

Basic formulas for electron rings

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Contents

1. Basics

- ◆ Variables
- ◆ Hamiltonian
- ◆ Transfer matrix
 - ◆ Symplecticity

2. Transformation of simple elements

- ◆ Drift space
 - ◆ Chromaticity
- ◆ Dipole magnet
- ◆ Thin multipole magnet
 - ◆ Thin quadrupole magnet
- ◆ RF cavity

3. FODO cells

- ◆ Typical structure of a collider
 - ◆ Various arc lattices
- ◆ Transfer matrix of a FODO cell
- ◆ Twiss parameters
- ◆ Twiss parameters along a beam line

3. FODO cells (cont'd)

- ◆ Dispersion
- ◆ Dispersion along a beam line
- ◆ Momentum compaction
- ◆ Synchrotron motion

4. Overview of synchrotron radiation

- ◆ Critical energy
- ◆ Some fundamental formulas
- ◆ Energy loss
- ◆ About $\langle N \rangle \langle u^2 \rangle$

5. Effects of radiation on beam motion

- ◆ basics
- ◆ Local radiation damping
- ◆ Effects of dispersion on damping
- ◆ Damping per revolution
- ◆ Momentum spread
- ◆ Transverse emittance
- ◆ Longitudinal emittance
- ◆ X-y coupling
- ◆ Emittance by opening angle of photons
- ◆ Synchrotron radiation integrals

Variables

In this lecture we use *canonical* variables:

$$(x, p_x, y, p_y, z, \delta), \quad (1)$$

where momenta are normalized by *design momentum* P_d as:

$$\begin{aligned} p_x &\equiv P_x/P_d, \\ p_y &\equiv P_y/P_d, \\ z &\equiv -v(t - t_0), \\ \delta &\equiv (P - P_d)/P_d, \end{aligned} \quad (2)$$

where we have introduced the total momentum P , the design arrival time $t_0 = t_0(s)$, and the total velocity $v = cP / \sqrt{m^2c^2 + P^2}$. For a storage ring, usually P_d is chosen constant over the entire ring.

These variables are functions of s , which is the length along the coordinate line.

Hamiltonian

For the variables defined above, the Hamiltonian is written as

$$H = -(1 + x/\rho) \sqrt{(1 + \delta + \varphi/c)^2 - (p_x - A_x)^2 - (p_y - A_y)^2} - (1 + x/\rho)A_z - (xp_y - yp_x)/\tau + \frac{E}{v_d} + \left(\frac{1}{v} + \frac{1}{c}\right)z \frac{\partial \varphi}{\partial s}, \quad (3)$$

where ρ , τ , and v_d are the bending radius, the torsion, and the design velocity, respectively, and $E = c \sqrt{(1 + \delta + \varphi/c)^2 + m^2 c^2 / P_d^2} + \varphi$ is the (normalized) energy.

The bending radius and the torsion are associated with the coordinate system and *not* with the motion of particles. Thus usually we do not have to use the torsion τ , even if particles are doing helical motions.

The external fields are expressed by (normalized) electromagnetic potentials (A_x, A_y, A_z, φ) , which are functions of $(x, y, z; s)$. In the case of magnets or RF cavities without a solenoid component B_z , only A_z is necessary to express the field. In such cases the Hamiltonian is simplified to

$$H = -(1 + x/\rho) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} - (1 + x/\rho)A_z + \frac{E}{v_d}. \quad (4)$$

Transfer Matrix

If a particle does a motion from s_0 to s , a transfer matrix M from s_0 to s is defined by

$$M = \frac{\partial(x, p_x, y, p_y, z, \delta)}{\partial(x_0, p_{x0}, y_0, p_{y0}, z_0, \delta_0)}, \quad (5)$$

where variables suffixed by 0 mean the initial value at $s = s_0$.

In general M depends on the initial values $(x_0, p_{x0}, y_0, p_{y0}, z_0, \delta_0)$.

