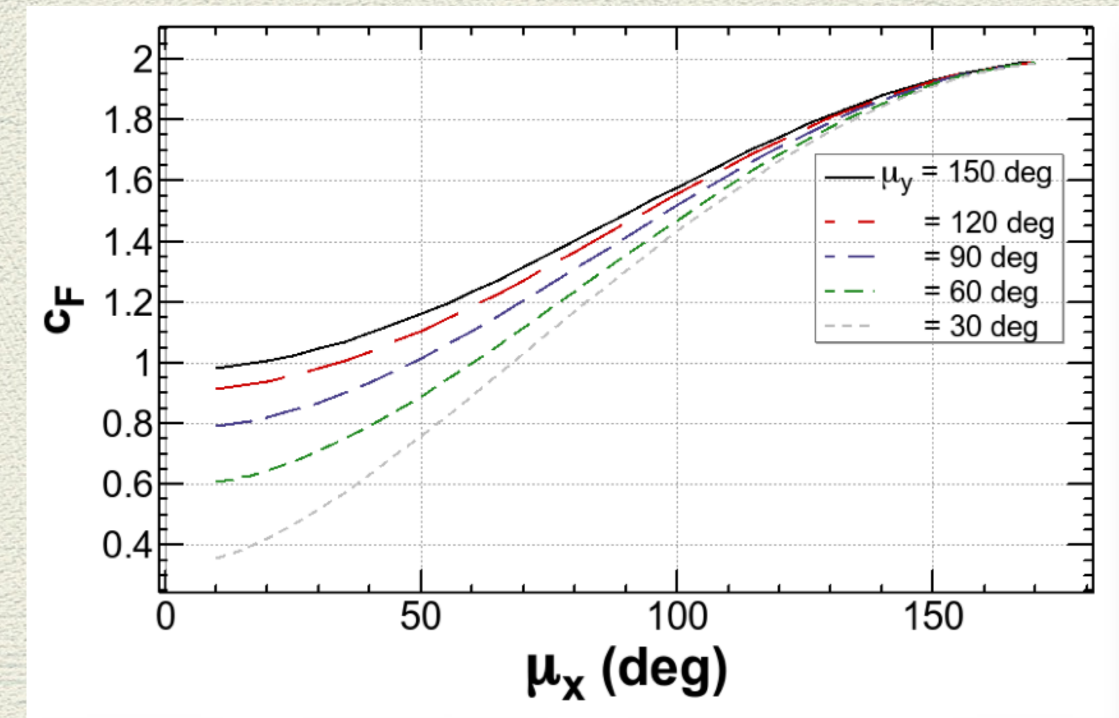
The value of β-function at QF is obtained by Eq. (48). Note that $0 < C_{F,D} < 2$.

Dependence of FODO cell parameters on $\mu_{x,y}$





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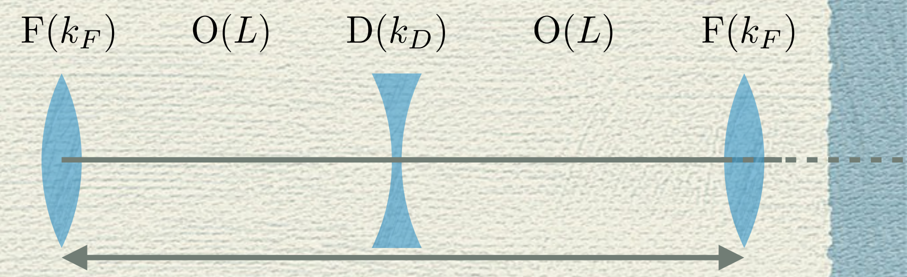

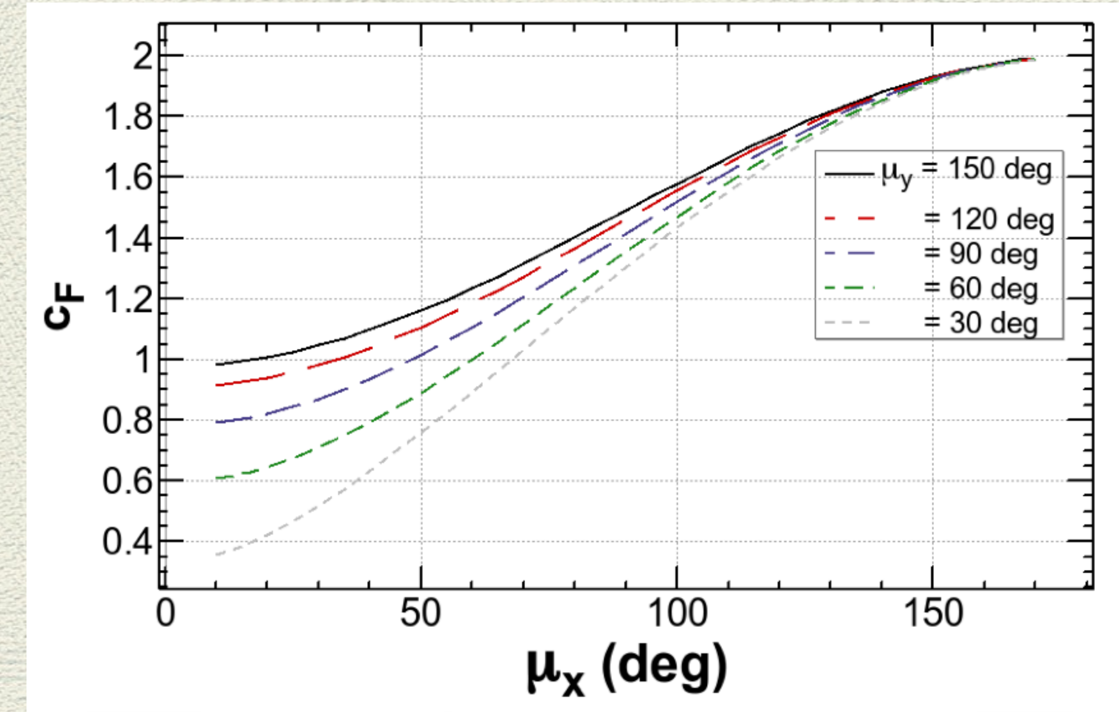

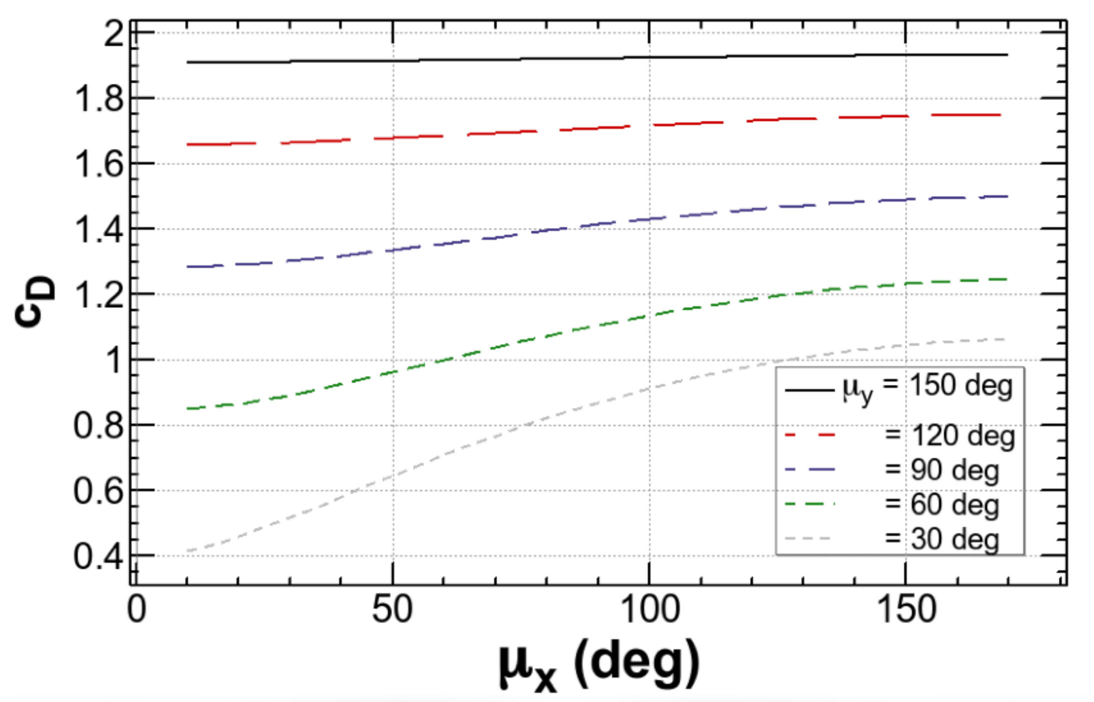

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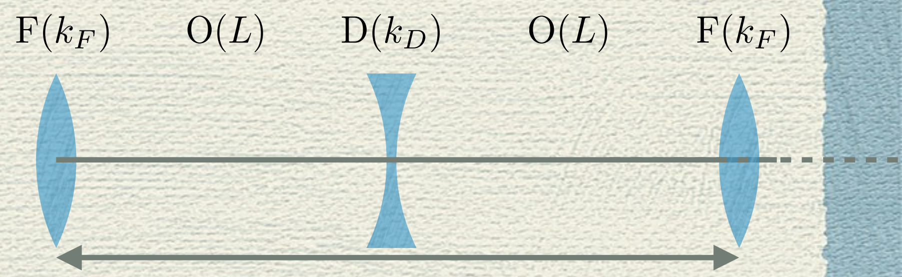
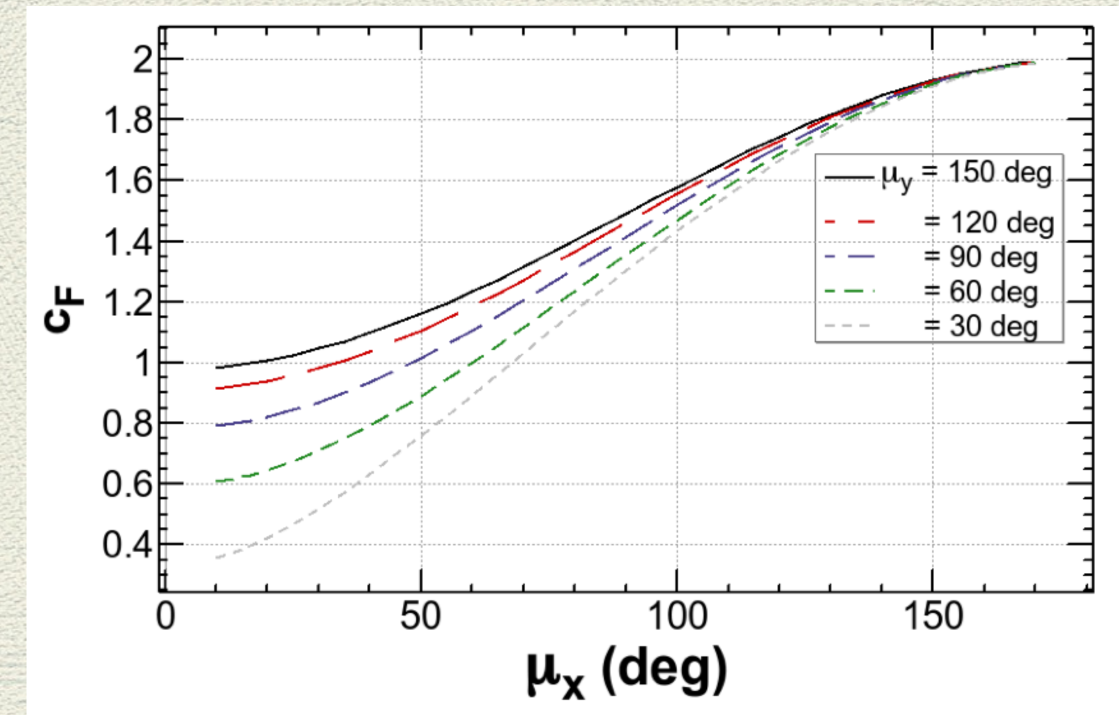
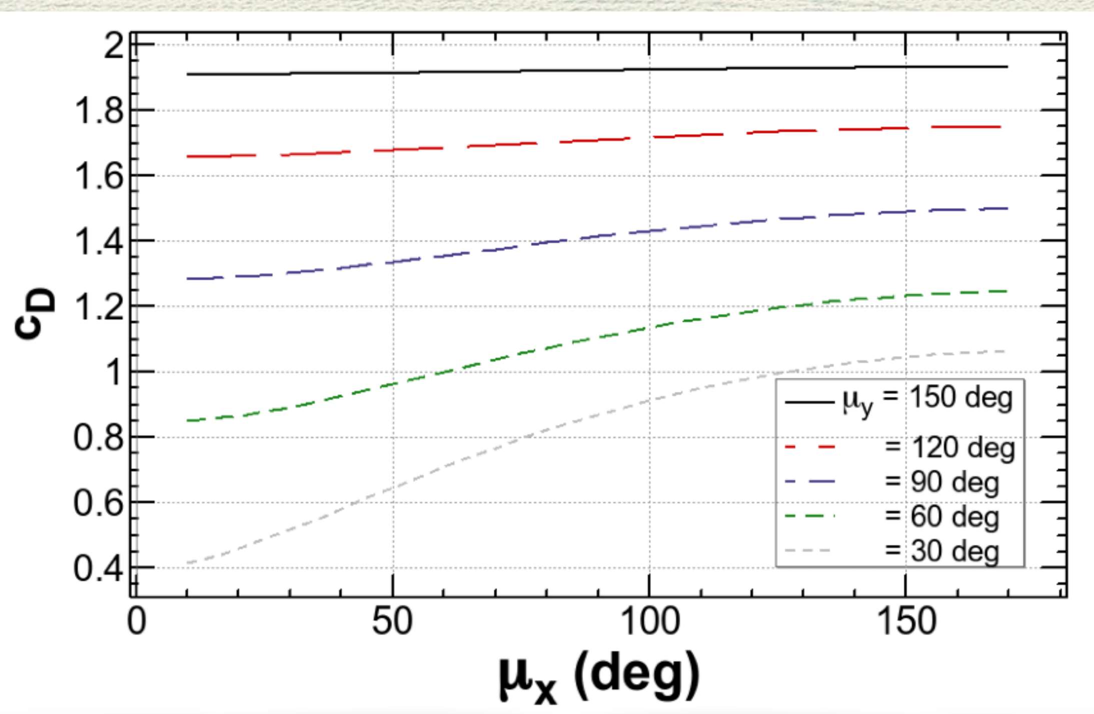
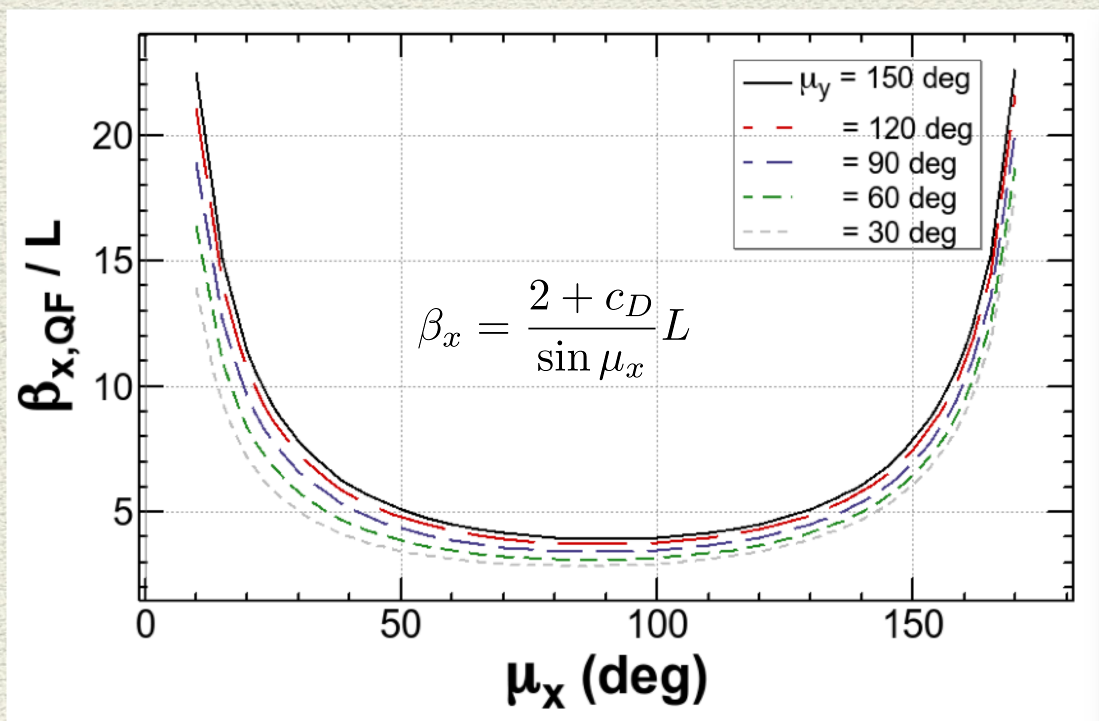
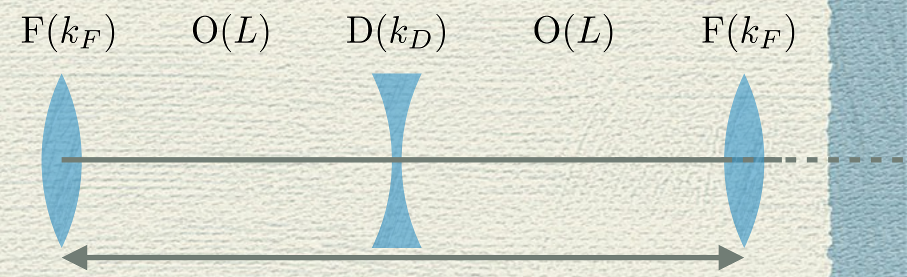


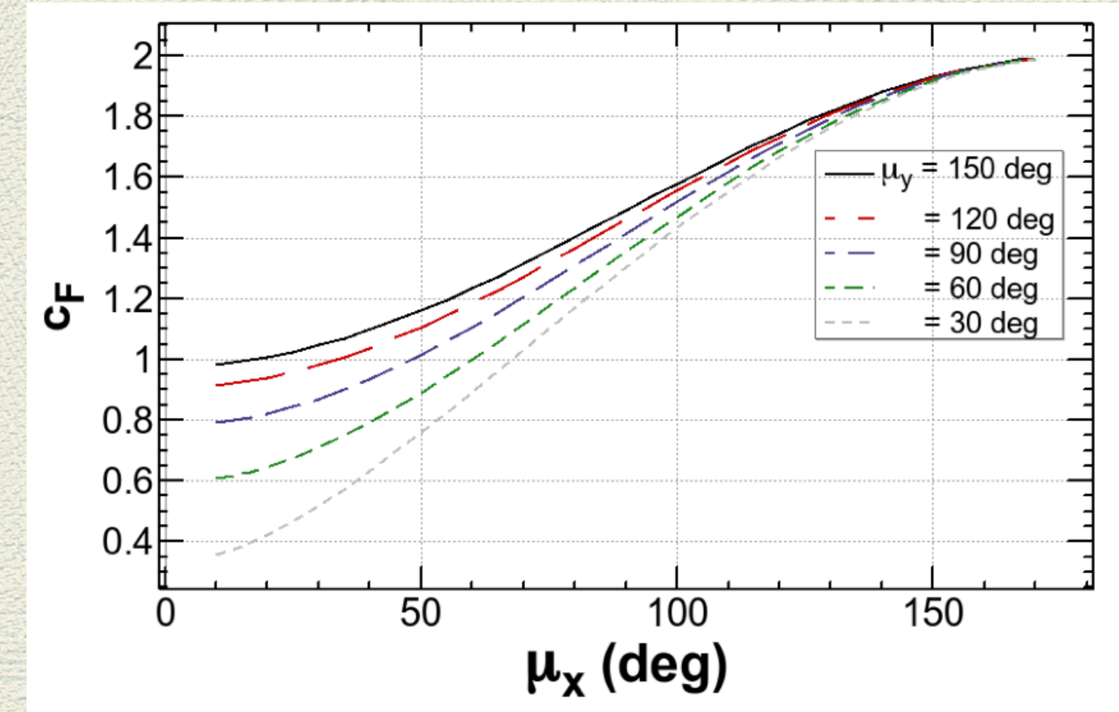


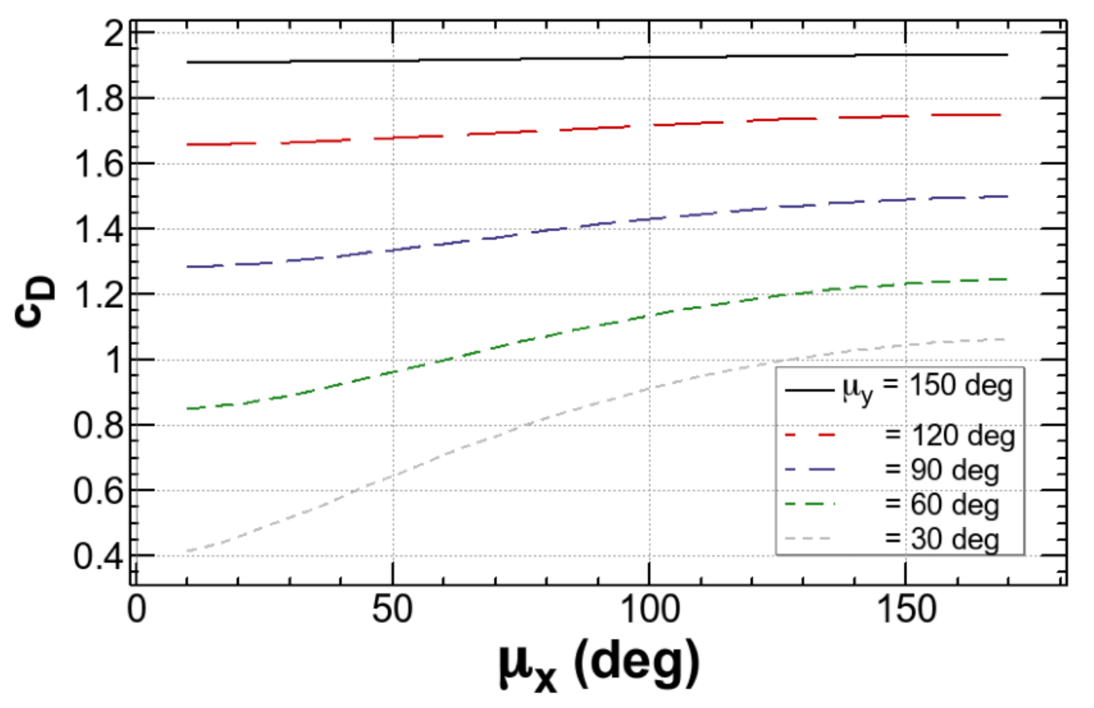


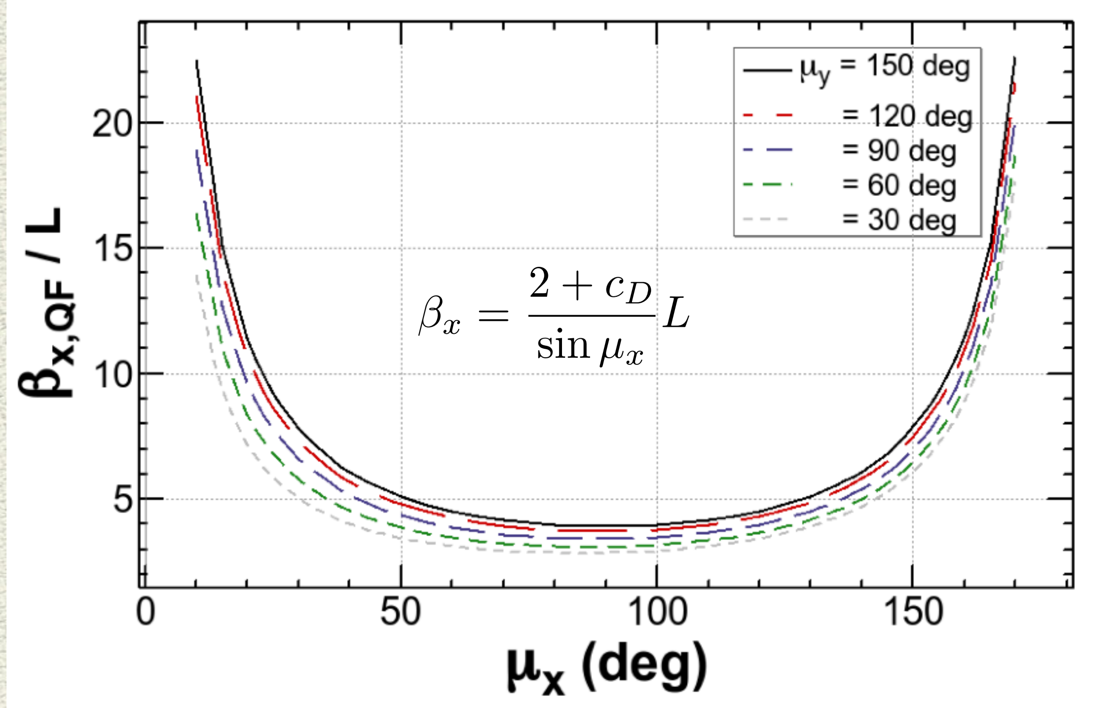


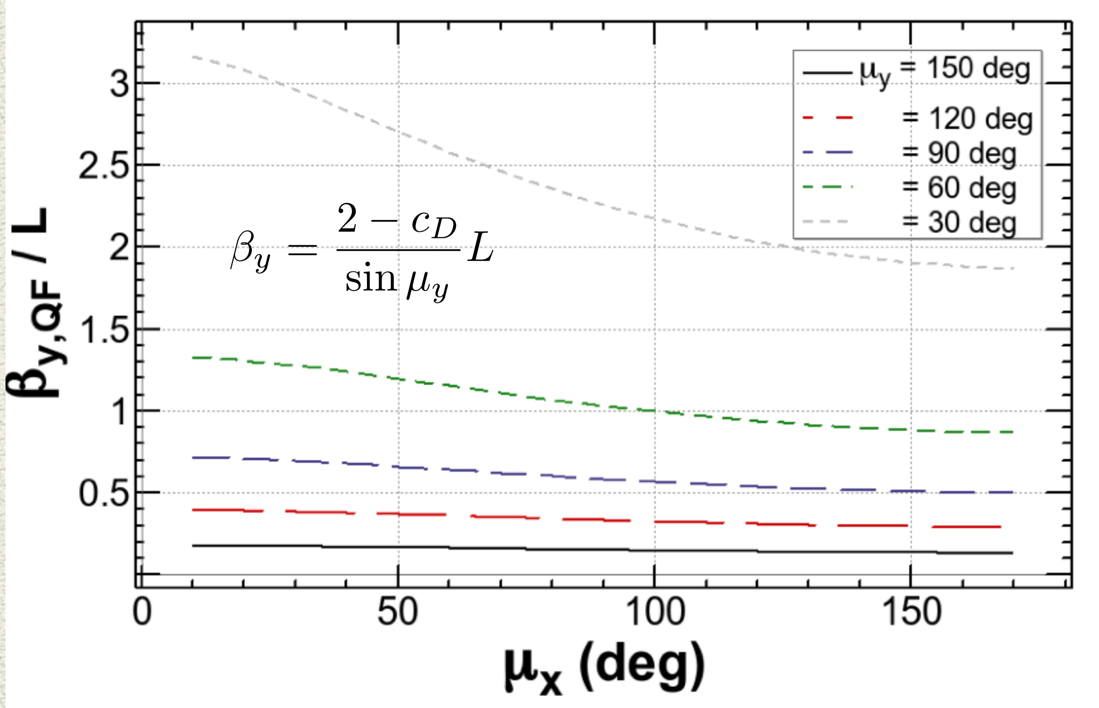
Dependence of FODO cell parameters on $\mu_{x,y}$











Twiss parameters along a beam line

Once we have the Twiss parameters α_1, β_1 at a particular location $s = s_1$ in a beam line, the 2 by 2 transfer matrix M from $s1$ to s is expressed as

$$\mathrm{M} = \left[\begin{pmatrix} 1 & 0 \\ \alpha(s) & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{\beta(s)} & 0 \\ 0 & \sqrt{\beta(s)} \end{pmatrix}\right]^{-1}$$

$$\times \begin{pmatrix} \cos\phi(s) & \sin\phi(s) \\ -\sin\phi(s) & \cos\phi(s) \end{pmatrix}$$

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Physical to normal conversion at entrance

where $\alpha(s)$, $\beta(s)$, and $\phi = \phi(s)$ are the Twiss parameters at s and the phase advance from s_1 to s, respectively.

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Rotation in the normal phase space

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$$= \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_1}}(\cos\phi + \alpha_1 \sin\phi) & \sqrt{\beta(s)\beta_1}\sin\phi \\ -\frac{(1+\alpha(s)\alpha_1)\sin\phi + (\alpha(s)-\alpha_1)\cos\phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\frac{\beta_1}{\beta(s)}}(\cos\phi - \alpha(s)\sin\phi) \end{pmatrix}, \quad (51)$$

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&\times \left[\begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{\beta_1} & 0 \\ 0 & \sqrt{\beta_1} \end{pmatrix}\right] \\
&= \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_1}}(\cos\phi + \alpha_1 \sin\phi) & \sqrt{\beta(s)\beta_1}\sin\phi \\ -\dfrac{(1+\alpha(s)\alpha_1)\sin\phi + (\alpha(s)-\alpha_1)\cos\phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\dfrac{\beta_1}{\beta(s)}}(\cos\phi - \alpha(s)\sin\phi) \end{pmatrix}, \quad (51)
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Normal to physical conversion at exit

Rotation in the normal phase space

Physical to normal conversion at entrance

where $\alpha(s)$, $\beta(s)$, and $\phi = \phi(s)$ are the Twiss parameters at s and the phase advance from s_1 to s, respectively.

The above means that a symplectic motion is reduced to a *circular motion* in the normalized phase space:

$$(u, p_u) = \left(\frac{x}{\sqrt{\beta}},\ p_x\sqrt{\beta} + x\frac{\alpha}{\sqrt{\beta}}\right). \quad (52)$$

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Normal to physical conversion at exit

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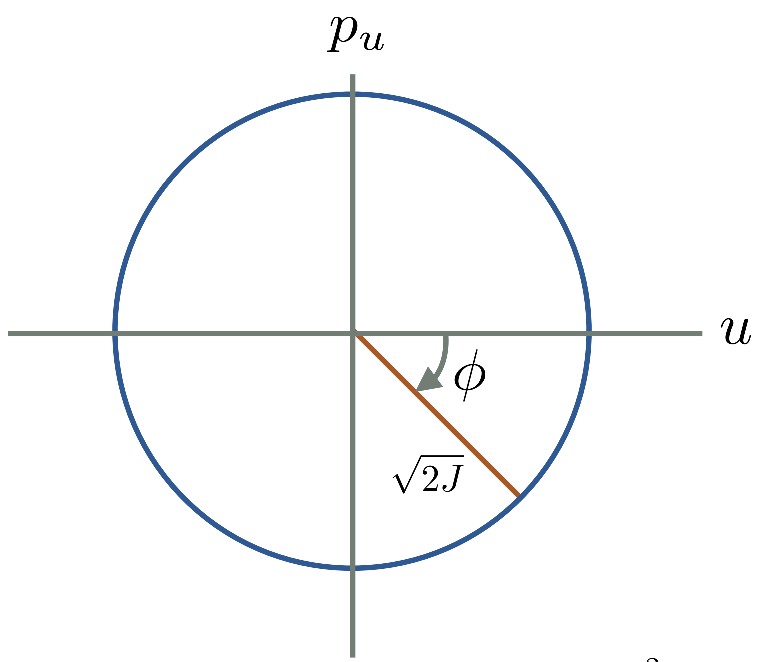

$$2J = u^2 + p_u^2 = \frac{x^2}{\beta} + \beta\left(p_x + \frac{\alpha}{\beta}x\right)^2$$

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= & \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_1}}(\cos\phi + \alpha_1 \sin\phi) & \sqrt{\beta(s)\beta_1}\sin\phi \\ -\dfrac{(1+\alpha(s)\alpha_1)\sin\phi + (\alpha(s)-\alpha_1)\cos\phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\dfrac{\beta_1}{\beta(s)}}(\cos\phi - \alpha(s)\sin\phi) \end{pmatrix}, \quad (51)
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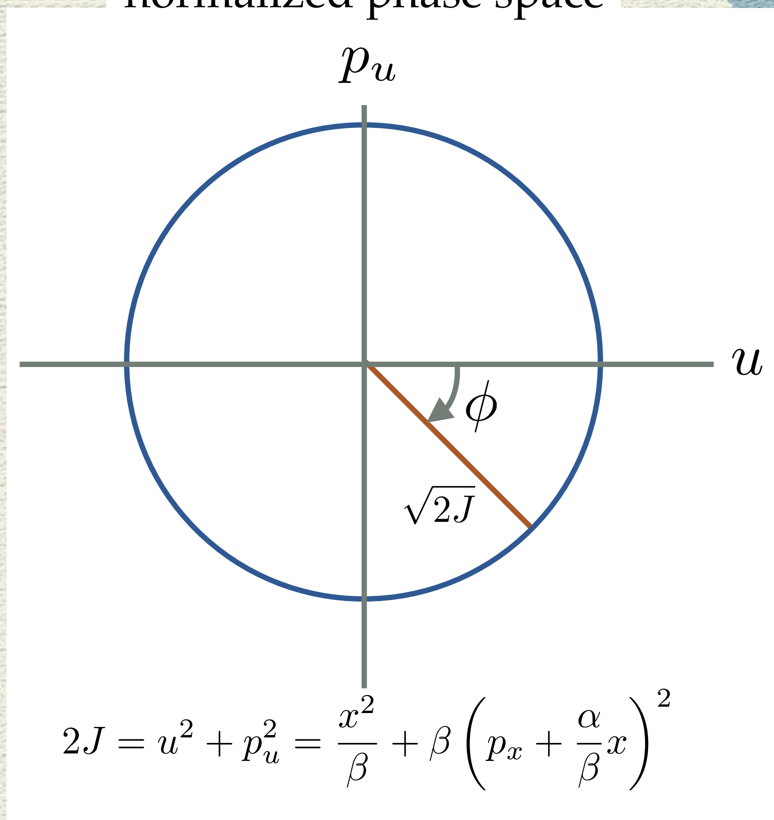
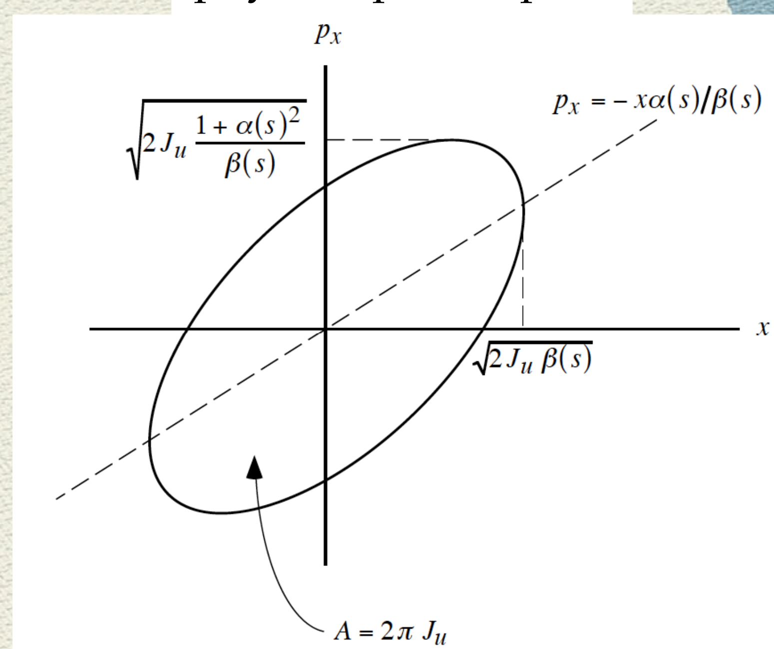


physical phase space



Twiss parameters along a beam line (2)

If we know the transfer matrix from s_1 to s

$$M(s) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} , \tag{53}$$

we can calculate the Twiss parameters at s using Eq, (51) as

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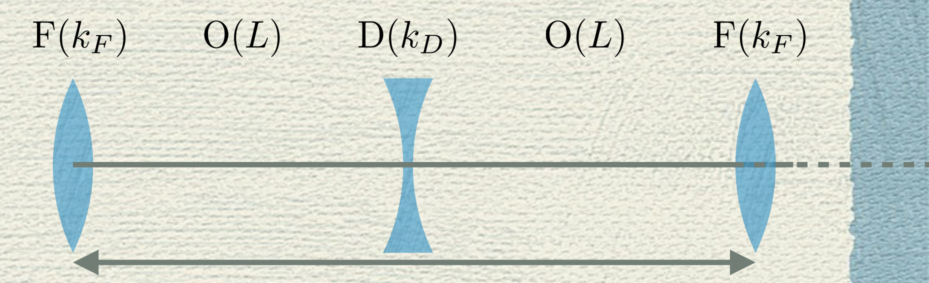
$$\alpha(s) = (M_{11}M_{22} + M_{12}M_{21})\alpha_1 - M_{11}M_{21}\beta_1 - M_{12}M_{22}\frac{1+\alpha_1^2}{\beta_1} \tag{54}$$

$$\beta(s) = -2M_{11}M_{12}\alpha_1 + M_{11}^2\beta_1 + M_{12}^2\frac{1+\alpha_1^2}{\beta_1} \tag{55}$$

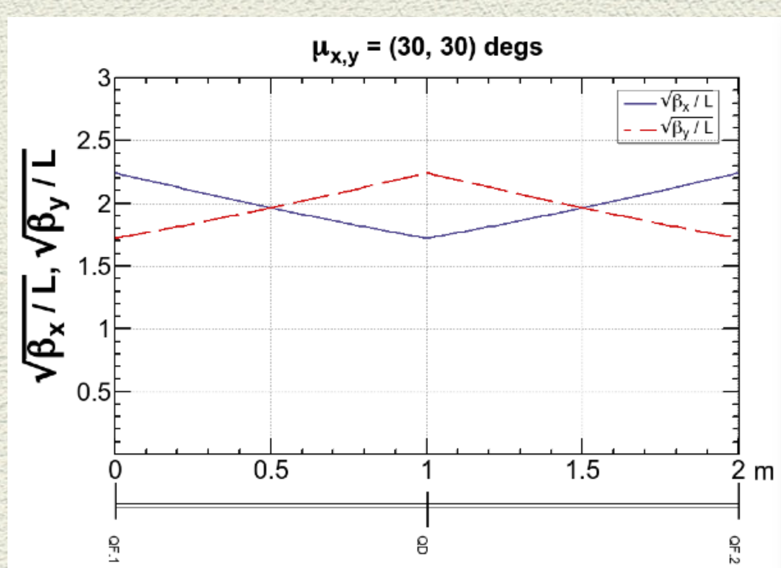
$$\phi(s) = \arg(-M_{12}\alpha_1 + M_{11}\beta_1 + iM_{12}) \tag{56}$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_1}}(\cos\phi + \alpha_1 \sin\phi) & \sqrt{\beta(s)\beta_1}\sin\phi \\ -\frac{(1+\alpha(s)\alpha_1)\sin\phi + (\alpha(s)-\alpha_1)\cos\phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\frac{\beta_1}{\beta(s)}}(\cos\phi - \alpha(s)\sin\phi) \end{pmatrix}, \tag{51}$$

FODO cell (4)



β-functions along a FODO cell with various phase advances ($\mu_x = \mu_y$)



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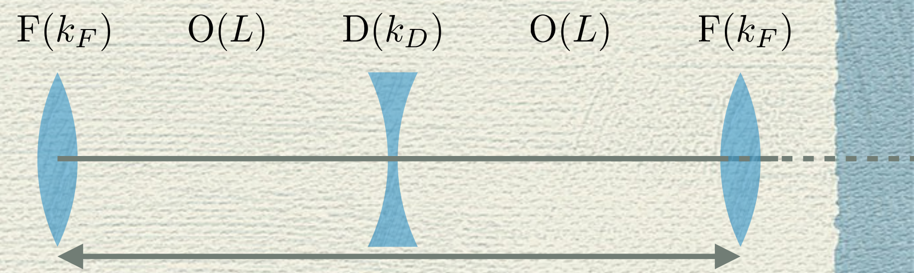

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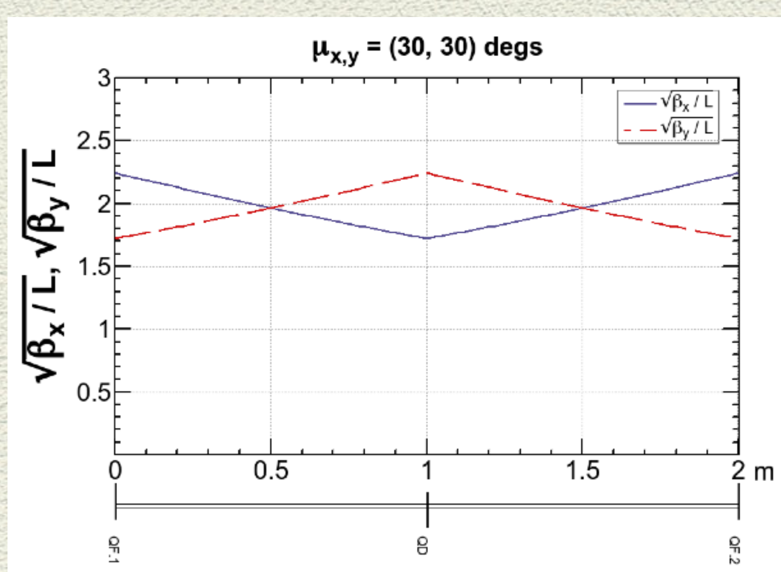
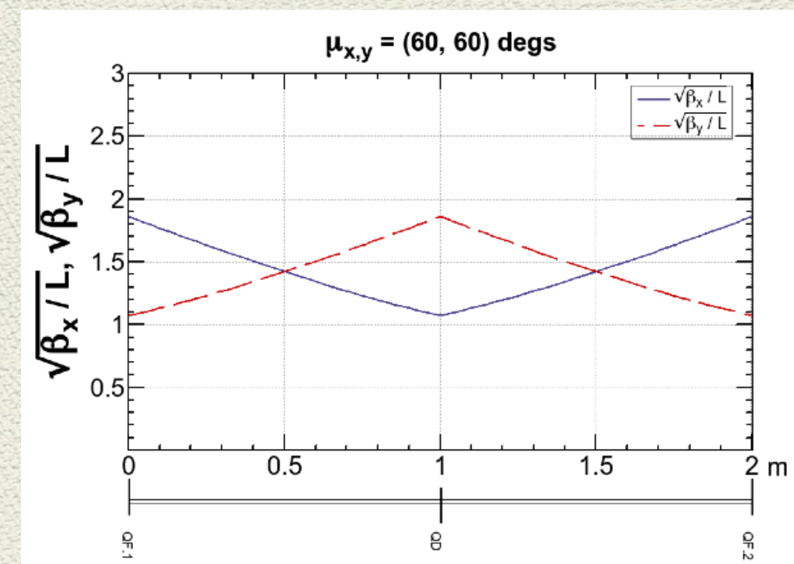
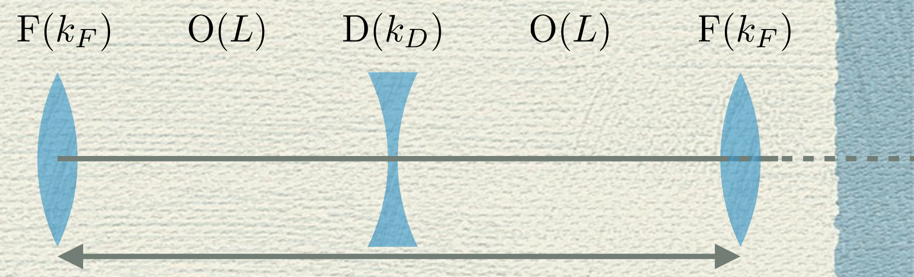
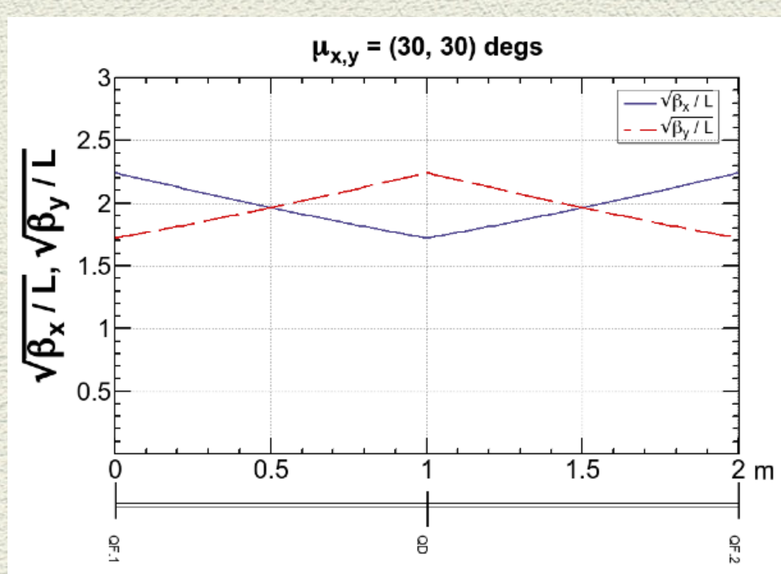