The most fundamental nature of the transfer matrix for a motion associated with a Hamiltonian is the *symplectic condition*:

“.” represents 0.

$${}^t M J M = J, \quad J \equiv \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \dots \\ -1 & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & 1 & \dots \\ \cdot & \cdot & -1 & \cdot & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (6)$$

It is known for a symplectic motion that the beam emittance, the spread of the beam in the phase space, is preserved *in each plane*, once the variables are chosen properly.

Symplectic condition

$$\begin{pmatrix}
 * & * & * & \dots \\
 * & * & * & \dots \\
 + & & & \\
 * & * & * & \dots \\
 * & * & * & \dots \\
 + & & & \\
 \dots & \dots & \dots & \dots \\
 \dots & & & \\
 = 1
 \end{pmatrix}$$

$$\begin{pmatrix}
 * & * & * & * & \dots \\
 * & * & * & * & \dots \\
 + & & & & \\
 * & * & * & * & \dots \\
 * & * & * & * & \dots \\
 + & & & & \\
 \dots & \dots & \dots & \dots & \dots \\
 \dots & & & & \\
 = 0
 \end{pmatrix}$$

✕ : 2 by 2 determinant

The sum of 2 by 2 determinants between two columns or two rows are:

$$\begin{cases}
 = 1 \text{ for } (u, p_u) . \\
 = 0 \text{ for } (u, v), (u, p_v), (p_u, p_v), \text{ when } u \neq v ,
 \end{cases}$$

Drift space

A *drift space* is a field-free region in the beam line. Its Hamiltonian is

$$H = -\sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} + \frac{E}{v_d}. \quad (7)$$

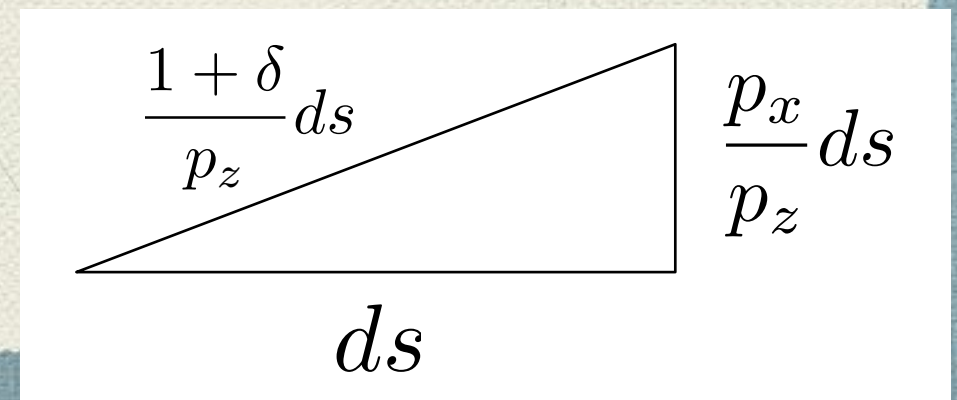
The solution of the motion for a drift space with a length L is

$$\begin{aligned} x &= x_0 + \frac{p_{x0}}{p_z} L, & p_x &= p_{x0} \\ y &= y_0 + \frac{p_{y0}}{p_z} L, & p_y &= p_{y0} \\ z &= z_0 + \left(\frac{v}{v_d} - \frac{1 + \delta_0}{p_z} \right) L, & \delta &= \delta_0, \end{aligned} \quad (8)$$

where we have used

$$p_z \equiv \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}. \quad (9)$$

Note that the motion in a drift space is *nonlinear* in the momenta due to the $1/p_z$ dependence.



Drift space (2)

For an on-axis particle $(x_0, p_{x0}, y_0, p_{y0}) = (0, 0, 0, 0)$, the transfer matrix of a drift space becomes

$$x = x_0 + \frac{p_{x0}}{p_z} L,$$

$$y = y_0 + \frac{p_{y0}}{p_z} L,$$

$$z = z_0 + \left(\frac{v}{v_d} - \frac{1 + \delta_0}{p_z} \right) L,$$

$$M = \begin{pmatrix} 1 & \frac{L}{1 + \delta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{L}{1 + \delta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \frac{v - v_d}{v_d} L \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (10)$$

since, for instance,

$$\begin{aligned} \frac{\partial x}{\partial p_{x0}} &= \frac{\partial}{\partial p_{x0}} \left(\frac{p_{x0}}{p_{z0}} L \right) = \left(\frac{1}{p_{z0}} - \frac{p_{x0}}{p_{z0}^2} \frac{\partial p_{z0}}{\partial p_{x0}} \right) L = \left(\frac{1}{p_{z0}} + \frac{p_{x0}^2}{p_{z0}^3} \right) L \\ &= \frac{L}{1 + \delta}, \quad \because p_{x0} = p_{y0} = 0, p_z = 1 + \delta. \end{aligned}$$