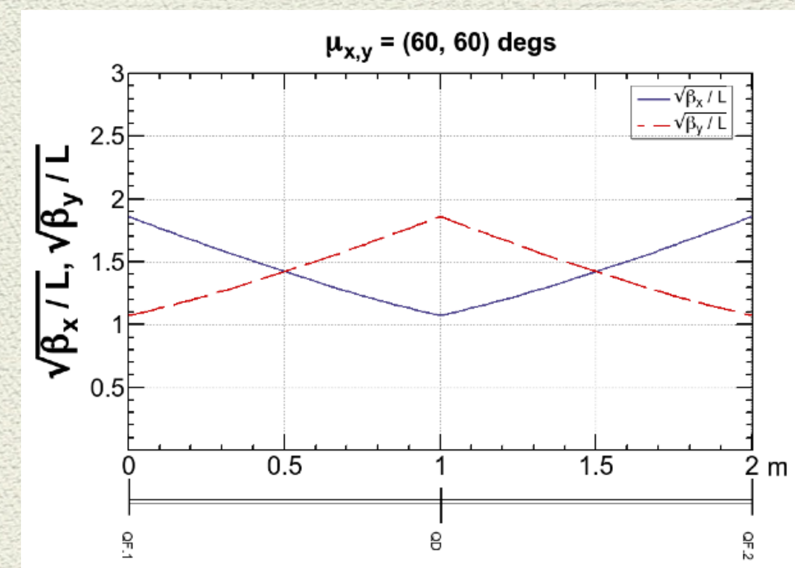


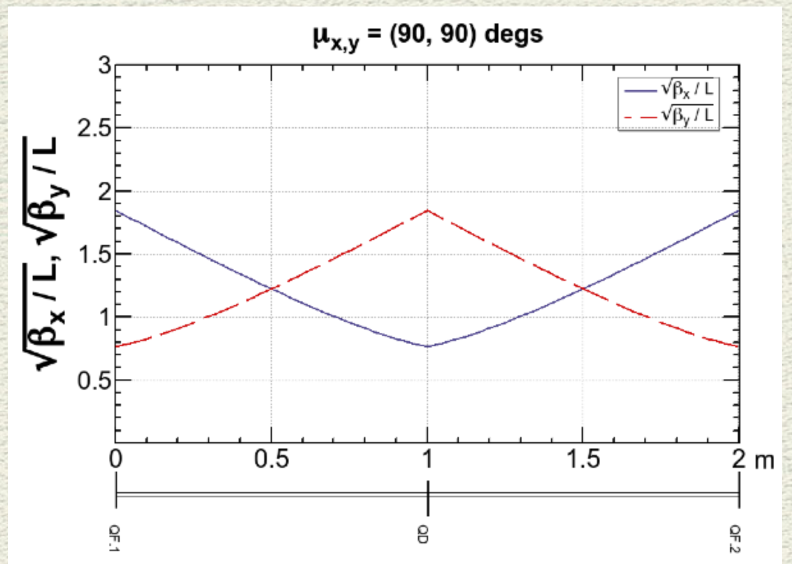
FODO cell (4)



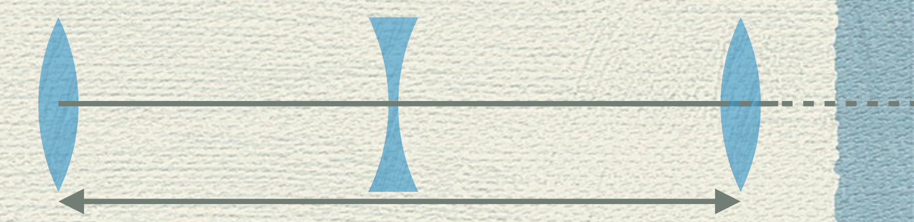
β-functions along a FODO cell with various phase advances ($\mu_x = \mu_y$)







FODO cell (4)



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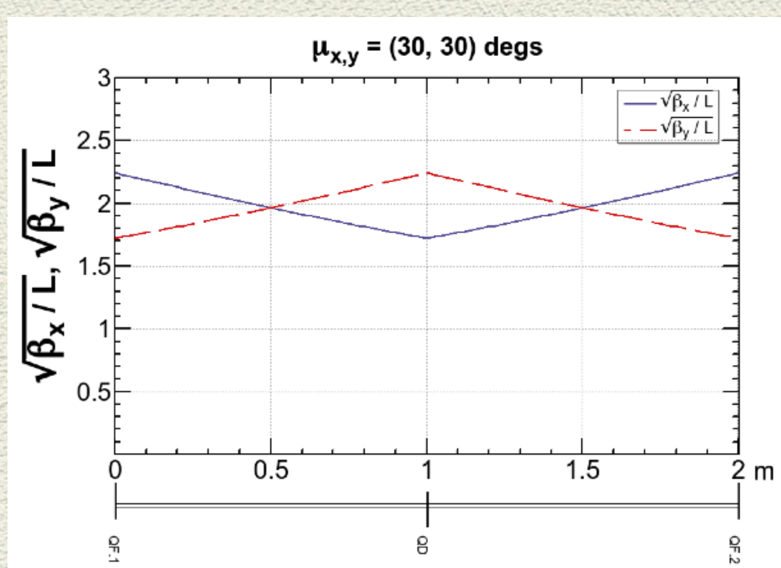

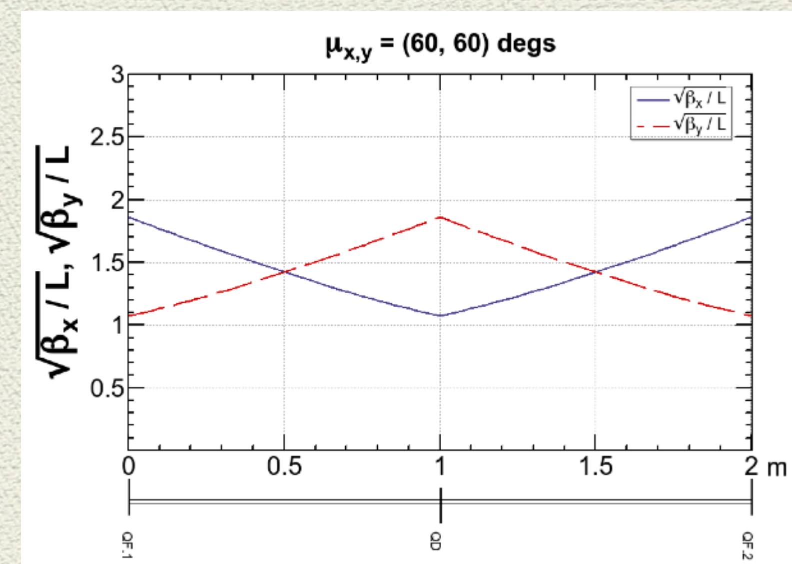

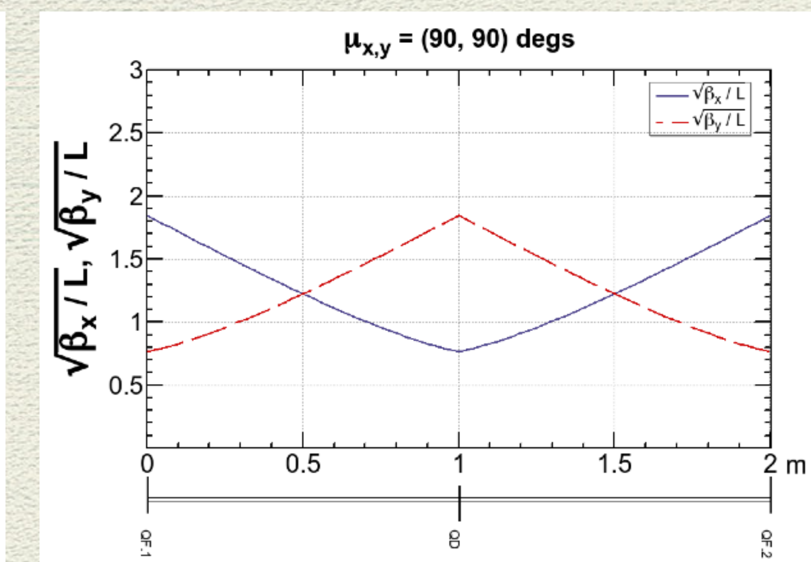

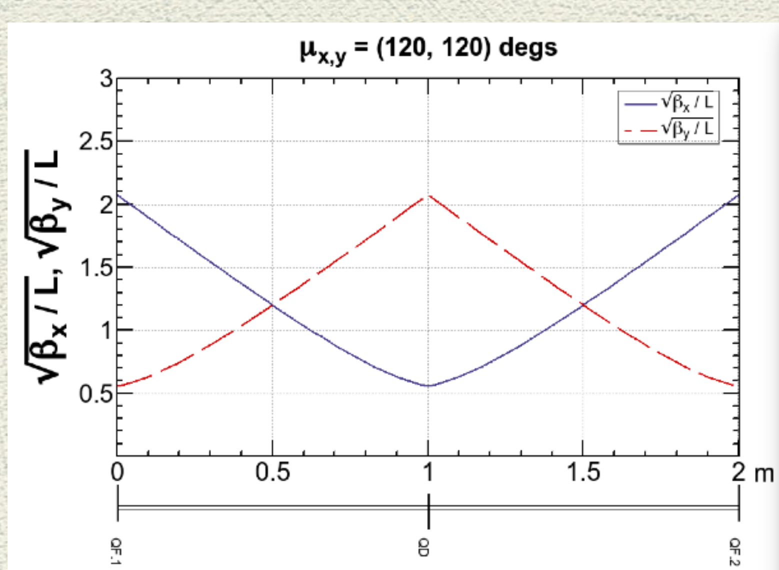

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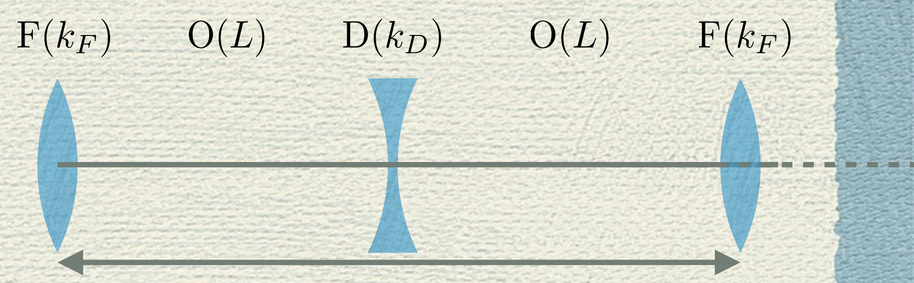

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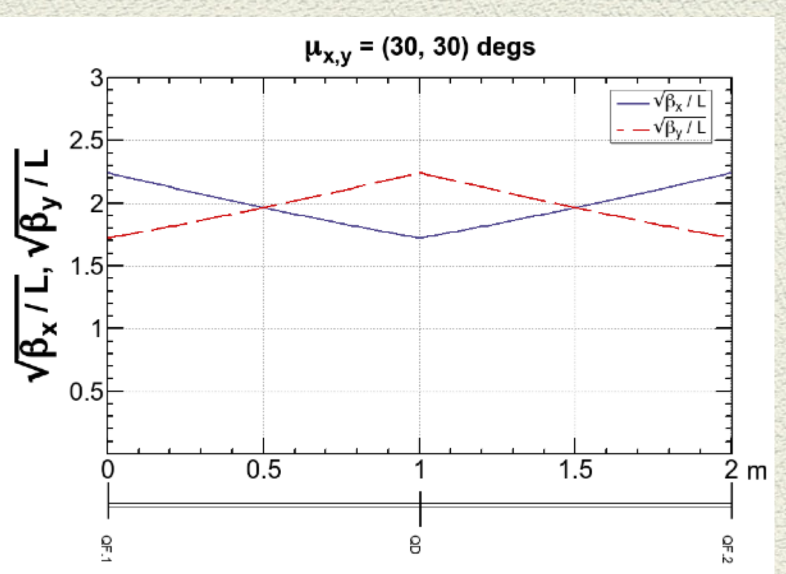

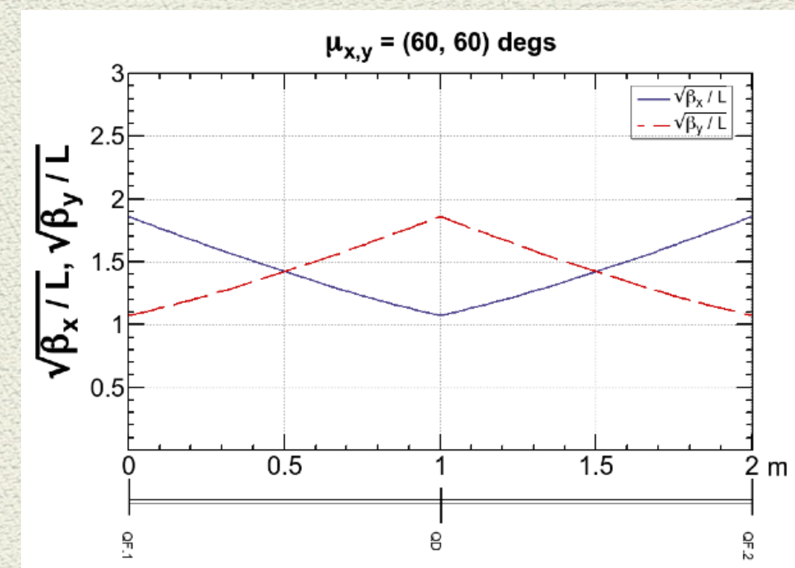

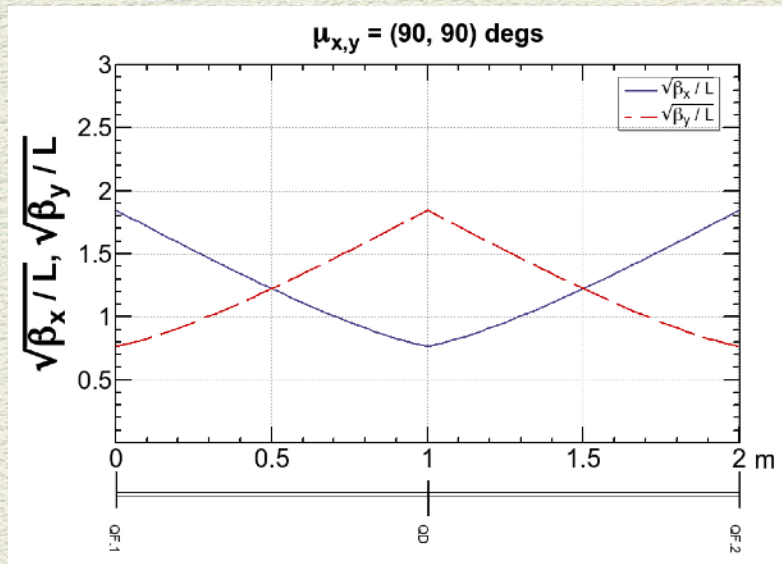

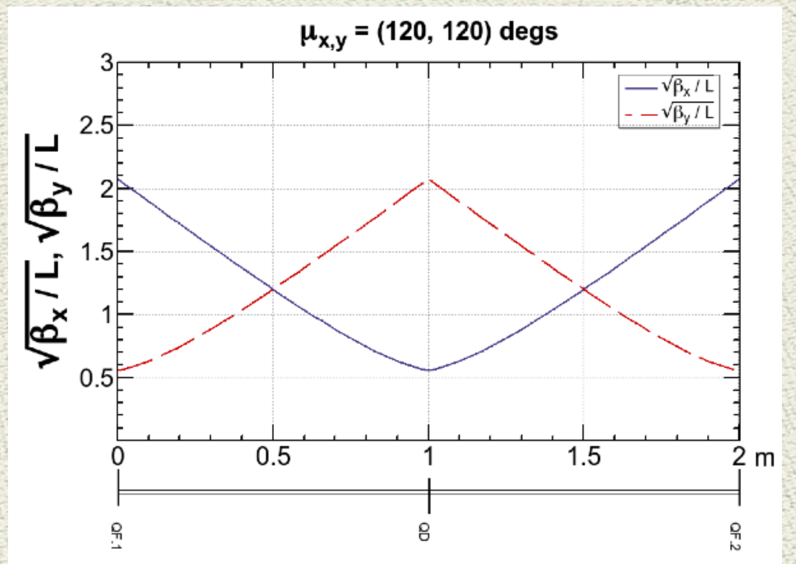

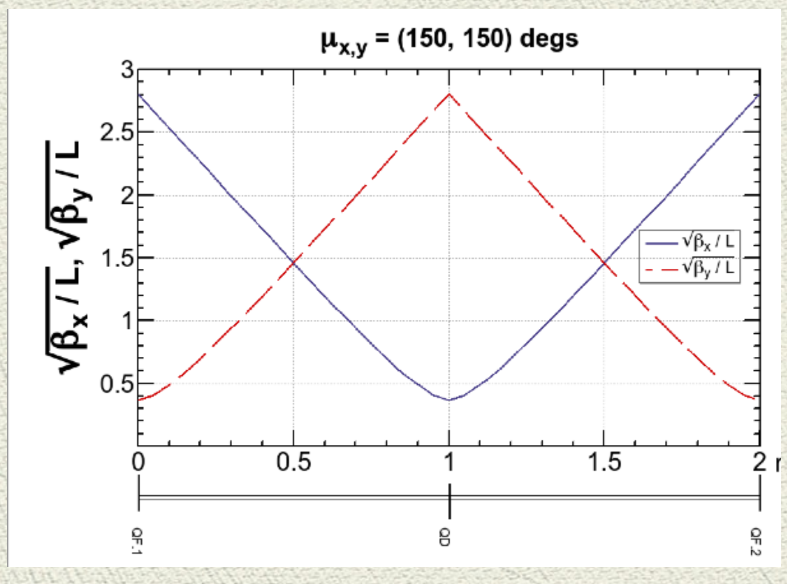

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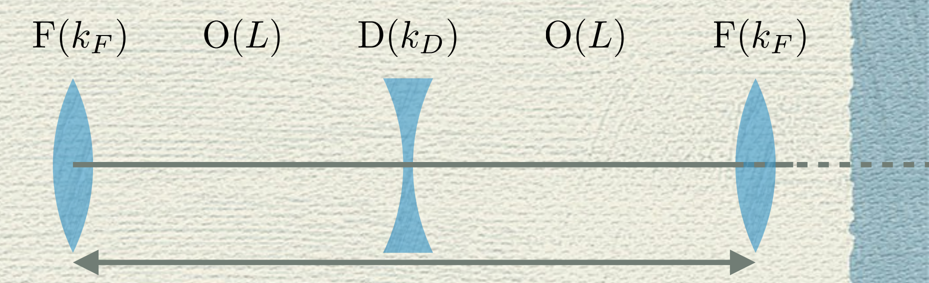

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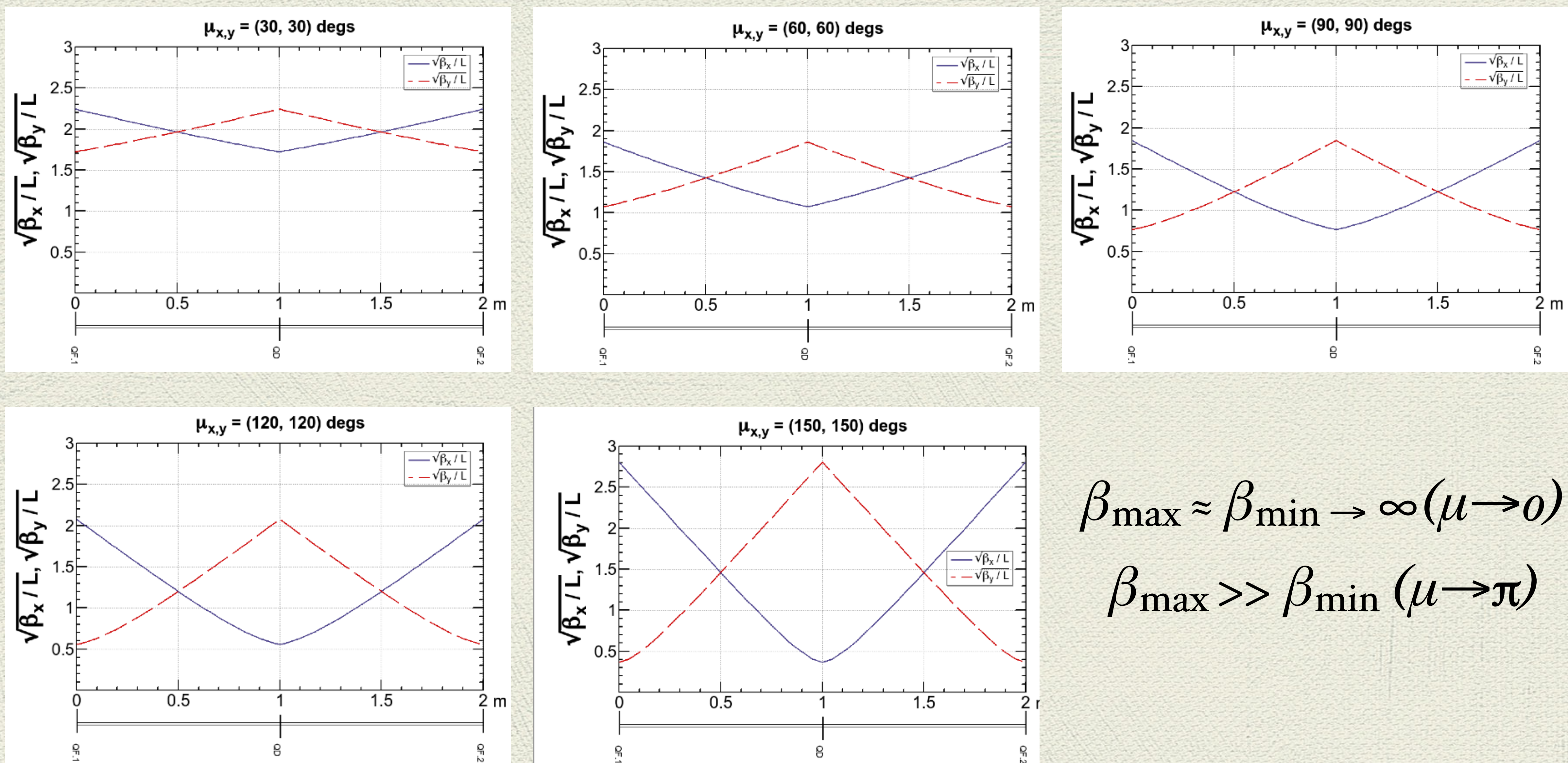
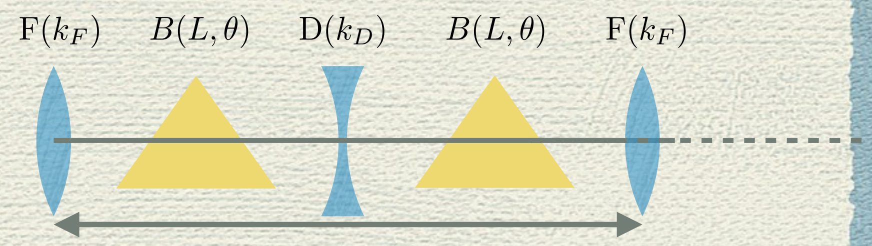


$\beta_{\max} \approx \beta_{\min} \to \infty (\mu \to 0)$

$\beta_{\max} >> \beta_{\min} \ (\mu \to \pi)$

Dispersion

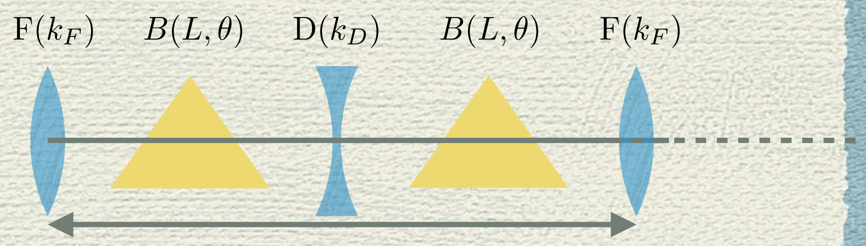


In a FODO cell, now let us consider a horizontal dipole with length L and the bending angle θ in place of the drift space. The on-axis, on-momentum transfer matrix between the center of QFs in (x, p_x, z, δ) phase space is written as

$$
\mathrm{M} = \begin{pmatrix} 1 & . & . & . \\ -k_F/2 & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & L & . & L\theta/2 \\ . & 1 & . & \theta \\ -\theta & -L\theta/2 & 1 & . \\ . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . \\ k_D & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}
$$

$$
\times \begin{pmatrix} 1 & L & . & L\theta/2 \\ . & 1 & . & \theta \\ -\theta & -L\theta/2 & 1 & . \\ . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . \\ -k_F/2 & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}
$$

Dispersion



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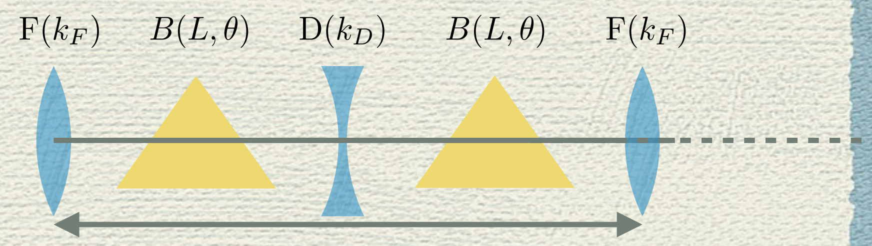
\begin{pmatrix} 1 & L & . & L\theta/2 \\ . & 1 & . & \theta \\ -\theta & -L\theta/2 & 1 & . \\ . & . & . & 1 \end{pmatrix}
\begin{pmatrix} 1 & . & . & . \\ k_D & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}
$$

$$
\times \begin{pmatrix} 1 & L & . & L\theta/2 \\ . & 1 & . & \theta \\ -\theta & -L\theta/2 & 1 & . \\ . & . & . & 1 \end{pmatrix}
\begin{pmatrix} 1 & . & . & . \\ -k_F/2 & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}
$$

dipole magnet

$$
\begin{pmatrix} 1 & \frac{L}{1+\delta} & . & . & . & \frac{L\theta}{2(1+\delta)^2} \\ . & 1 & . & . & . & \theta \\ . & . & 1 & \frac{L}{1+\delta} & . & . \\ . & . & . & 1 & . & . \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & . & . & 1 & \frac{v-v_d}{v_d}L \\ . & . & . & . & . & 1 \end{pmatrix} + O(\theta)^2
$$

Dispersion



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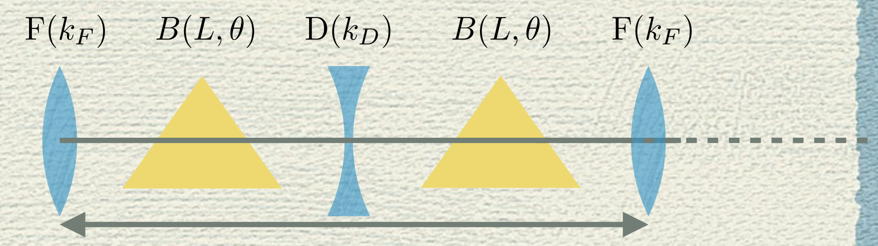
$$= \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{M_x}} & . & \dfrac{(c_D+4)L\theta}{2} \\ & & . & -\dfrac{(c_D+4)(c_F-2)\theta}{4} \\ \dfrac{(c_D+4)(c_F-2)\theta}{4} & -\dfrac{(c_D+4)L\theta}{2} & 1 & -\dfrac{(c_D+4)L\theta^2}{4} \\ . & . & . & 1 \end{pmatrix}, \qquad (57)$$

dipole magnet

$$\begin{pmatrix} 1 & \dfrac{L}{1+\delta} & . & . & . & \dfrac{L\theta}{2(1+\delta)^2} \\ . & 1 & . & . & . & \theta \\ . & . & 1 & \dfrac{L}{1+\delta} & . & . \\ . & . & . & 1 & . & . \\ -\theta & -\dfrac{L\theta(1+2\delta)}{2(1+\delta)^2} & . & . & 1 & \dfrac{v-v_d}{v_d}L \\ . & . & . & . & . & 1 \end{pmatrix} + O(\theta)^2$$

Dispersion



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dipole magnet

$$\begin{pmatrix} 1 & \frac{L}{1+\delta} & . & . & . & \frac{L\theta}{2(1+\delta)^2} \\ . & 1 & . & . & . & \theta \\ . & . & 1 & \frac{L}{1+\delta} & . & . \\ . & . & . & 1 & . & . \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & . & . & 1 & \frac{v-v_d}{v_d}L \\ . & . & . & . & . & 1 \end{pmatrix} + O(\theta)^2$$

where M_x is the 2 by 2 matrix of FODO given by Eq.(44).

$$\mathrm{M}_x = \begin{pmatrix} 1 \pm (c_D - c_F) - \frac{c_D c_F}{2} & L(2 \pm c_D) \\ \frac{\pm(c_D - c_F) - c_F c_D + c_F^2(1 \pm c_D/2)}{L} & 1 \pm (c_D - c_F) - \frac{c_F c_D}{2} \end{pmatrix}, \qquad (44)$$

$$\begin{pmatrix} & \mathrm{M}_x & \cdot & \frac{(c_D+4)L\theta}{2} \\ & & \cdot & -\frac{(c_D+4)(c_F-2)\theta}{4} \\ \frac{(c_D+4)(c_F-2)\theta}{4} & -\frac{(c_D+4)L\theta}{2} & 1 & -\frac{(4+c_D)L\theta^2}{4} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Dispersion (2)

This transfer matrix has nonzero $M_{x,\delta}$, $M_{px,\delta}$, $M_{z,x}$,$M_{z,px}$ components, which means a *coupled motion* between x- and z- planes.

$$\begin{pmatrix} & \mathrm{M}_x & & . & \frac{(c_D+4)L\theta}{2} \\ & & & . & -\frac{(c_D+4)(c_F-2)\theta}{4} \\ \frac{(c_D+4)(c_F-2)\theta}{4} & -\frac{(c_D+4)L\theta}{2} & & 1 & -\frac{(4+c_D)L\theta^2}{4} \\ . & . & & . & 1 \end{pmatrix}$$

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This transfer matrix has nonzero $M_{x,\delta}$, $M_{px,\delta}$, $M_{z,x}$,$M_{z,px}$ components, which means a *coupled motion* between *x*- and *z*- planes.

In general, any *stable* coupled symplectic matrix M can be decomposed into a non-coupled semi-diagonal matrix consisting of 2 by 2 matrices $\mathrm{M}_{x,y,z}$ by

$$\mathrm{D}^{-1}\mathrm{MD} = \begin{pmatrix} \mathrm{M}_x & . & . \\ . & \mathrm{M}_y & . \\ . & . & \mathrm{M}_z \end{pmatrix}, \qquad (58)$$

where D is a symplectic matrix.

This means that any stable coupled motion can be decoupled into 1D motions, by the choice of variables by the matrix D by

$$
\begin{pmatrix}
\multicolumn{2}{c}{\multirow{2}{*}{M_x}} & . & \frac{(c_D+4)L\theta}{2} \\
 & & . & -\frac{(c_D+4)(c_F-2)\theta}{4} \\
\frac{(c_D+4)(c_F-2)\theta}{4} & -\frac{(c_D+4)L\theta}{2} & 1 & -\frac{(4+c_D)L\theta^2}{4} \\
. & . & . & 1
\end{pmatrix}
$$

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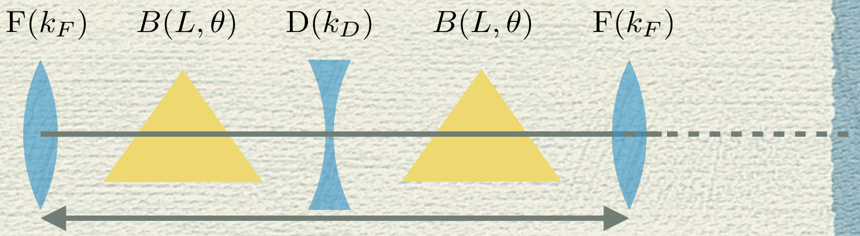
$$
\begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta_\beta \end{pmatrix} = \mathrm{D}^{-1} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \tag{59}
$$

where the lhs are the uncoupled coordinates, or *betatron coordinates*, for which we can define the Twiss parameters in a usual way of a 1D motion.



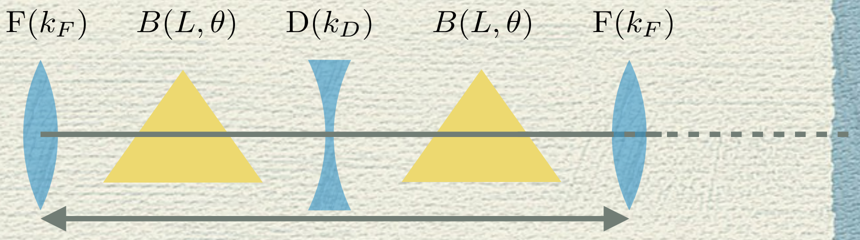
Dispersion (3)

Consider a FODO with dipoles having the cell transfer matrix for x, z planes:

$$\mathrm{M} = \begin{pmatrix} M_{11} & M_{12} & . & M_{16} \\ M_{21} & M_{22} & . & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ . & . & . & 1 \end{pmatrix}. \qquad (60)$$

The matrix M is semi-diagonalized by another matrix

Dispersion (3)



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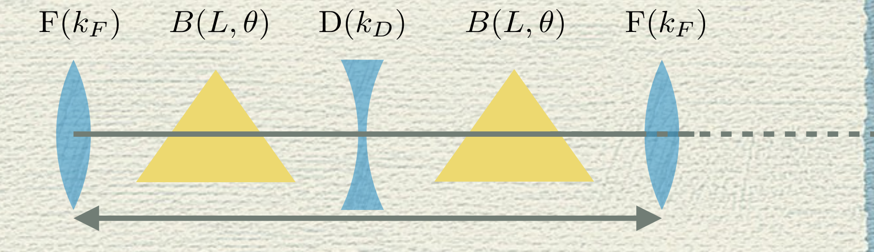
The matrix M is semi-diagonalized by another matrix

$$\mathrm{D} = \begin{pmatrix} 1 & . & . & \eta_x \\ . & 1 & . & \eta_{px} \\ -\eta_{px} & \eta_x & 1 & . \\ . & . & . & 1 \end{pmatrix} \text{ as } \mathrm{D}^{-1}\mathrm{MD} = \begin{pmatrix} M_{11} & M_{12} & . & . \\ M_{21} & M_{22} & . & . \\ . & . & 1 & M_{D56} \\ . & . & . & 1 \end{pmatrix} \tag{61}$$

with

$$\mathrm{D}^{-1} = \mathrm{D}/.\{\eta_x \to -\eta_x,\ \eta_{px} \to -\eta_{px}\}$$

Dispersion (3)



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normal to physical

with

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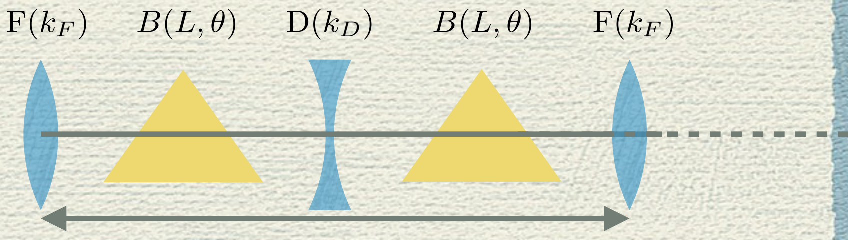
Dispersion (3)



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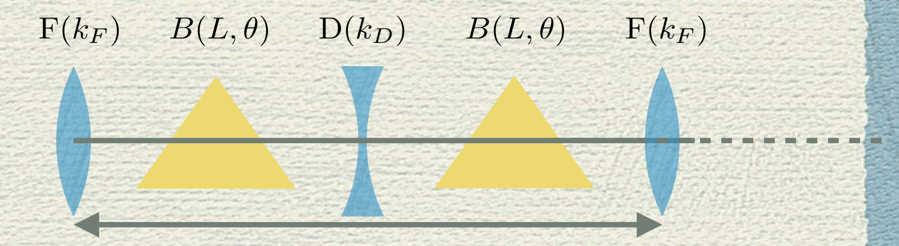
physical to normal

normal to physical

with

$$\mathrm{D}^{-1} = \mathrm{D}/.\{\eta_x \to -\eta_x,\ \eta_{px} \to -\eta_{px}\}$$

Dispersion (3)



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The matrix M is semi-diagonalized by another matrix

$$\mathrm{D} = \begin{pmatrix} 1 & . & . & \eta_x \\ . & 1 & . & \eta_{px} \\ -\eta_{px} & \eta_x & 1 & . \\ . & . & . & 1 \end{pmatrix} \quad \text{as} \quad \mathrm{D}^{-1}\mathrm{M}\mathrm{D} = \begin{pmatrix} M_{11} & M_{12} & . & . \\ M_{21} & M_{22} & . & . \\ . & . & 1 & M_{D56} \\ . & . & . & 1 \end{pmatrix} \qquad (61)$$

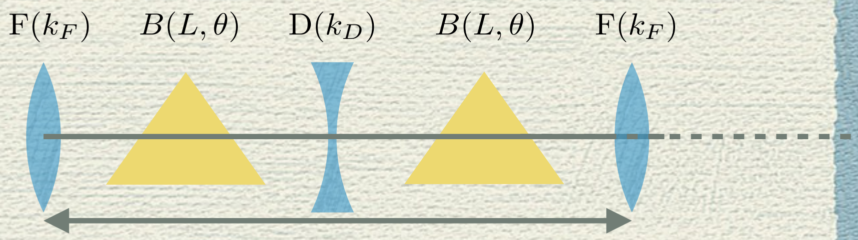
physical to normal (D^{-1}) — normal to physical (D)

with

$$\mathrm{D}^{-1} = \mathrm{D}/.\{\eta_x \to -\eta_x,\ \eta_{px} \to -\eta_{px}\}$$

$$\eta_x = \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4\sin^2(\mu_x/2)} ,$$
$$\eta_{px} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4\sin^2(\mu_x/2)} , \qquad (62)$$

Dispersion (3)



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$$\mathrm{M} = \begin{pmatrix} M_{11} & M_{12} & . & M_{16} \\ M_{21} & M_{22} & . & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ . & . & . & 1 \end{pmatrix} . \qquad (60)$$

The matrix M is semi-diagonalized by another matrix

$$\mathrm{D} = \begin{pmatrix} 1 & . & . & \eta_x \\ . & 1 & . & \eta_{px} \\ -\eta_{px} & \eta_x & 1 & . \\ . & . & . & 1 \end{pmatrix} \quad \text{as} \quad \mathrm{D}^{-1}\mathrm{MD} = \begin{pmatrix} M_{11} & M_{12} & . & . \\ M_{21} & M_{22} & . & . \\ . & . & 1 & M_{D56} \\ . & . & . & 1 \end{pmatrix} \qquad (61)$$

physical to normal

normal to physical

$$\mathrm{D}^{-1} = \mathrm{D}/.\{\eta_x \to -\eta_x,\ \eta_{px} \to -\eta_{px}\}$$

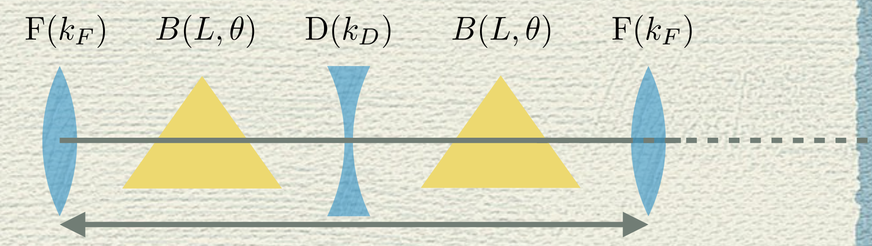
with

$$\eta_x = \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4\sin^2(\mu_x/2)} , \qquad \eta_{px} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4\sin^2(\mu_x/2)} , \qquad (62)$$

where we have used $M_{11} + M_{22} = 2\cos\mu_x$ from Eq. (45).

$$\mathrm{M}_{x,y} = \begin{pmatrix} \cos\mu_{x,y} + \alpha_{x,y}\sin\mu_{x,y} & \beta_{x,y}\sin\mu_{x,y} \\ \dfrac{1+\alpha_{x,y}^2}{\beta_{x,y}}\sin\mu_{x,y} & \cos\mu_{x,y} - \alpha_{x,y}\sin\mu_{x,y} \end{pmatrix} . \qquad (45)$$

Dispersion (4)



In the case of the FODO transfer matrix Eq. (57), the solution Eq, (62) is

$$\eta_x = \frac{(4 + c_D)L\theta}{2(c_F - c_D) + c_F c_D} \tag{63}$$

$$= \frac{2L + \beta_x \sin\mu_x}{4\sin^2(\mu_x/2)^2}\theta\,, \tag{64}$$

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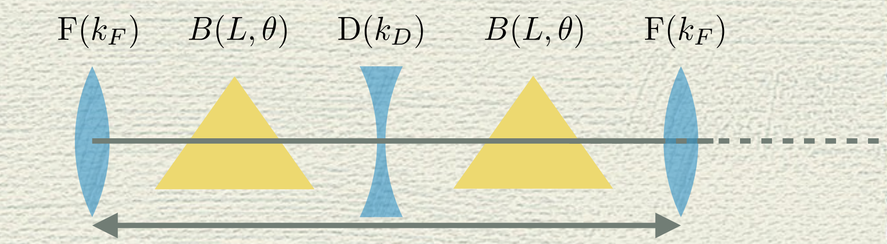

In the case of the FODO transfer matrix Eq. (57), the solution Eq, (62) is

$$= \begin{pmatrix} \mathrm{M}_x & & \cdot & \frac{(c_D+4)L\theta}{2} \\ & & \cdot & -\frac{(c_D+4)(c_F-2)\theta}{4} \\ \frac{(c_D+4)(c_F-2)\theta}{4} & -\frac{(c_D+4)L\theta}{2} & 1 & -\frac{(4+c_D)L\theta^2}{4} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (57)$$

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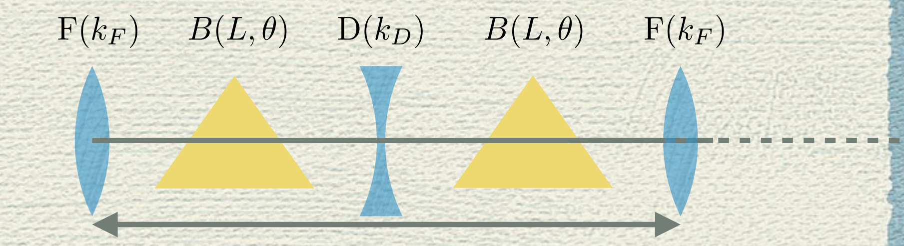

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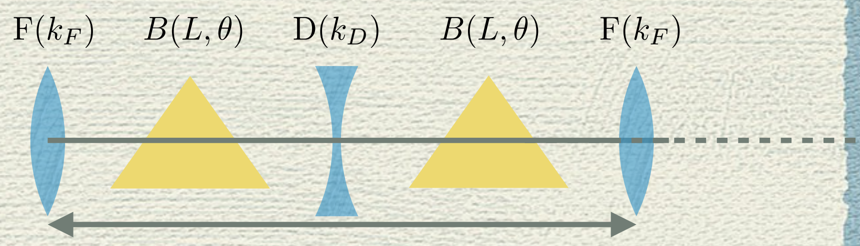

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then the matrix M is semi-diagonalized as

$$D^{-1}MD = \begin{pmatrix} M_x & & \cdot & \cdot \\ & & \cdot & \cdot \\ \cdot \quad \cdot & & 1 & \frac{(c_F-4)\eta_x\theta}{2} \\ \cdot & & \cdot & 1 \end{pmatrix} \quad (66)$$

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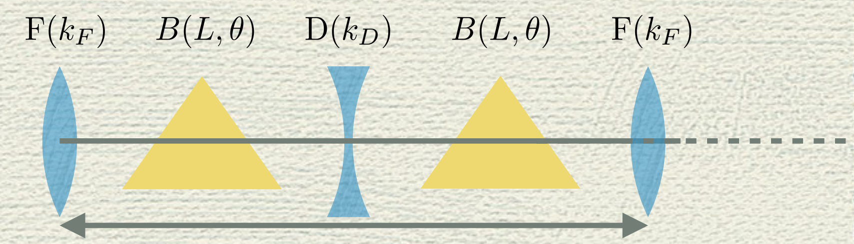

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We will discuss the nature of M_{D56} in the matrix Eq. (61) later.