Thus the "length" of a drift has a $1/(1 + \delta)$ dependence on the momentum offset δ . This is the source of *natural chromaticity*.

Chromaticity

$$H = -\sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} + \frac{E}{v_d}.$$

If we approximate the Hamiltonian of a drift space up to the second order of p_x and p_y :

$$H \approx -(1 + \delta) \left(1 - \frac{p_x^2}{2(1 + \delta)^2} - \frac{p_y^2}{2(1 + \delta)^2} \right) + \frac{E}{v_d}, \quad (11)$$

the change of path length dz/ds due to the transverse momenta is expressed as

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = -\frac{p_x^2}{2(1 + \delta)^2} - \frac{p_y^2}{2(1 + \delta)^2} + \frac{v - v_d}{v_d}. \quad (12)$$

It means that the change of the path length should be proportional to the transverse actions (another canonical variable, square of the amplitudes in the normalized phase space, to be introduced later):

$$\Delta z = 2\pi\xi_x J_x + 2\pi\xi_y J_y$$

with coefficients ξ_x, ξ_y .

Then since

$$2\pi\xi_x = \frac{\partial \Delta z}{\partial J_x} = \frac{\partial}{\partial J_x} \frac{\partial H}{\partial \delta} = \frac{\partial}{\partial \delta} \frac{\partial H}{\partial J_x} = \frac{\partial \mu_x}{\partial \delta}, \quad (14)$$

we have realized that the coefficients $\xi_{x,y}$ correspond to the momentum derivatives of tunes $\frac{1}{2\pi} \frac{\partial \mu_{x,y}}{\partial \delta}$.

$$\frac{1 + \delta}{p_z} ds \approx \left(1 + \frac{p_x^2}{2(1 + \delta)^2} \right) ds$$

Chromaticity (2)

For an off-momentum particle having $d\delta$, a small section of a drift space ds has an effective length $ds/(1 + d\delta) \approx -dsd\delta$. Thus the entire transfer matrix of a stable ring changes to

$$\begin{aligned} M' &= \begin{pmatrix} \cos \mu' + \alpha' \sin \mu' & \beta' \sin \mu' \\ -\frac{1 + \alpha'^2}{\beta'} \sin \mu' & \cos \mu' - \alpha' \sin \mu' \end{pmatrix} = \begin{pmatrix} 1 & -dsd\delta \\ & 1 \end{pmatrix} \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\frac{1 + \alpha^2}{\beta} \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix} \\ &= \begin{pmatrix} \cos \mu + \left(\alpha + \frac{1 + \alpha^2}{\beta} dsd\delta \right) \sin \mu & \beta \sin \mu - (\cos \mu - \alpha \sin \mu) dsd\delta \\ \frac{1 + \alpha^2}{\beta} \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}. \end{aligned} \quad (15)$$

By looking at the trace of M' above we obtain:

$$2 \cos \mu' = 2 \cos \mu + \frac{1 + \alpha^2}{\beta} \sin \mu dsd\delta. \quad (16)$$

By putting $\mu' = \mu + d^2\mu$ into above:

$$d^2\mu = -\frac{1 + \alpha^2}{2\beta} dsd\delta, \quad \text{and} \quad d\mu = -d\delta \oint \frac{1 + \alpha^2}{2\beta} ds \quad (17)$$

by integrating over the ring. Thus the *natural chromaticity* is expressed as

$$\xi_{x,y} \equiv \frac{1}{2\pi} \frac{\partial \mu_{x,y}}{\partial \delta} = -\frac{1}{4\pi} \oint \frac{1 + \alpha_{x,y}^2}{\beta_{x,y}} ds. \quad (18)$$

Note that $\xi_{x,y}$ is always negative.