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$$\mathrm{D}^{-1}\mathrm{MD} = \begin{pmatrix} M_{11} & M_{12} & \cdot & \cdot \\ M_{21} & M_{22} & \cdot & \cdot \\ \cdot & \cdot & 1 & M_{D56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (61)$$

Dispersion (5)

$$\begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta_\beta \end{pmatrix} = \mathrm{D}^{-1} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \quad (59)$$

In this case, the betatron coordinates by Eq. (59) are written as

$$x_\beta = x - \eta_x \delta \ , \quad (67)$$

$$p_{x\beta} = p_x - \eta_{px} \delta \ , \quad (68)$$

$$z_\beta = z + \eta_{px} x - \eta_x p_x \ , \quad (69)$$

and other variables are unchanged.

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In this special case, the resulting 2 by 2 transfer matrix M_x in Eq. (66) did not change, but in the z-plane, the 2 by 2 matrix or $M_{z\delta}$ in Eq. (66) has changed.

$$
\mathrm{D}^{-1}\mathrm{M}\mathrm{D} = \begin{pmatrix} \mathrm{M}_x & & \cdot & \cdot \\ & & \cdot & \cdot \\ \cdot & \cdot & 1 & \dfrac{(c_F - 4)\eta_x \theta}{2} \\ \cdot & & \cdot & 1 \end{pmatrix} \qquad (66)
$$

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Note that such a simple decoupling by dispersions as Eqs. (67–69) is only possible when there is no acceleration in the beam line. If there is an RF cavity in a non-dispersive location in a beam line, the x-z coupling requires more parameters to decouple. The resulting 2 by 2 matrices will not preserve the forms of Eq. (66), in such a general case.

$$\mathrm{D}^{-1}\mathrm{MD} = \begin{pmatrix} \mathrm{M}_x & & \cdot & \cdot \\ & & \cdot & \cdot \\ \cdot & \cdot & 1 & \dfrac{(c_F - 4)\eta_x \theta}{2} \\ \cdot & & \cdot & 1 \end{pmatrix} \qquad (66)$$

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After obtained the dispersions at a particular location $s = s_1$ of a beam line, it is possible to transfer the dispersion at s_1 to another location s in the beam line.

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The decoupling matrix $\mathrm{D}(s)$ should be chosen to semi-diagonalize $\mathrm{M}(s)$ to $\mathrm{M}_D(s)$ as:

$$\mathrm{D}(s)^{-1}\mathrm{M}(s)\mathrm{D} = \mathrm{M}_D(s) = \begin{pmatrix} \mathrm{M}_x(s) & . & . \\ . & \mathrm{M}_y(s) & . \\ . & . & \mathrm{M}_z(s) \end{pmatrix}, \tag{70}$$

where $\mathrm{M}_{x,y,z}(s)$ are the 2 by 2 transfer matrices from s_1 to s.

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Dispersion along a beam line (2)

In the case of a simple dispersion in *x*-*z* plane, M(*s*) is expressed as

$$\mathrm{M}(s) = \begin{pmatrix} M_{11} & M_{12} & . & M_{16} \\ M_{21} & M_{22} & . & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ . & . & . & 1 \end{pmatrix}. \qquad (71)$$

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Then the matrix D(*s*) is written as

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$$\mathrm{D}(s) = \begin{pmatrix} 1 & . & . & \eta_x(s) \\ . & 1 & . & \eta_{px}(s) \\ -\eta_{px}(s) & \eta_x(s) & 1 & . \\ . & . & . & 1 \end{pmatrix} \quad (72)$$

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with dispersions at *s*:

$$\begin{aligned} \eta_x(s) &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16} \, , \\ \eta_{px}(s) &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26} \, . \end{aligned} \qquad (73)$$

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In the case of a simple dispersion in x-z plane, M(s) is expressed as

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Then the matrix D(s) is written as

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with dispersions at s:

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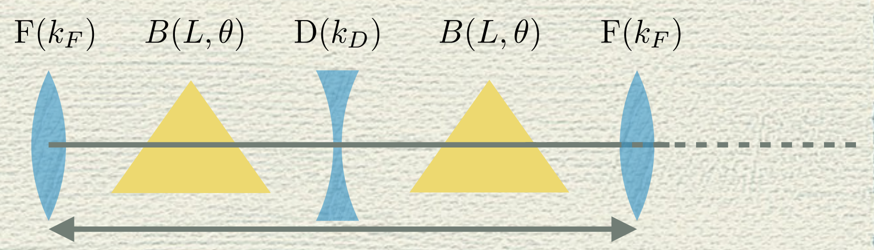
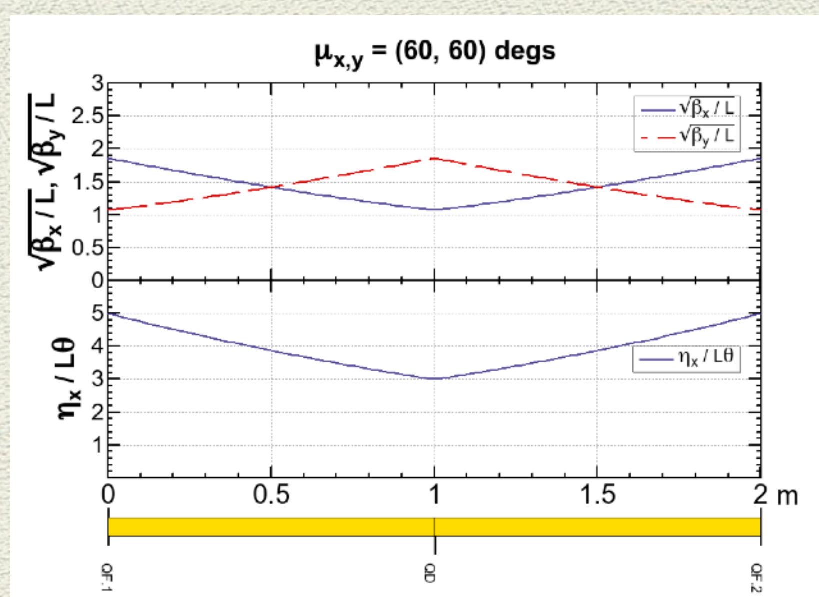
Above means that the the dispersions $(\eta_x(s), \eta_{px}(s))$ behave in the same way as variables (x, p_x) when the inhomogeneous terms M_{16} and M_{26} are zero.

Dispersion in a FODO cell



β-functions and dispersions with various phase advances ($\mu_x = \mu_y$)



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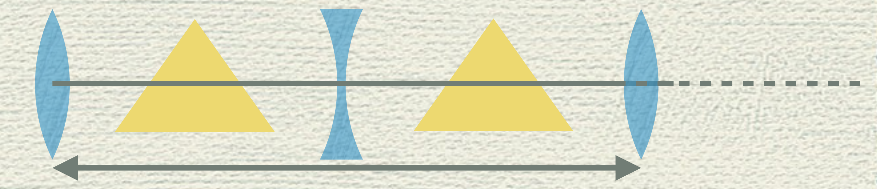

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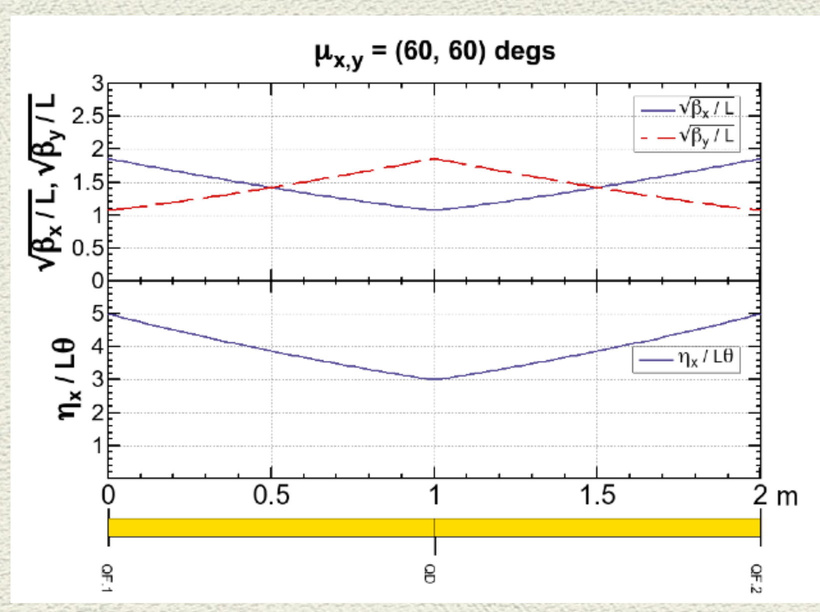

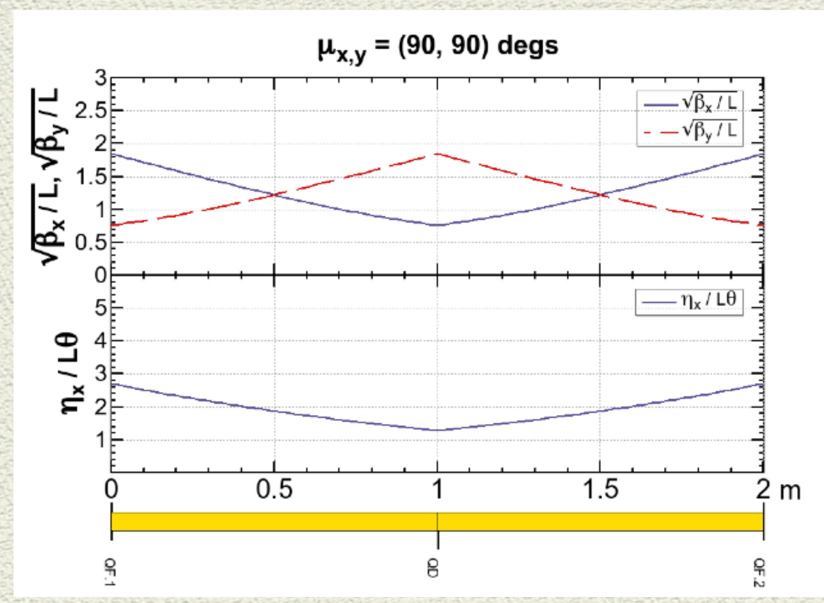

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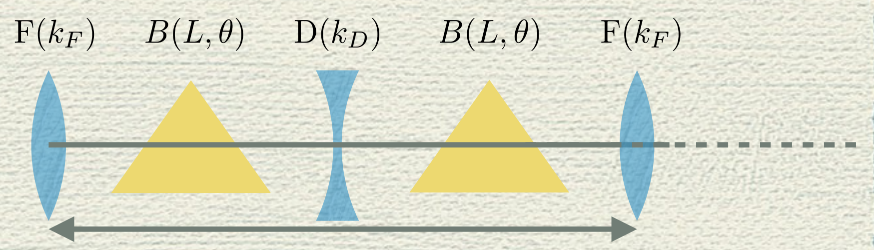

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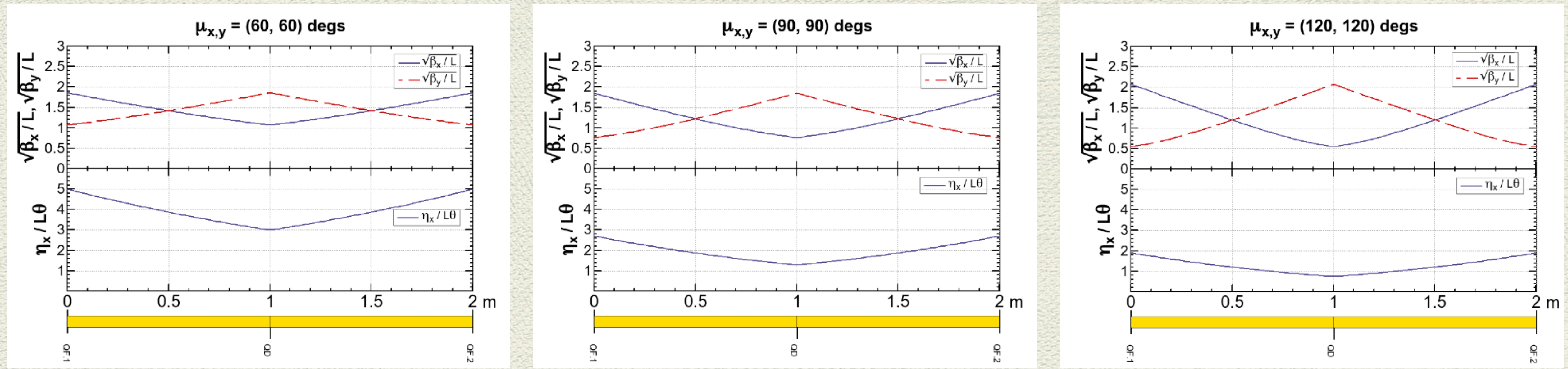

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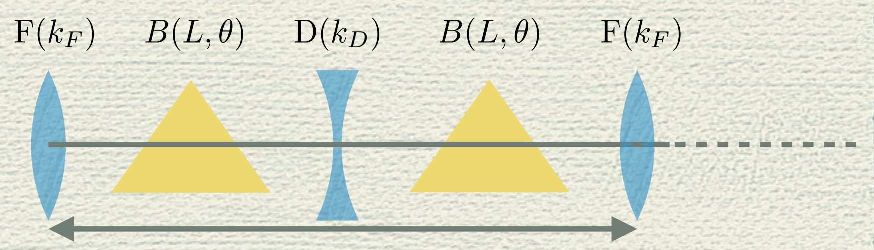

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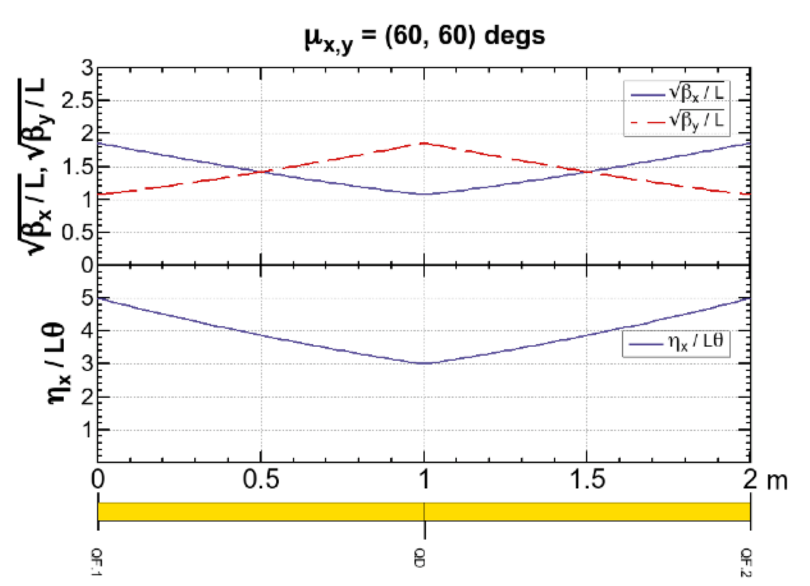

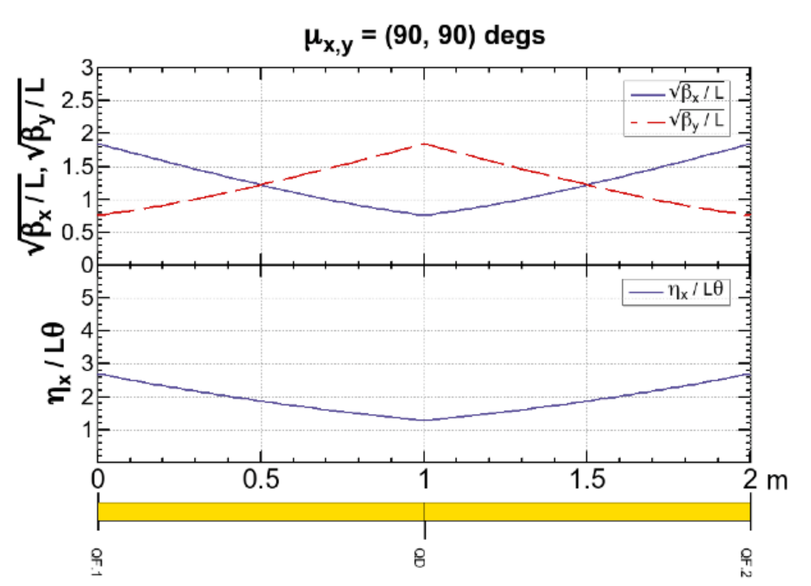

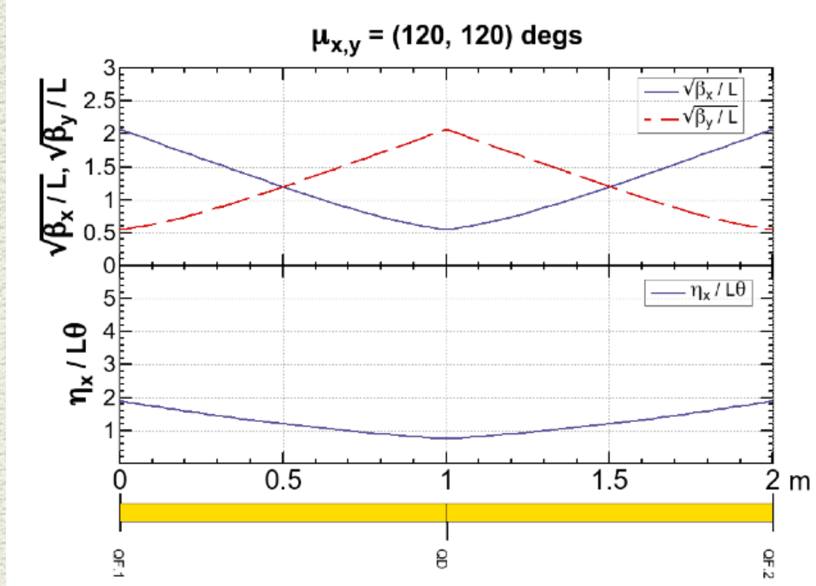

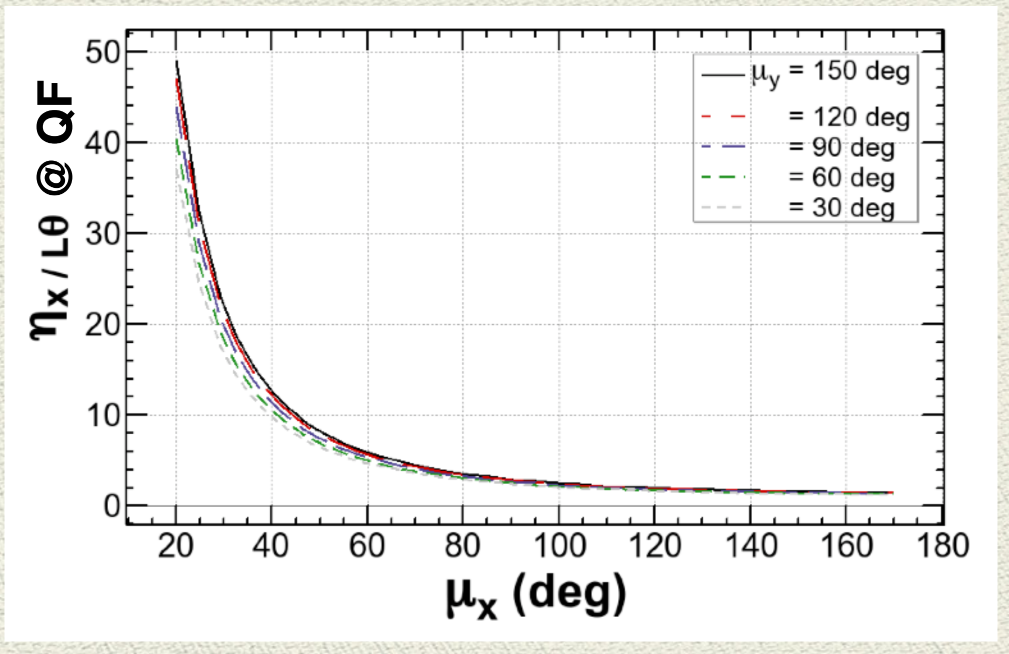

Dispersion in a ring

If we put $s = C + s_1$ in Eq. (73), then M(s) becomes the one-turn transfer matrix M in Eq. (60), and

$$\eta_x(s) = M_{11}\eta_x + M_{12}\eta_{px} + M_{16} \, , \qquad (73)$$
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by the periodic condition $(\eta_x(s), \eta_{px}(s)) = (\eta_x, \eta_{px})$.

$$\mathrm{M} = \begin{pmatrix} M_{11} & M_{12} & . & M_{16} \\ M_{21} & M_{22} & . & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ . & . & . & 1 \end{pmatrix} . \qquad (60)$$

Dispersion in a ring

If we put $s = C + s_1$ in Eq. (73), then M(s) becomes the one-turn transfer matrix M in Eq. (60), and

$$\eta_x(s) = M_{11}\eta_x + M_{12}\eta_{px} + M_{16} \, , \quad (73)$$
$$\eta_{px}(s) = M_{21}\eta_x + M_{22}\eta_{px} + M_{26} \, .$$

$$\eta_x = M_{11}\eta_x + M_{12}\eta_{px} + M_{16} \, , \quad (74)$$
$$\eta_{px} = M_{21}\eta_x + M_{22}\eta_{px} + M_{26} \, ,$$

by the periodic condition $(\eta_x(s), \eta_{px}(s)) = (\eta_x, \eta_{px})$.