Dipole magnet

A horizontally bending dipole magnet with a uniform magnetic field

$$\mathbf{B} = (0, B, 0), \quad B = \frac{P_d}{e\rho} \quad (19)$$

has a Hamiltonian:

$$H = -\left(1 + \frac{x}{\rho}\right) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} + \frac{1}{2} \left(1 + \frac{x}{\rho}\right)^2 + \frac{E}{v_d}. \quad (20)$$

The equations of motion are:

$$\begin{aligned} \frac{dx}{ds} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{p_z} \left(1 + \frac{x}{\rho}\right), & \frac{dp_x}{ds} &= -\frac{\partial H}{\partial x} = \left(p_z - 1 - \frac{x}{\rho}\right) \frac{1}{\rho}, \\ \frac{dy}{ds} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{p_z} \left(1 + \frac{x}{\rho}\right), & \frac{dp_y}{ds} &= -\frac{\partial H}{\partial y} = 0, \\ \frac{dz}{ds} &= \frac{\partial H}{\partial \delta} = -\frac{1 + \delta}{p_z} \left(1 + \frac{x}{\rho}\right) + \frac{v}{v_d}, & \frac{d\delta}{ds} &= -\frac{\partial H}{\partial z} = 0. \end{aligned} \quad (21)$$

$$H = -(1 + x/\rho) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} - (1 + x/\rho)A_z + \frac{E}{v_d}.$$

$$\begin{aligned} A_z &= -\frac{1}{2} \left(1 + \frac{x}{\rho}\right), \\ \frac{eB_y}{P_d} &= -\frac{1}{1 + x/\rho} \frac{\partial}{\partial x} [(1 + x/\rho)A_z] \\ &= \frac{1}{\rho} \end{aligned}$$

Dipole magnet (2)

The motion in a uniform magnetic field is helical motion along the field line, which is in y-direction in this case. First we can derive

$$\frac{d^2 p_x}{ds^2} = -\frac{p_x}{\rho^2}, \quad (22)$$

then the solution has the form:

$$p_x = a \cos \theta + b \sin \theta, \quad (23)$$

where $\theta \equiv s/\rho$, and a and b are constants determined by the initial conditions.

Thus we obtain

$$p_x = p_{x0} \cos \theta + \left(p_{z0} - 1 - \frac{x_0}{\rho} \right) \sin \theta, \quad (24)$$

by comparing with the initial values of p_x and $\frac{dp_x}{ds}$.

Then we continue to obtain the solution of the other variables:

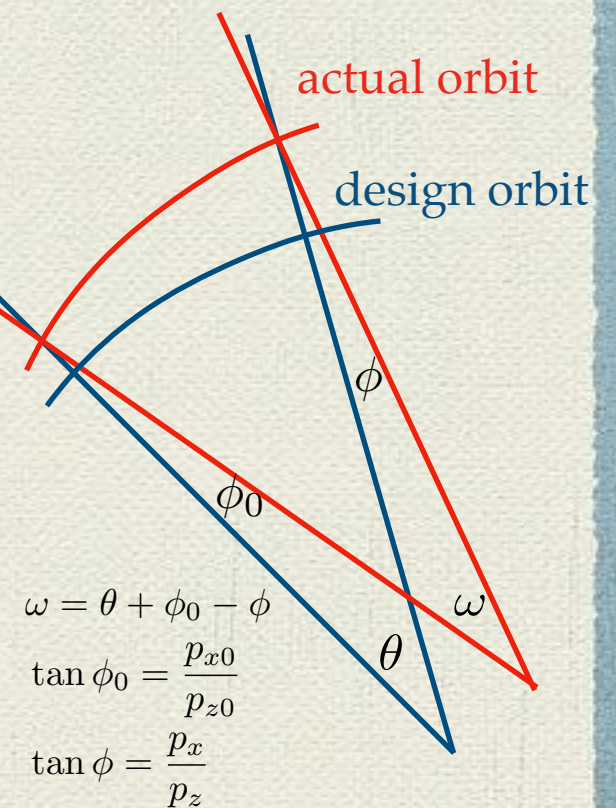
$$x = \rho \left\{ p_{x0} \sin \theta - \left(p_{z0} - 1 - \frac{x_0}{\rho} \right) \cos \theta + p_z - 1 \right\}, \quad (25)$$

$$y = y_0 + \rho \omega p_{y0}, \quad (26)$$

$$z = z_0 - \rho \omega (1 + \delta) + \frac{v}{v_d} \rho \theta \quad (27)$$

with the angle of the arc of the actual orbit:

$$\omega = \theta + \tan^{-1} \frac{p_{x0}}{p_{z0}} - \tan^{-1} \frac{p_x}{p_z}. \quad (28)$$



$$\omega = \theta + \phi_0 - \phi$$

$$\tan \phi_0 = \frac{p_{x0}}{p_{z0}}$$

$$\tan \phi = \frac{p_x}{p_z}$$

Dipole magnet (3)

The transfer matrix for a flat dipole can be obtained by differentiating above. For an on-axis particle $(x_0, p_{x0}, y_0, p_{y0}) = (0, 0, 0, 0)$ with a small bending angle $\theta \ll 1$, it is written as

$$M = \begin{pmatrix} 1 & \frac{L}{1+\delta} & \cdot & \cdot & \cdot & \frac{L\theta}{2(1+\delta)^2} \\ \cdot & 1 & \cdot & \cdot & \cdot & \theta \\ \cdot & \cdot & 1 & \frac{L}{1+\delta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & \cdot & \cdot & 1 & \frac{v-v_d}{v_d}L \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + O(\theta)^2, \quad (29)$$

where $L \equiv \rho\theta$. If we compare this with the transfer matrix for a drift, the differences are the non-zero $M_{16}, M_{26}, M_{51}, M_{52}$ components, which are related to *dispersions*, and will be discussed later.

Except for these terms, the transfer matrix of a flat, small-bending dipole is just equal to that of the drift space with the length L , including the chromatic behavior.

Thin multipole magnets

A $2n$ -pole multipole magnet has a magnetic field

$$B_x + iB_y = \frac{B_n}{(n-1)!} (x + iy)^{n-1}. \quad (30)$$

The associated Hamiltonian is

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{K_{n-1}}{n!} (x + iy)^n + \frac{E}{v_d}, \quad (31)$$

where $K_{n-1} = \frac{1}{B\rho} \frac{d^{n-1}B_y}{dx^{n-1}}$.

There is no analytic solution of the motion with the Hamiltonian above, due to the $\sqrt{\quad}$ term. The simplest way of approximation is to concentrate the field into a *thin lens*. In this case the resulting Hamiltonian with the thin field at $s = 0$ is expressed as

$$H = \frac{k_{n-1}}{n!} (x + iy)^n \delta(s), \quad (32)$$

where

$$k_{n-1} \equiv \int K_{n-1} ds \approx K_{n-1} L. \quad (33)$$

Thin multipole magnets (2)

Then the solution is

$$p_x + ip_y = p_{x0} + ip_{y0} - \frac{k_{n-1}}{(n-1)!} (x_0 + iy_0)^{n-1}, \quad (34)$$

$$x = x_0, \quad y = y_0, \quad z = z_0, \quad \delta = \delta_0. \quad (35)$$

Note that the transformation is independent on δ , which means there is no chromaticity in a thin multipole.

Then the associated transfer matrix is written as

$$M = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\Re\left(\frac{k_{n-1}}{(n-2)!} (x_0 + iy_0)^{n-2}\right) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \Im\left(\frac{ik_{n-1}}{(n-2)!} (x_0 + iy_0)^{n-2}\right) & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad (36)$$