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The solution of Eqs. (74) is identical to the solution of semi-diagonalization, Eqs. (62).

$$\eta_x = \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4\sin^2(\mu_x/2)} \, , \quad (62)$$
$$\eta_{px} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4\sin^2(\mu_x/2)} \, ,$$

Momentum compaction

Then the resulting semi-diagonal matrix $\mathrm{M}_D(s)$ becomes

$$\mathrm{D}(s)^{-1}\mathrm{M}(s)\mathrm{D} = \mathrm{M}_D(s)$$

$$\mathrm{M}_D(s) = \begin{pmatrix} M_{11} & M_{12} & . & . \\ M_{21} & M_{22} & . & . \\ . & . & 1 & M_{D56} \\ . & . & . & 1 \end{pmatrix}, \qquad (75)$$

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$$\begin{aligned} M_{D56} &= (M_{21}\eta_x + M_{22}\eta_{px})M_{16} - (M_{11}\eta_x + M_{12}\eta_{px})M_{26} + M_{56} , \\ &= \eta_{px}(s)M_{16} - \eta_x(s)M_{26} + M_{56} , \end{aligned} \tag{76}$$

where we have used Eq. (73).

$$\begin{aligned} \eta_x(s) &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16} , \\ \eta_{px}(s) &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26} . \end{aligned} \tag{73}$$

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Let us consider a small portion of a beam line whose transfer matrix is given by $\Delta\mathrm{M}$. According to Eq. (76), the change of M_{D56} is written as

$$\Delta M_{D56} = \eta_{px}(s)\Delta M_{16} - \eta_x(s)\Delta M_{26} + \Delta M_{56} . \tag{77}$$

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Thus the component M_{D56} changes only when there are non-zero ΔM_{16}, ΔM_{26}, ΔM_{56}. Thus M_{D56} does not change in drift spaces nor multipole magnets at least for an on-momentum & on-axis orbit, where $M_{*6} = 0$.

Momentum compaction (2)

$$\mathrm{M} = \begin{pmatrix} 1 & \frac{L}{1+\delta} & . & . & . & \frac{L\theta}{2(1+\delta)^2} \\ . & 1 & . & . & . & \theta \\ . & . & 1 & \frac{L}{1+\delta} & . & . \\ . & . & . & 1 & . & . \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & . & . & 1 & \frac{v-v_d}{v_d}L \\ . & . & . & . & . & 1 \end{pmatrix} + O(\theta)^2 , \qquad (29)$$

In the case of a dipole with a small length $L = ds$ and a bending angle $\theta = ds/\rho$, the transfer matrix Eq. (29) becomes

$$\varDelta \mathrm{M} = \begin{pmatrix} 1 & ds & . & . \\ . & 1 & . & \frac{ds}{\rho} \\ -\frac{ds}{\rho} & . & 1 & . \\ . & . & . & 1 \end{pmatrix} + O(ds)^2 . \qquad (78)$$

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Then if we plug in above into Eq. (77), the change of ΔM_{D56} is written as

$$\Delta M_{D56} = \eta_{px}(s)\Delta M_{16} - \eta_x(s)\Delta M_{26} + \Delta M_{56} \,. \quad (77)$$

$$dM_{D56} = -\eta_{xs}\frac{ds}{\rho} \,. \qquad (79)$$

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By integrating above, we obtain an expression for M_{D56}:

$$M_{D56} = -\int \frac{\eta_x(s)}{\rho(s)} ds . \qquad (80)$$

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For a storage ring, the quantity M_{56} is sometimes expressed in terms of the ratio to the circumference C as

$$\alpha_p \equiv -\frac{M_{D56}}{C} = \frac{1}{C}\oint \frac{\eta_x(s)}{\rho(s)}ds\,, \tag{81}$$

where the ratio α_p is called *momentum compaction factor*.

Synchrotron motion

Let us consider a ring with an RF cavity with the voltage V_c and the phase ϕ_{RF} at a dispersion-free location. Using Eq. (40), the one-turn, on-axis transfer matrix M_z in z-plane from the center of the cavity is written as:

$$\mathrm{M}_z = \begin{pmatrix} 1 & \cdot \\ k_z/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha_p C \\ \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ k_z/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\alpha_p C k_z}{2} & -\alpha_p C \\ k_z - \dfrac{\alpha_p C k_z^2}{4} & 1 - \dfrac{\alpha_p C k_z}{2} \end{pmatrix}, \qquad (82)$$

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$$\mathrm{M}_z = \begin{pmatrix} \cos\mu_z + \alpha_z \sin\mu_z & \beta_z \sin\mu_z \\ \dfrac{1+\alpha_z^2}{\beta_z}\sin\mu_z & \cos\mu_z - \alpha_z\sin\mu_z \end{pmatrix} , \quad (84)$$

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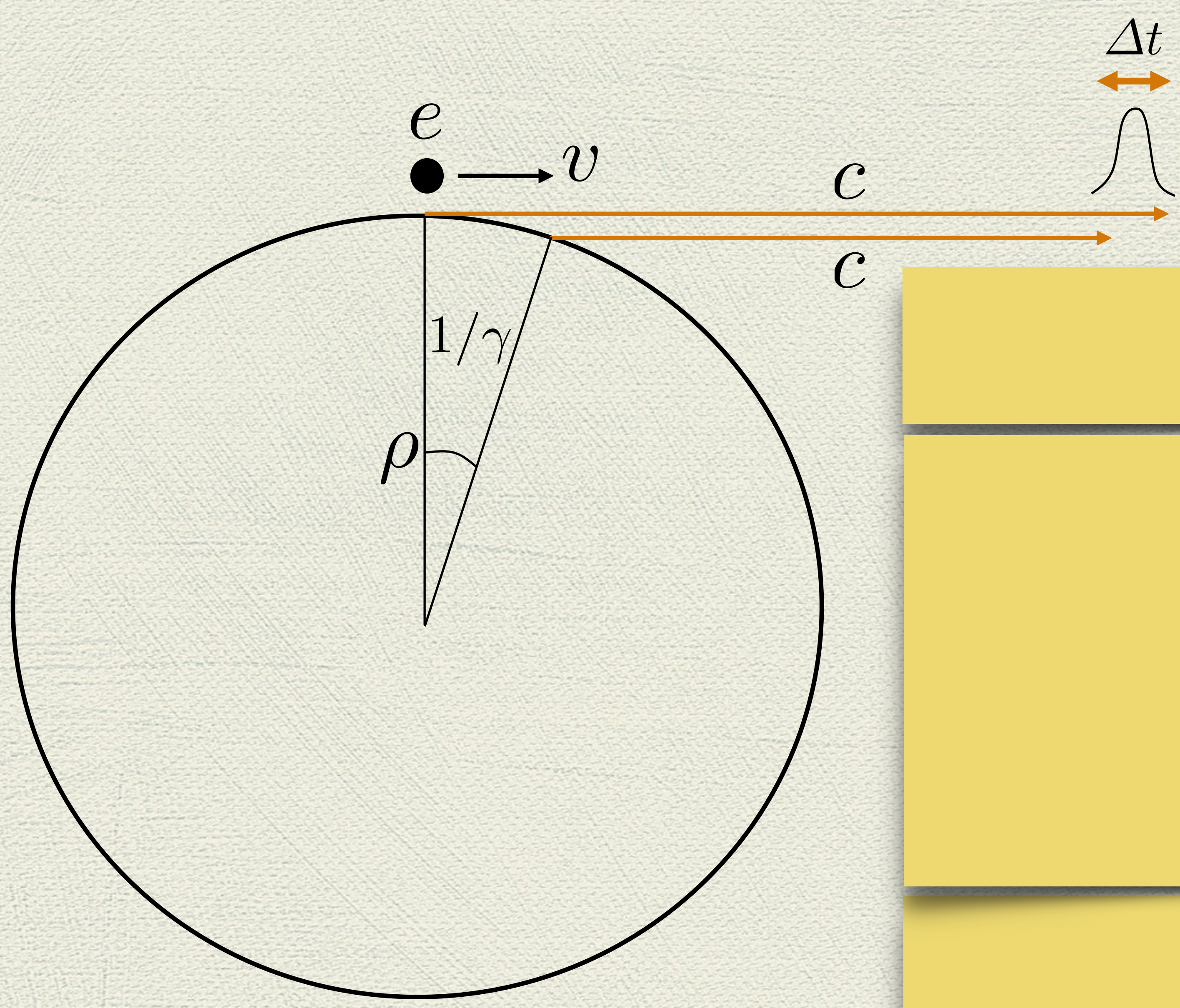
Note that $\alpha_p k_z > 0$ for the stability, and $\mu_z < 0$ for $\alpha_p > 0$.

In the case of $|\mu_z| \ll 1$, α_z and β_z are nealy constant over the ring, and:

$$\mu_z^2 \approx \alpha_p C k_z , \qquad \beta_z \approx -\frac{\alpha_p}{\mu_z} C . \quad (86)$$

Overview of synchrotron radiation



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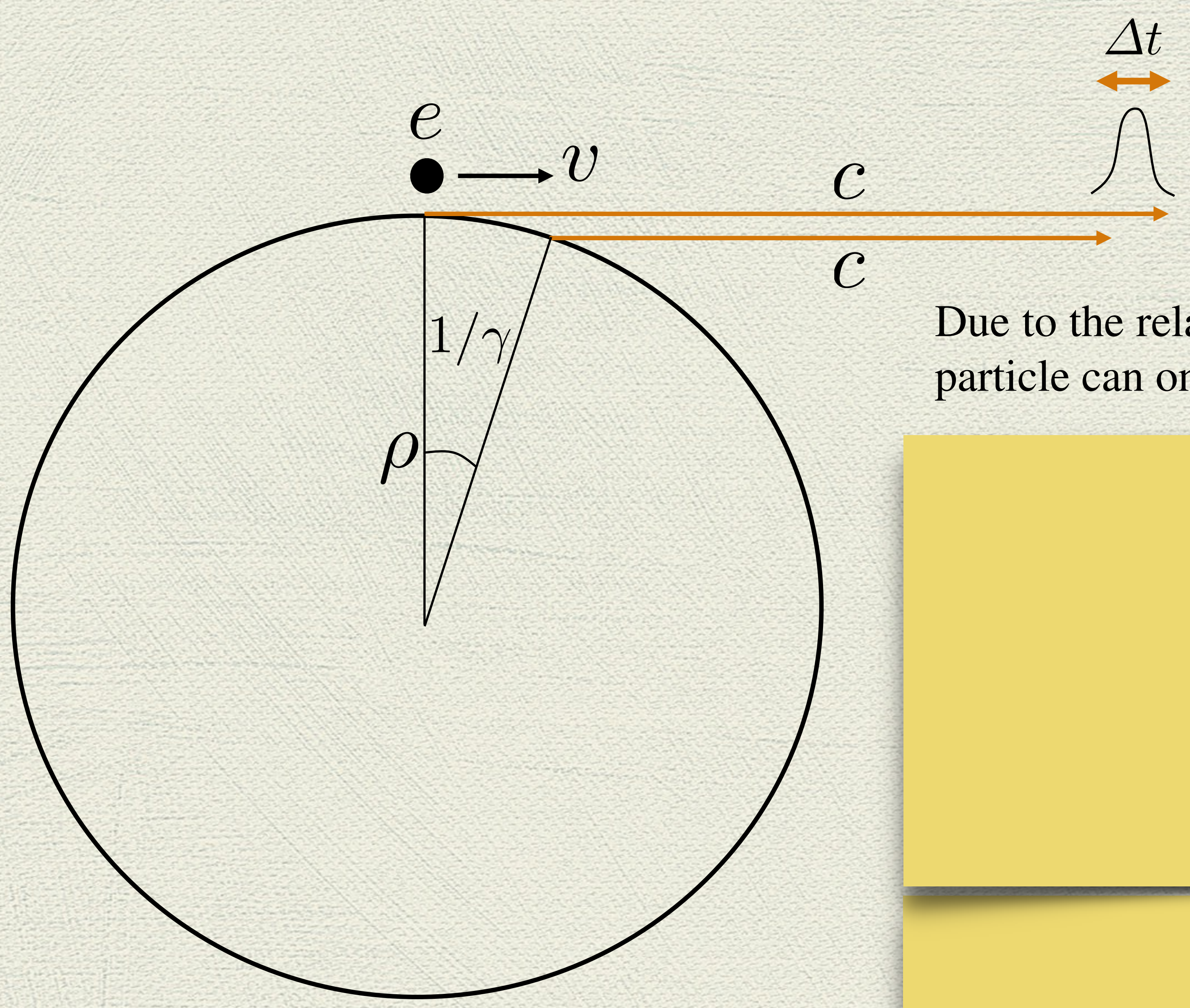

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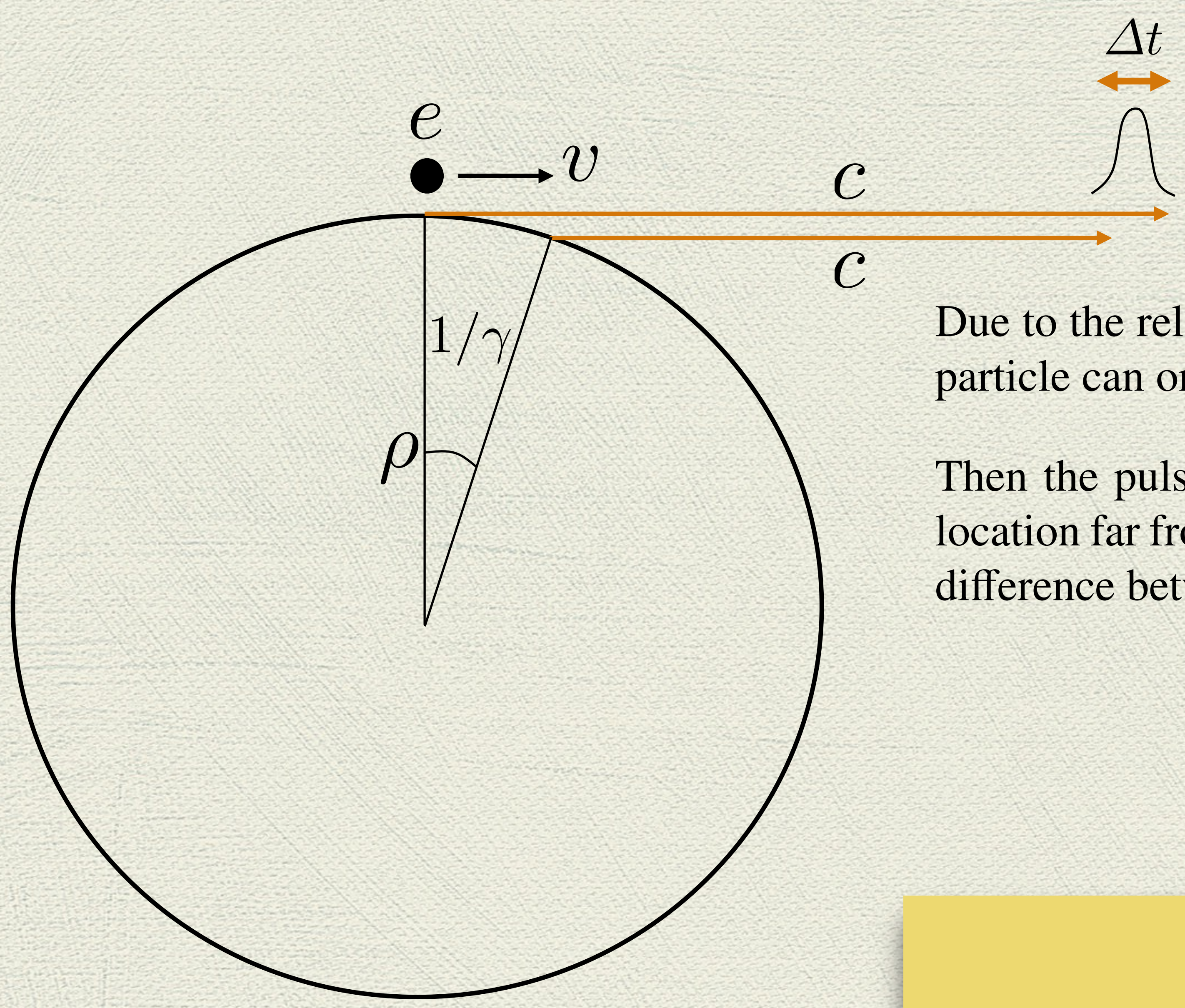

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Then the pulse length of the radiation observed at a location far from the ring can be estimated by the time difference between the particle and the light:

$$\Delta t \approx \frac{\rho/\gamma}{v} - \frac{\rho \sin 1/\gamma}{c} \tag{87}$$

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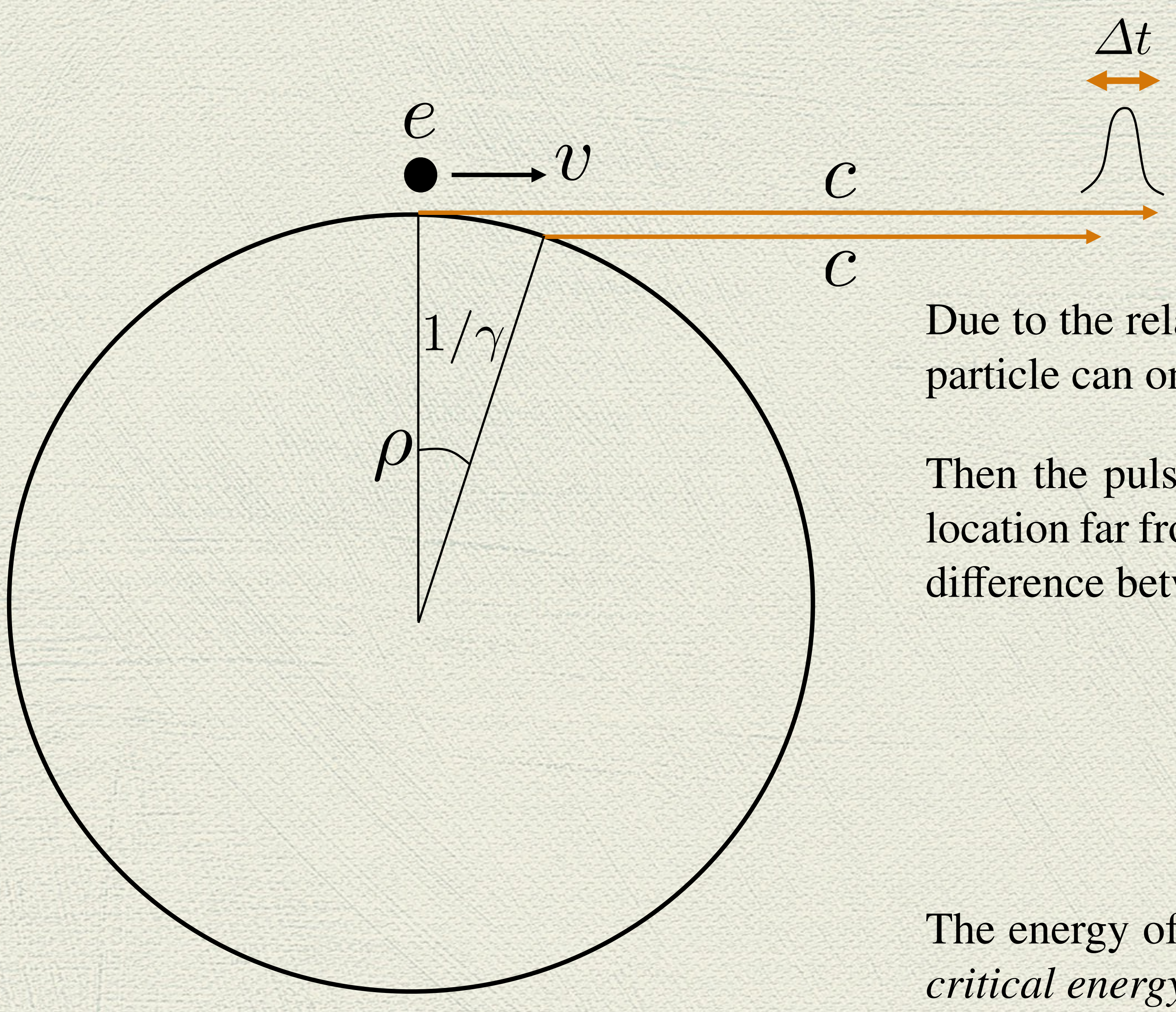

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The energy of the photon should be characterized by *critical energy* using the uncertainty principle:

$$u_c = \frac{\hbar}{\Delta t} = \frac{3}{2}\gamma^3 \frac{\hbar c}{\rho} = \frac{3}{2}\gamma^3 mc^2 \left(\frac{r_e}{\alpha \rho}\right) \,, \tag{89}$$

where α and r_e are the fine structure constant and the classical electron radius, respectively.