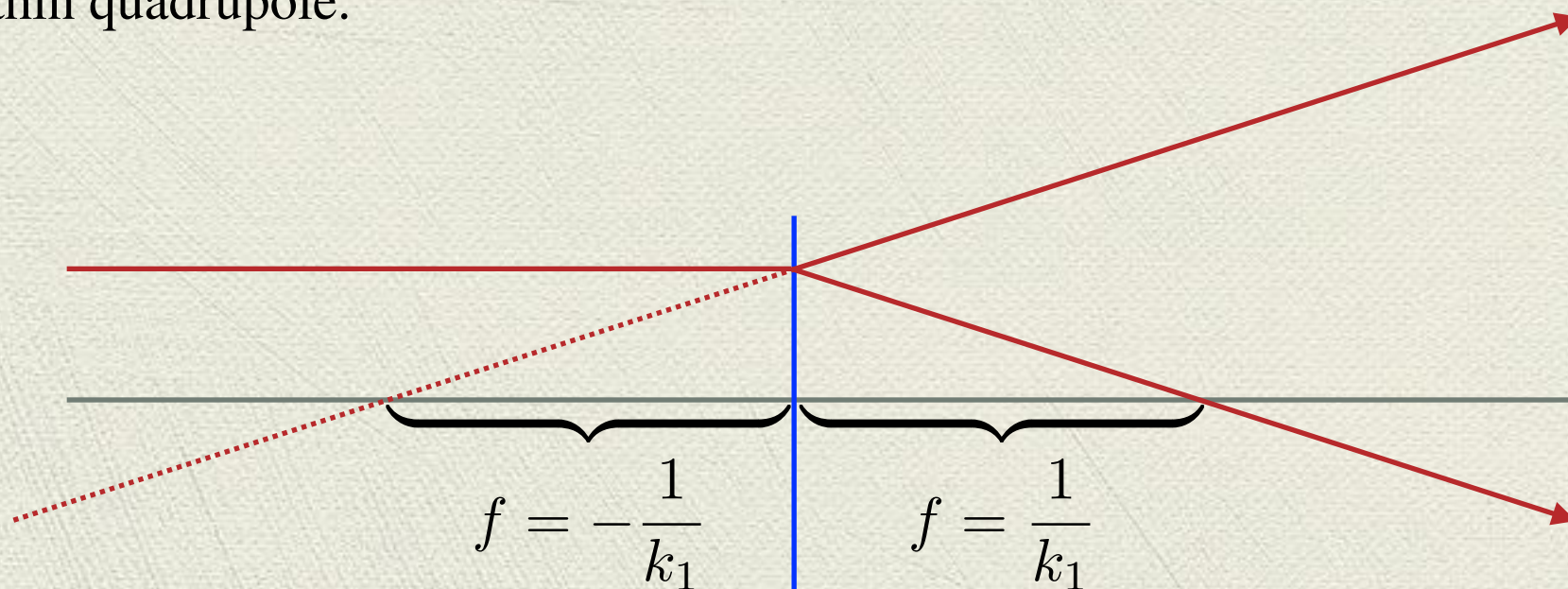
Thin quadrupole magnet

In the case of a quadrupole, M is obtained by setting $n = 2$ in Eq. (36):

$$M = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -k_1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & k_1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (37)$$

where we have assumed k_1 is real.

The physical meaning of above is that k_1 is the inverse of the focal length of the thin quadrupole.



RF cavity

A thin accelerating RF cavity can be expressed by a Hamiltonian, consisting of only A_z :

$$H = \frac{eV_c}{cP_d\omega_{\text{RF}}} \cos\left(-\omega_{\text{RF}}\frac{z}{v} + \phi_{\text{RF}}\right) \delta(s) \quad (38)$$

$$= \frac{eV_c}{cP_d\omega_{\text{RF}}} \cos(\omega_{\text{RF}}(t - t_{0\text{RF}}) + \phi_{\text{RF}}) \delta(s), \quad (39)$$

where V_c , ω_{RF} , ϕ_{RF} , and $t_{0\text{RF}}$ are the peak accelerating voltage, angular frequency, RF phase, and the design arrival time of the cavity.