Overview of synchrotron radiation (2)

Most basic characteristics of the radiation, concerning to the beam, are represented by the following formulas: The expected number of photons in a bending angle ϕ:

$$\langle N \rangle = \frac{5}{2\sqrt{3}} \alpha \gamma \phi \, . \tag{90}$$

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The expected energy spread per an angle ϕ:

$$\langle \mathit{\Delta} E^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = \langle N \rangle \langle u^2 \rangle = \frac{55}{24\sqrt{3}} \frac{\gamma^7}{\alpha} m^2 c^4 \left(\frac{r_e}{\rho} \right)^2 \phi \, . \tag{94}$$

Overview of synchrotron radiation (3)

Let us denote the amount of the momentum loss, normalized by the design momentum P_d, in a small section of a beam line by $-dW(x, y, \delta; s)$. As we have seen, dW is expressed by

$$dW = \frac{\langle N \rangle \langle u \rangle}{cP_d} \,. \tag{95}$$

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We obtain the energy loss per revolution by integrating dW at the standard orbit:

$$U_0 = cP_d \oint dW_d = \frac{C_\gamma}{2\pi} E_d^4 \oint \frac{1}{\rho^2} ds \,, \tag{96}$$

where $E_d \approx cP_d$ and

$$C_\gamma \equiv \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} \,. \tag{97}$$

Overview of synchrotron radiation (3)

Let us denote the amount of the momentum loss, normalized by the design momentum P_d, in a small section of a beam line by $-dW(x, y, \delta; s)$. As we have seen, dW is expressed by

$$dW = \frac{\langle N \rangle \langle u \rangle}{cP_d} . \tag{95}$$

We obtain the energy loss per revolution by integrating dW at the standard orbit:

$$U_0 = cP_d \oint dW_d = \frac{C_\gamma}{2\pi} E_d^4 \oint \frac{1}{\rho^2} ds , \tag{96}$$

where $E_d \approx cP_d$ and

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If the bending radius is constant, $\rho = \rho_0$, along the ring,

$$U_0 = C_\gamma \frac{E_d^4}{\rho_0} = 88.5[\text{keV}] \times \frac{E_d[\text{GeV}]^4}{\rho_0[\text{m}]} . \tag{98}$$

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Table 1: One-turn energy loss U_0 for FCC-ee.

Beam Energy E_d	[GeV]	45.6	182.5
Bending radius ρ_0	[m]	10760	
Energy loss / turn U_0	[MeV]	35.6	9124

Justification of $\langle E^2 \rangle - \langle E \rangle^2 = \langle N \rangle \langle u^2 \rangle$

If the system emits k uncorrelated photons randomly, its probability is given by

$$\lambda_k = \frac{\langle N \rangle^k}{k!} \exp(-\langle N \rangle) . \tag{99}$$

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$$\begin{aligned}\langle E^2\rangle &= \sum_{k=0}^{\infty}\lambda_k\langle E_k^2\rangle\\ &= E_0^2 - 2E_0\langle N\rangle\langle u\rangle + \sum_{k=0}^{\infty}k(k-1)\frac{\langle N\rangle^k}{k!}\exp(-\langle N\rangle)\langle u^2\rangle + \langle N\rangle\langle u^2\rangle\\ &= E_0^2 - 2E_0\langle N\rangle\langle u\rangle + \langle N\rangle^2\langle u\rangle^2 + \langle N\rangle\langle u^2\rangle\\ &= \langle E\rangle^2 + \langle N\rangle\langle u^2\rangle\ .\end{aligned} \tag{103}$$

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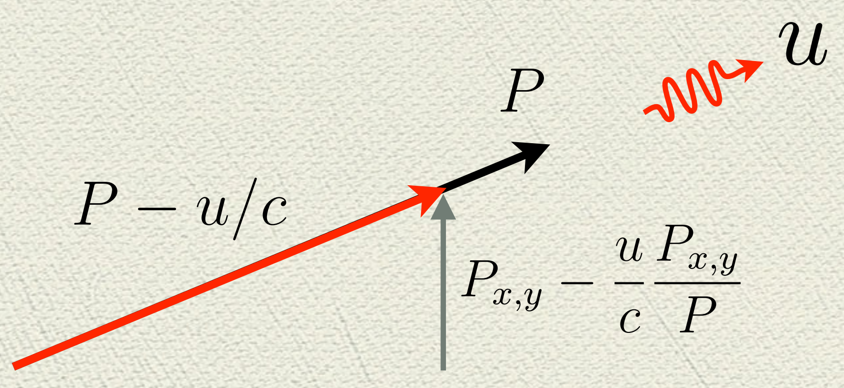
- The radiation is almost in the same direction of the particle momenta, within an angular deviation $\lesssim 1/\gamma$, The angular divergence will be omitted below for the time being.
- Thus the particle almost does not change the direction of motion. The changes of the momenta are expressed as

$$P \to P - u/c \,, \qquad\qquad P_{x,y} \to P_{x,y} - \frac{u}{c}\frac{P_{x,y}}{P} \,. \qquad (104)$$



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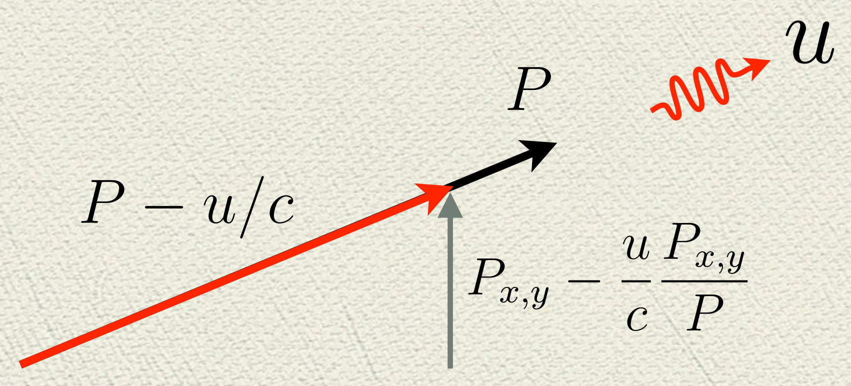

- The emission of photons is a totally random process. It is not possible to tell when and where a particle emits a photon. Each amount of photon is also stochastic under a given probability distribution.

Local radiation damping

Now let us consider a particle having small deviations $(x, p_x, y, p_y, z, \delta)$ from the *standard orbit*, where the radiation is given by dW_d.

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Then the change of momenta relative to the standard orbit due to radiation are written as

$$\begin{aligned} d\delta &= -(dW - dW_d) \\ &= -\frac{\partial dW}{\partial \delta}\delta - \frac{\partial dW}{\partial x}x - \frac{\partial dW}{\partial y}y\,, \end{aligned} \qquad (105)$$

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where we have assumed the momentum components at the standard orbit $p_{x,y;d}$ are zero.

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where we have assumed the momentum components at the standard orbit $p_{x,y;d}$ are zero.

In a simple case with $\frac{\partial dW}{\partial x} = \frac{\partial dW}{\partial y} = 0$, the equations above become

$$
d\delta = -\frac{\partial dW}{\partial \delta}\delta\,, \qquad dp_{x,y} = -dW p_{x,y}\,, \tag{108}
$$

which means a *local damping* of the momenta around the standard orbit.

Local radiation damping (2)

In the case of a uniform (transverse) magnetic field B, the momentum loss in a length ds is expressed as

$$dW = -\frac{\langle N \rangle \langle u \rangle}{c P_d} \tag{109}$$

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$$dW = -\frac{\langle N\rangle\langle u\rangle}{cP_d} \qquad (109)$$

$$\propto P^2B^2ds' \propto (1+\delta)^2 \qquad (110)$$

using Eq.(93) with $\phi = ds'/\rho$ and $P = (1+\delta)P_d = eB\rho$. Here we have introduced the orbit length ds', which will be described later.

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using Eq.(93) with $\phi = ds'/\rho$ and $P = (1+\delta)P_d = eB\rho$. Here we have introduced the orbit length ds', which will be described later.

Then the momentum damping coefficient in Eq. (108) is written as

$$\frac{\partial dW}{\partial \delta} = 2dW\ , \tag{111}$$

which means the longitudinal damping rate is twice faster than the transverse ones in this simple case.

$$d\delta = -\frac{\partial dW}{\partial \delta}\delta\ , \qquad dp_{x,y} = -dW\,p_{x,y}\ , \tag{103}$$

Effects of dispersion on damping

If there are dispersions, the motion should be considered in the betatron coordinates instead of physical coordinates:

$$\begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \end{pmatrix} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} - \begin{pmatrix} \eta_x \\ \eta_{p_x} \\ \eta_y \\ \eta_{p_y} \end{pmatrix} \delta \tag{112}$$

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$$z_\beta = z + \eta_{p_x} x - \eta_x p_x + \eta_{p_y} y - \eta_y p_y \, . \tag{113}$$

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we obtain

$$\begin{aligned} d\delta &= -\frac{\partial dW}{\partial x}(x_\beta + \eta_x \delta) - \frac{\partial dW}{\partial y}(y_\beta + \eta_y \delta) - \frac{\partial dW}{\partial \delta}\delta \ , \\ dx_\beta &= -\eta_x d\delta \ , \\ dp_{x\beta} &= -dW(p_{x\beta} + \eta_{px}\delta) - \eta_{px} d\delta \ , \\ dy_\beta &= -\eta_y d\delta \ , \\ dp_{y\beta} &= -dW(p_{y\beta} + \eta_{py}\delta) - \eta_{py} d\delta \ , \\ dz_\beta &= \eta_{p_x} dx_\beta - \eta_x dp_{x\beta} + \eta_{p_y} dy_\beta - \eta_y dp_{y\beta} \ . \end{aligned} \tag{115}$$

Effects of dispersion on damping (2)

The effect of radiation in the previous equation can be expressed by a matrix:

$$\begin{pmatrix} dx_\beta \\ dp_{x\beta} \\ dy_\beta \\ dp_{y\beta} \\ dz_\beta \\ d\delta \end{pmatrix} = d\mathrm{R}_D \begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial dW}{\partial x}\eta_x & . & \frac{\partial dW}{\partial y}\eta_x & . & . & -dR_{D66}\eta_x \\ \frac{\partial dW}{\partial x}\eta_{p_x} & -dW & \frac{\partial dW}{\partial y}\eta_{p_x} & . & . & -(dR_{D66}+dW)\eta_{p_x} \\ \frac{\partial dW}{\partial x}\eta_y & . & \frac{\partial dW}{\partial y}\eta_y & . & . & -dR_{D66}\eta_y \\ \frac{\partial dW}{\partial x}\eta_{p_y} & . & \frac{\partial dW}{\partial y}\eta_{p_y} & -dW & . & -(dR_{D66}+dW)\eta_{p_y} \\ . & dW\eta_x & . & dW\eta_y & . & dW(\eta_x\eta_{px}+\eta_y\eta_{py}) \\ -\frac{\partial dW}{\partial x} & . & -\frac{\partial dW}{\partial y} & . & . & dR_{D66} \end{pmatrix} \begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta \end{pmatrix}, \quad (116)$$

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Then the local damping rates $d\kappa_{x,y,z}$ are calculated by the diagonal parts of $d\mathrm{R}_D$:

$$\begin{aligned} d\kappa_x &= -\frac{dR_{D11}+dR_{D22}}{2} = \frac{1}{2}\left(dW - \frac{\partial dW}{\partial x}\eta_x\right) , \\ d\kappa_y &= -\frac{dR_{D33}+dR_{D44}}{2} = \frac{1}{2}\left(dW - \frac{\partial dW}{\partial y}\eta_y\right) , \\ d\kappa_z &= -\frac{dR_{D55}+dR_{D66}}{2} = \frac{1}{2}\left(\frac{\partial dW}{\partial x}\eta_x + \frac{\partial dW}{\partial y}\eta_y + 2dW\right) , \end{aligned} \tag{118}$$

where we have used Eq. (111).

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$$\frac{\partial dW}{\partial \delta} = 2dW\ , \tag{111}$$

Note that $d\kappa_x + d\kappa_y + d\kappa_z = 2dW$.

Radiation damping per revolution

We can integrate the local damping decrements Eq. (118) all over the ring to obtain the damping rate per revolution. We also assume there is no *x*-*y* coupling nor more complicated betatron-synchrotron couplings beyond that expressed by the dispersions.

$$
\begin{aligned}
d\kappa_x &= -\frac{dR_{D11} + dR_{D22}}{2} = \frac{1}{2}\left(dW - \frac{\partial dW}{\partial x}\eta_x\right) , \\
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Radiation damping per revolution

We can integrate the local damping decrements Eq. (118) all over the ring to obtain the damping rate per revolution. We also assume there is no *x*-*y* coupling nor more complicated betatron-synchrotron couplings beyond that expressed by the dispersions.

The local momentum loss dW is expressed as

$$-\langle N\rangle\langle u\rangle = -\frac{2}{3}\gamma^4 mc^2\left(\frac{r_e}{\rho}\right)\phi \qquad (93)$$

$$dW = \frac{\langle N\rangle\langle u\rangle}{cP_d} = \frac{2}{3}\gamma_0^3 r_e e^2(1+\delta)^2 B^2 ds' \,, \qquad (119)$$

$$\begin{aligned} d\kappa_x &= -\frac{dR_{D11}+dR_{D22}}{2} = \frac{1}{2}\left(dW - \frac{\partial dW}{\partial x}\eta_x\right) \,, \\ d\kappa_y &= -\frac{dR_{D33}+dR_{D44}}{2} = \frac{1}{2}\left(dW - \frac{\partial dW}{\partial y}\eta_y\right) \,, \qquad (118) \\ d\kappa_z &= -\frac{dR_{D55}+dR_{D66}}{2} = \frac{1}{2}\left(\frac{\partial dW}{\partial x}\eta_x + \frac{\partial dW}{\partial y}\eta_y + 2dW\right) \,, \end{aligned}$$

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with the orbit length ds'

$$ds' = \left(1 + \frac{x}{\rho_x} + \frac{y}{\rho_y}\right) ds \ , \qquad (120)$$

where $\rho_x = -P_d/eB_y$ and $\rho_y = P_d/eB_y$ are the bending radius in each plane, and $B^2 = B_x^2 + B_y^2$.

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We obtain the derivatives of dW in Eq. (118), on the standard orbit where $(x, p_x, y, p_y, z, \delta) = 0$. Using Eqs. (119,120), they are written as:

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Thus the damping decrement per revolution at the standard orbit becomes

$$\kappa_x = \oint d\kappa_x = \oint \left(1 - \frac{\eta_x}{\rho_x} - 2\eta_x \frac{B_y}{B^2}\frac{\partial B_y}{\partial x}\right)\frac{dW_d}{2} , \quad (124)$$

$$\kappa_y = \oint d\kappa_y = \oint \left(1 - \frac{\eta_y}{\rho_y} - 2\eta_y \frac{B_x}{B^2}\frac{\partial B_x}{\partial y}\right)\frac{dW_d}{2} , \quad (125)$$

$$\kappa_z = \oint d\kappa_y = \oint \left(2 + \frac{\eta_x}{\rho_x} + \frac{\eta_y}{\rho_y} + 2\eta_x \frac{B_y}{B^2}\frac{\partial B_y}{\partial x} + 2\eta_y \frac{B_x}{B^2}\frac{\partial B_x}{\partial y}\right)\frac{dW_d}{2} . \quad (126)$$

where $dW_d = \frac{C_\gamma}{2\pi c}E_d^3 \frac{1}{\rho^2} ds$, as shown in Eq. (96).

$$U_0 = cP_d \oint dW_d = \frac{C_\gamma}{2\pi}E_d^4 \oint \frac{1}{\rho^2} ds , \quad (91)$$

Radiation damping per revolution (3)

These damping decrements are expressed in terms of *damping partitions* $J_{x,y,z}$ as:

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$$J_x \approx J_y \approx 1, \qquad J_z \approx 2 \, . \tag{129}$$

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More terms in the damping partitions can arise from the edge angle of a dipole. Also a non-zero closed orbit will change them.

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Momentum spread

The stochastic nature of the synchrotron radiation generates beam emittance, which is the spread of particles in the phase space.

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$$\langle N\rangle\langle u^2\rangle = \frac{55}{24\sqrt{3}}\frac{\gamma^7}{\alpha}m^2c^4\left(\frac{r_e}{\rho}\right)^2\phi\,. \qquad (94)$$

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First let us look at the z-plane. From Eq. (94), we can write the increment of the square of momentum spread in a small section ds as

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The momentum spread of the beam follows an equation of damping and excitation:

$$\frac{d\sigma_\delta^2}{dn} = -2\kappa_z\sigma_\delta^2 + \frac{1}{2}\oint dA_z\,, \tag{131}$$

where n is the number of turns.

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Using Eq. (127), at the equilibrium the momentum spread becomes

$$\sigma_\delta^2 = \frac{1}{4\kappa_z}\oint dA_0 = \frac{1}{2J_z}\frac{cP_d}{U_0}\oint dA_z \qquad (132)$$

$$= \frac{C_q}{J_z}\gamma_0^2\frac{\oint 1/|\rho|^3 ds}{\oint 1/\rho^2 ds}\,, \qquad (133)$$

where $C_q \equiv \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 0.9923\lambda\!\!\!^{-}_e$.

$$\kappa_{x,y,z} = \frac{J_{x,y,z}}{2}\oint dW_d = \frac{J_{x,y,z}}{2}\frac{U_0}{cP_d}\,. \qquad (122)$$

Transverse emittance

The transverse equilibrium emittance can be estimated in a similar way. In the normalized coordinate

$$(u_x, p_{ux}) = \left(\frac{x}{\sqrt{\beta_x}}, \sqrt{\beta_x}p_x + \frac{\alpha_x}{\sqrt{\beta_x}}x\right), \tag{134}$$

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where $\gamma_x = (1 + \alpha_x^2)/\beta_x$, and we have used $dx_\beta = -\eta_x d\delta$ and $dp_{x\beta} = -\eta_{px} d\delta$. Then in a similar way as the z-plane, the equilibrium emittance can be obtained as

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$$\varepsilon_{x,y} = \frac{1}{4\kappa_{x,y}} \oint dA_{x,y} = \frac{1}{2J_{x,y}} \frac{cP_d}{U_0} \oint dA_{x,y}$$

$$= \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{\oint \mathcal{H}_{x,y}/|\rho|^3 ds}{\oint 1/\rho^2 ds} , \tag{139}$$

where

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y} \eta_{x,y}^2 + 2\alpha_{x,y} \eta_{x,y} \eta_{px,py} + \beta_{x,y} \eta_{px,py}^2 . \tag{140}$$

Longitudinal emittance

The result for the transverse equilibrium emittance Eq. (139) is extendable to z-plane as:

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$$\mathcal{H}_z \equiv \gamma_z \eta_z^2 + 2\alpha_{x,y}\eta_z\eta_\delta + \beta_z\eta_\delta^2 \tag{142}$$

$$\approx \beta_z\,, \tag{143}$$

where we have used $\eta_\delta = 1$ and $\eta_z \approx 0$.

Longitudinal emittance

The result for the transverse equilibrium emittance Eq. (139) is extendable to z-plane as:

$$\varepsilon_z = \frac{C_q}{J_z}\gamma_0^2 \frac{\oint \mathcal{H}_z/|\rho|^3 ds}{\oint 1/\rho^2 ds} \ , \tag{141}$$

with

$$\mathcal{H}_z \equiv \gamma_z \eta_z^2 + 2\alpha_{x,y}\eta_z\eta_\delta + \beta_z\eta_\delta^2 \tag{142}$$

$$\approx \beta_z \ , \tag{143}$$

where we have used $\eta_\delta = 1$ and $\eta_z \approx 0$.

From the emittance, the resulting bunch length and the energy spread are written as

$$\sigma_z^2 = \beta_z \varepsilon_z \ , \tag{144}$$

$$\sigma_\delta^2 = \gamma_z \varepsilon_z = \frac{1+\alpha_z^2}{\beta_z}\varepsilon_z \ . \tag{145}$$

In the case of a slow synchrotron motion, above agree with previous results by setting $\beta_z \approx$ const. $= -\dfrac{\alpha_p}{\mu_z}C$ as in Eq. (86).

x-y coupling

- The expression for the vertical equilibrium emittance Eq. (139) is only true when the x-y coupling is zero. It is no longer correct under x-y coupling.

$$\varepsilon_{x,y} = \frac{C_q}{J_{x,y}}\gamma_0^2\frac{\oint \mathcal{H}_{x,y}/|\rho|^3 ds}{\oint 1/\rho^2 ds}\,, \tag{139}$$

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y}\eta_{x,y}^2 + 2\alpha_{x,y}\eta_{x,y}\eta_{px,py} + \beta_{x,y}\eta_{px,py}^2$$

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- On the other hand, the vertical emittance can be increased depending on the magnitude of the skew quadrupole.

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There are several ways to express such betatron coordinates (u, p_u, v, p_v). One way is

$$\begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = \mathrm{R} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} \mu & . & -r_4 & r_2 \\ . & \mu & r_3 & -r_1 \\ r_1 & r_2 & \mu & . \\ r_3 & r_4 & . & \mu \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} , \tag{146}$$

where $r_{1,2,3,4}$ are the coupling coefficients at each location s and $\mu^2 + (r_1 r_4 - r_2 r_3) = 1$.

$$\varepsilon_{x,y} = \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{\oint \mathcal{H}_{x,y}/|\rho|^3 ds}{\oint 1/\rho^2 ds} , \tag{139}$$

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y}\eta_{x,y}^2 + 2\alpha_{x,y}\eta_{x,y}\eta_{px,py} + \beta_{x,y}\eta_{px,py}^2$$

x-y coupling (2)

Then for the function $\mathcal{H}$ we should use dispersions for (u, p_u, v, p_v)., it ie.,

$$\begin{pmatrix}\eta_u\\ \eta_{pu}\\ \eta_v\\ \eta_{pv}\end{pmatrix} = \mathrm{R}\begin{pmatrix}\eta_x\\ \eta_{px}\\ \eta_y\\ \eta_{py}\end{pmatrix} = \begin{pmatrix}\mu & . & -r_4 & r_2\\ . & \mu & r_3 & -r_1\\ r_1 & r_2 & \mu & .\\ r_3 & r_4 & . & \mu\end{pmatrix}\begin{pmatrix}\eta_x\\ \eta_{px}\\ \eta_y\\ \eta_{py}\end{pmatrix}, \tag{147}$$

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Thus even with $\eta_y = \eta_{py} = 0$, the dispersions in the normal coordinates can be nonzero due to the x-y coupling.

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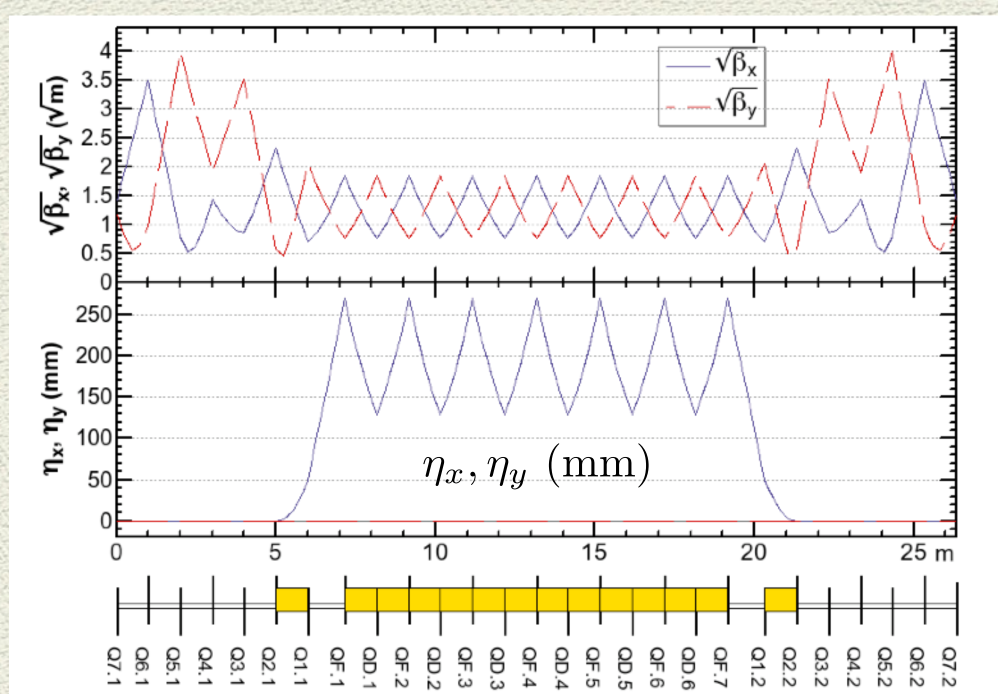
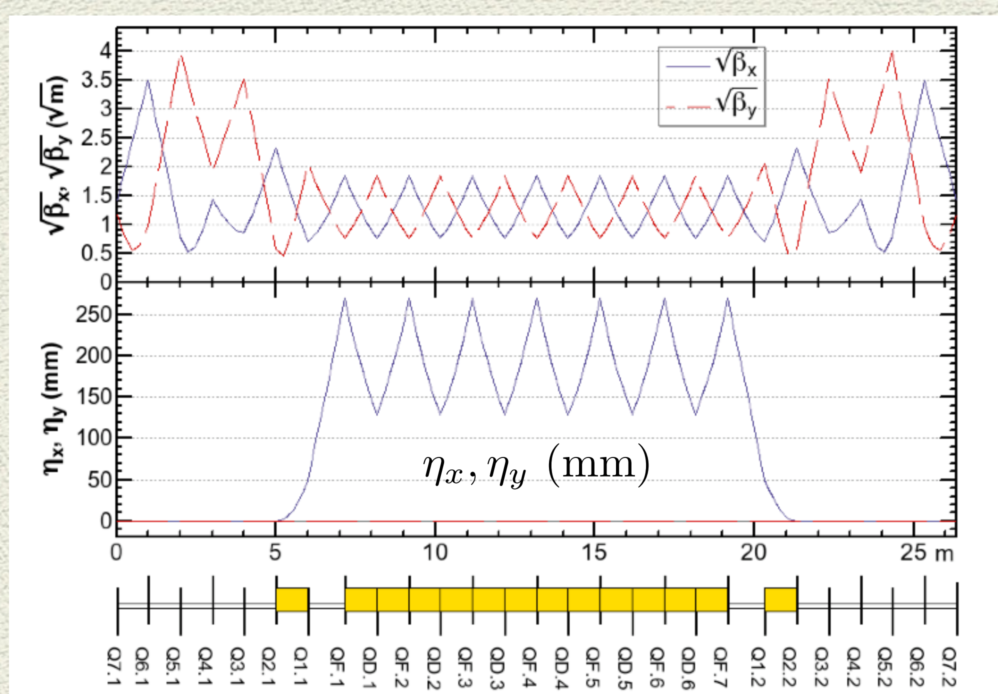
- Practically such an x-y coupling can arise from the vertical offset of sextupoles, which generates skew quadrupole components.
- Usually a sextupole is placed at a location with non-zero horizontal dispersion for the chromaticity correction, then such a vertical offset also produces the vertical dispersion.
- Thus to reduce the vertical emittance, correcting the vertical dispersion is not enough and x-y coupling correction through the ring is necessary.

x-y coupling (3)



Under the presence of x-y coupling, the vertical emittance can increase even with zero (physical) vertical dispersion.

x-y coupling (3)



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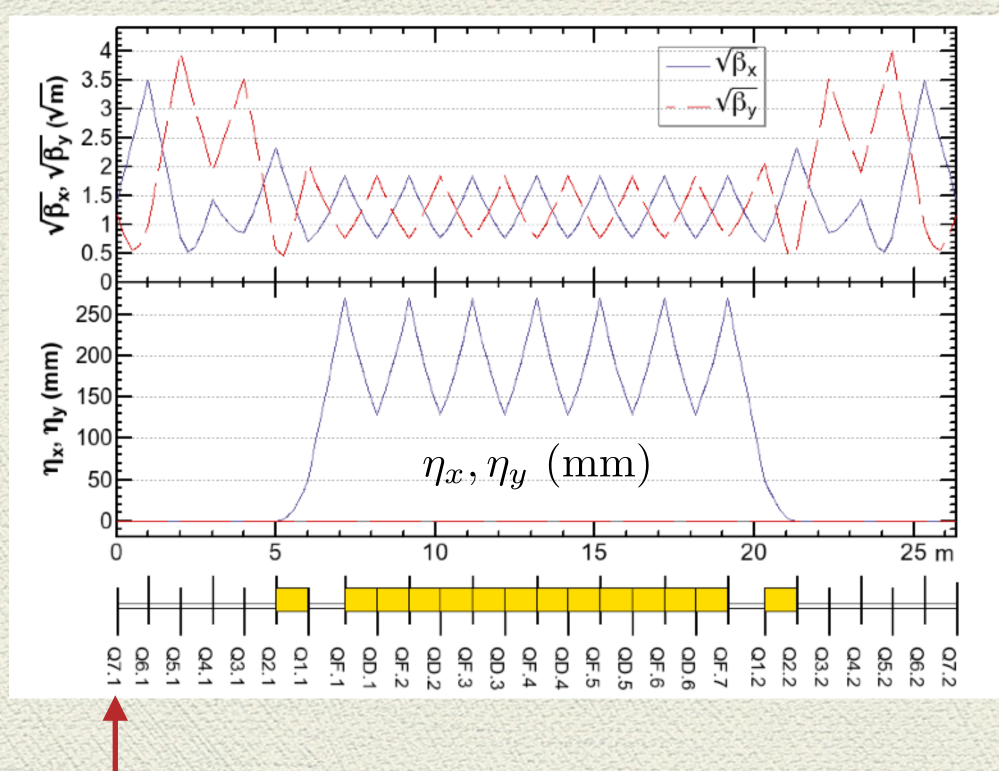
Skew quadrupole (SK1)

x-y coupling (3)



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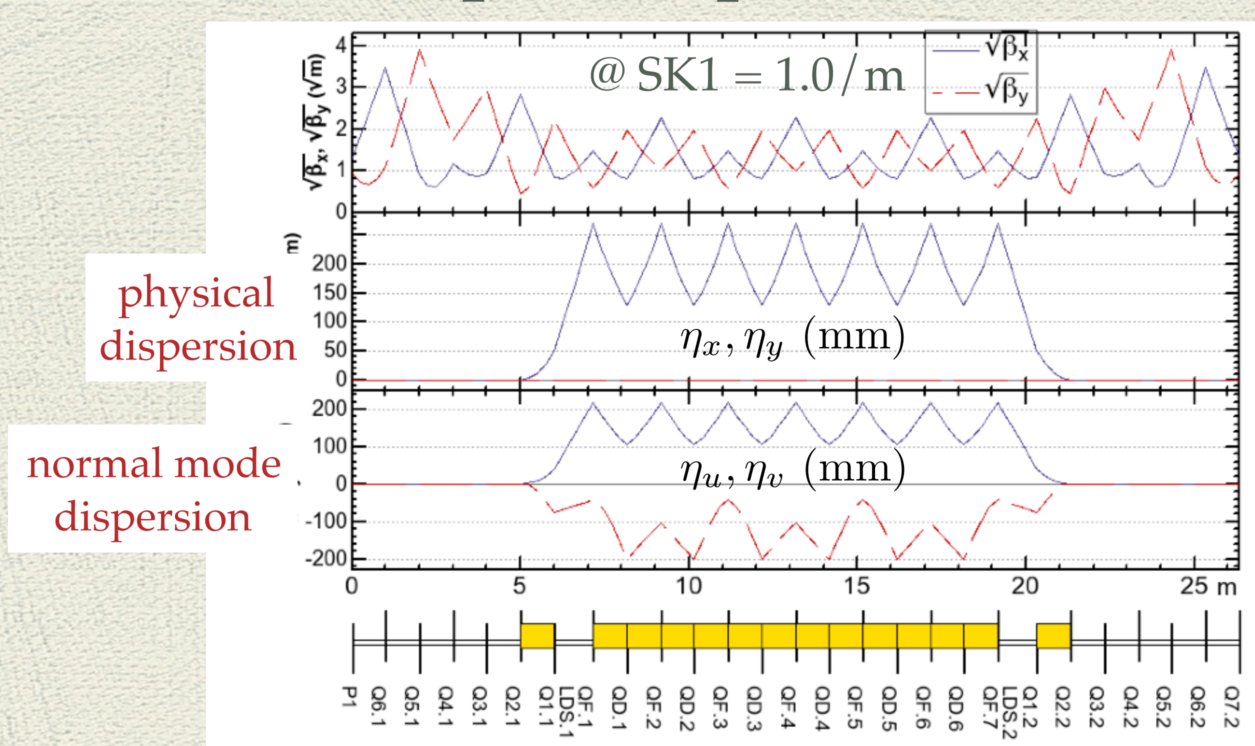

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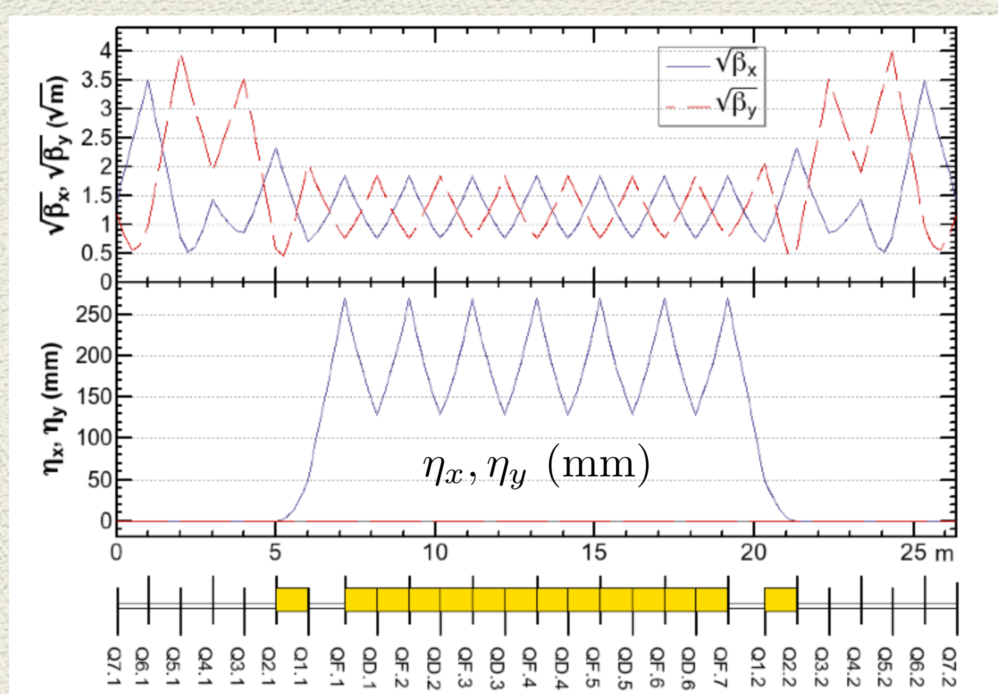

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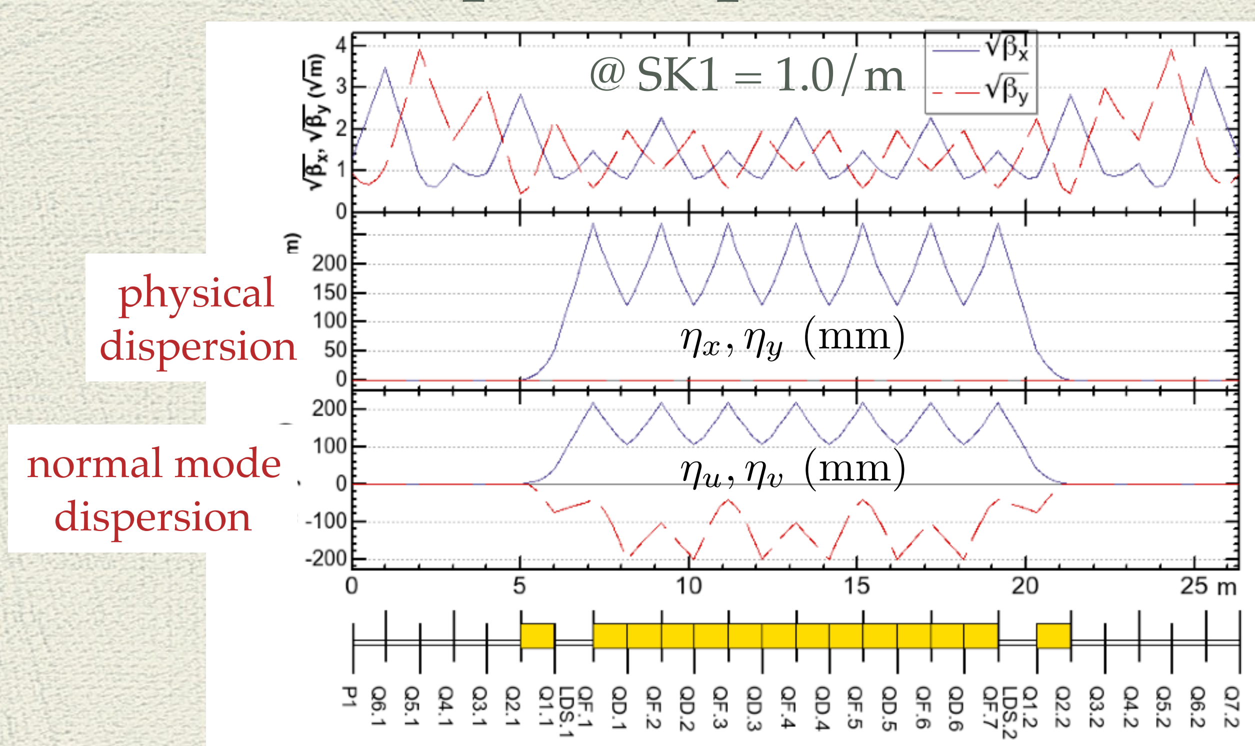
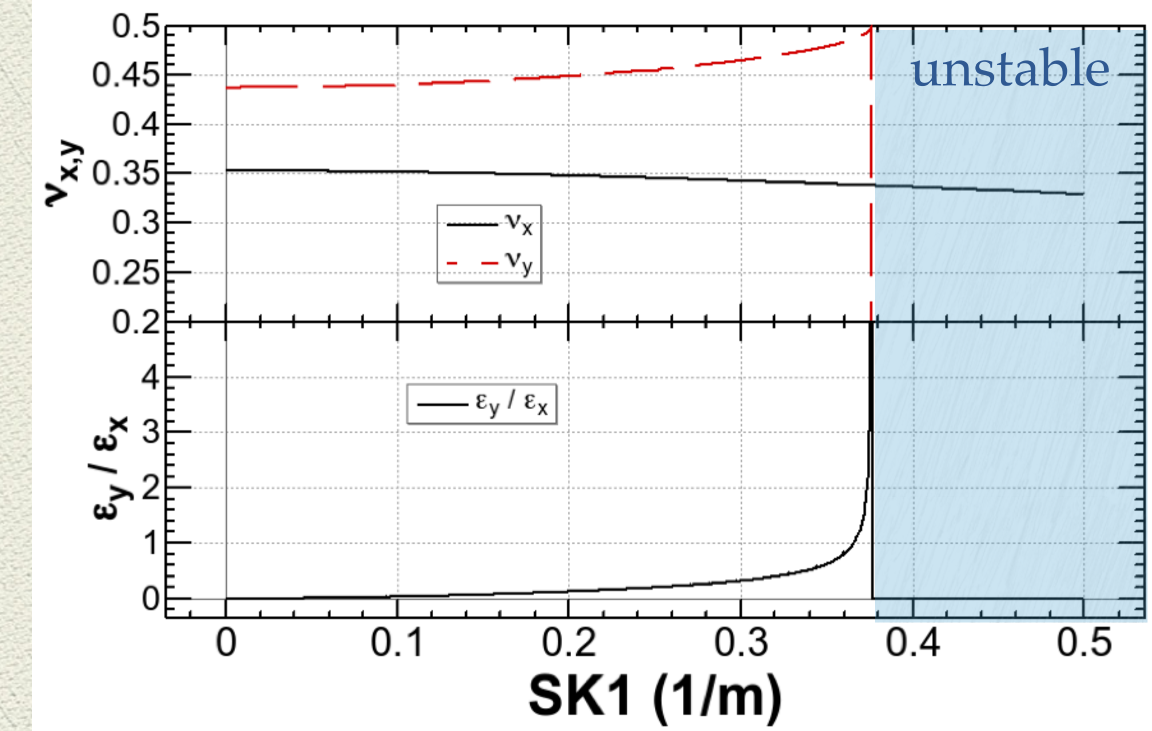


Emittance ratio



Emittance by the opening angle of photons

Usually for a vertical direction in a flat storage ring, the design vertical dispersion is zero and there is no *x*-*y* coupling in the ring. Then the vertical emittance given by Eq. (139) becomes zero.

$$\varepsilon_{x,y} = \frac{C_q}{J_{x,y}}\gamma_0^2\frac{\oint \mathcal{H}_{x,y}/|\rho|^3 ds}{\oint 1/\rho^2 ds}\,, \qquad (139)$$

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$$dA_{y,1/\gamma} = \beta_y \langle d\delta^2/\gamma^2 \rangle \,, \tag{148}$$

assuming the emitted photons have angular divergence $\sim 1/\gamma$.

$$dA_x = \gamma_x \langle dx_\beta^2 \rangle + \alpha_x \langle dx_\beta dp_{x\beta} \rangle + \beta_x \langle dp_{x\beta}^2 \rangle \,, \tag{136}$$

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In the case of uniform bending radius $\rho = \rho_0$,

$$\varepsilon_{y,1/\gamma} = \frac{C_q}{J_y} \frac{\langle \beta_y \rangle}{\rho_0} \approx \frac{\lambda_e}{J_y} \frac{\langle \beta_y \rangle}{\rho_0} \,. \tag{150}$$

Interestingly, this limit on the vertical emittance does not explicitly depend on the beam energy. Anyway the value is more important for a small ring.

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Synchrotron radiation integrals

We can summarize these formulas related to synchrotron radiation in terms of *radiation integrals* defined by:

$$I_1 \equiv \oint \left(\frac{\eta_x}{\rho_x} + \frac{\eta_y}{\rho_y}\right) ds\,, \qquad I_2 \equiv \oint \frac{1}{\rho^2} ds\,, \qquad I_3 \equiv \oint \frac{1}{|\rho|^3} ds\,,$$
$$I_{4x} \equiv \oint \eta_x \left(\frac{1}{\rho_x} + 2\frac{B_y}{B^2}\frac{\partial B_y}{\partial x}\right) ds\,, \quad I_{4y} \equiv \oint \eta_y \left(\frac{1}{\rho_y} + 2\frac{B_x}{B^2}\frac{\partial B_x}{\partial y}\right) ds\,, \quad I_{5x,y} \equiv \oint \frac{\mathcal{H}_{x,y}}{|\rho|^3} ds\,. \tag{144}$$

Then we can write as:

$$\alpha_p = \frac{I_1}{C}\,, \qquad U_0 = \frac{C_\gamma}{2\pi} E_d^4 I_2\,, \tag{145}$$
$$J_{x,y} = 1 - \frac{I_{4x,y}}{I_2}\,, \qquad J_z = 4 - J_x - J_y = 2 + \frac{I_{4x} + I_{4y}}{I_2}\,, \tag{146}$$
$$\kappa_{x,y,z} = \frac{J_{x,y,z}}{2}\frac{U_0}{cP_d}\,, \tag{147}$$
$$\sigma_\delta^2 = \frac{C_q}{J_z}\gamma_0^2 \frac{I_3}{I_2}\,, \qquad \varepsilon_{x,y} = \frac{C_q}{J_{x,y}}\gamma_0^2 \frac{I_{5x,y}}{I_2}\,. \tag{148}$$

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