The resulting transformation for the RF cavity is:

$$E = E_0 + \frac{eV_c}{cP_d} \sin\left(-\omega_{\text{RF}}\frac{z}{v} + \phi_{\text{RF}}\right), \quad (40)$$

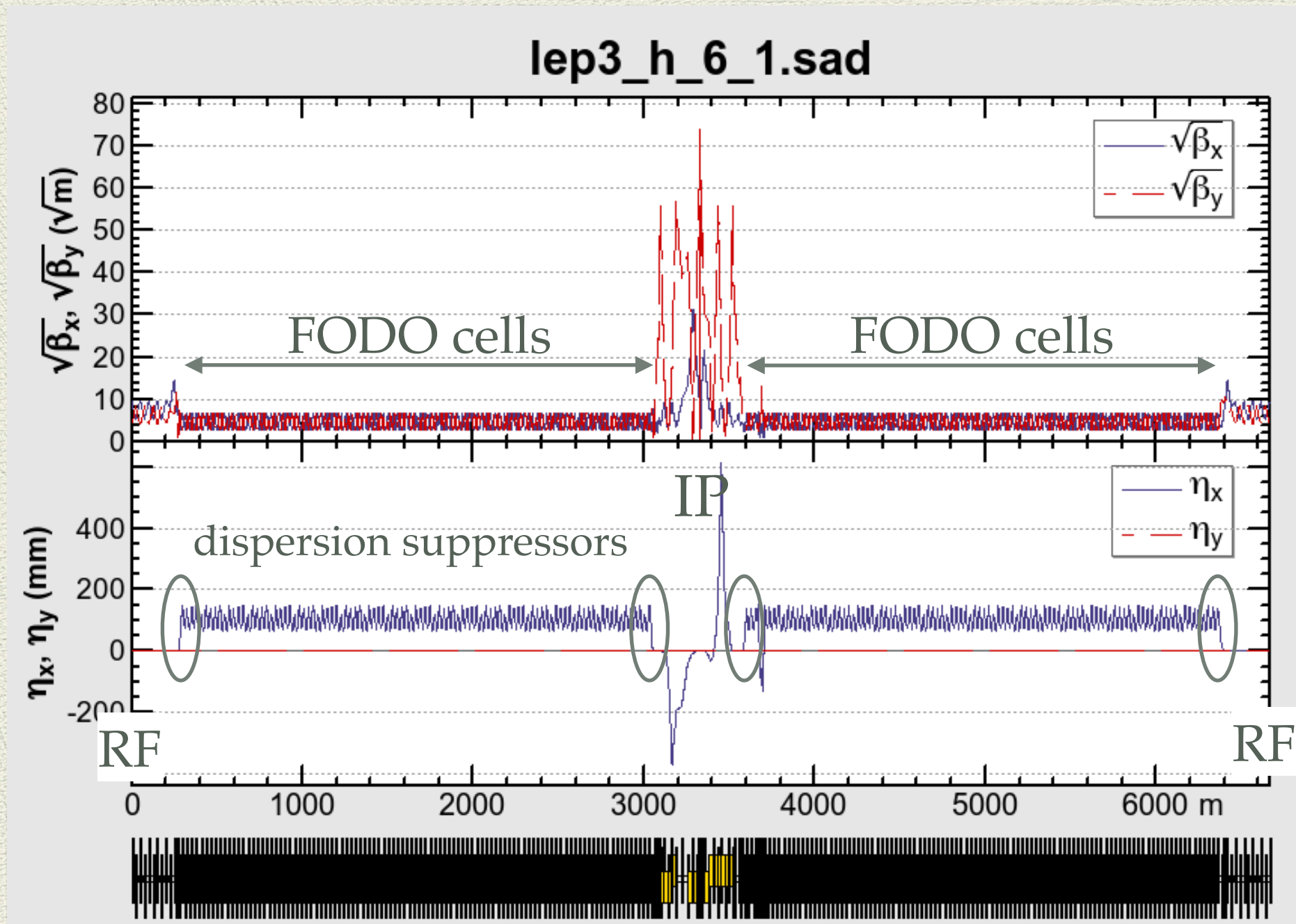
and the change of z , and δ are calculated from their definitions:

$$E^2 = \frac{m^2c^4}{c^2p_d^2} + (1 + \delta)^2, \quad z = -\frac{1 + \delta}{E}c(t - t_{0\text{RF}}). \quad (41)$$

Since the RF cavity is expressed by Hamiltonian, the emittance is preserved in each plane.

FODO cell

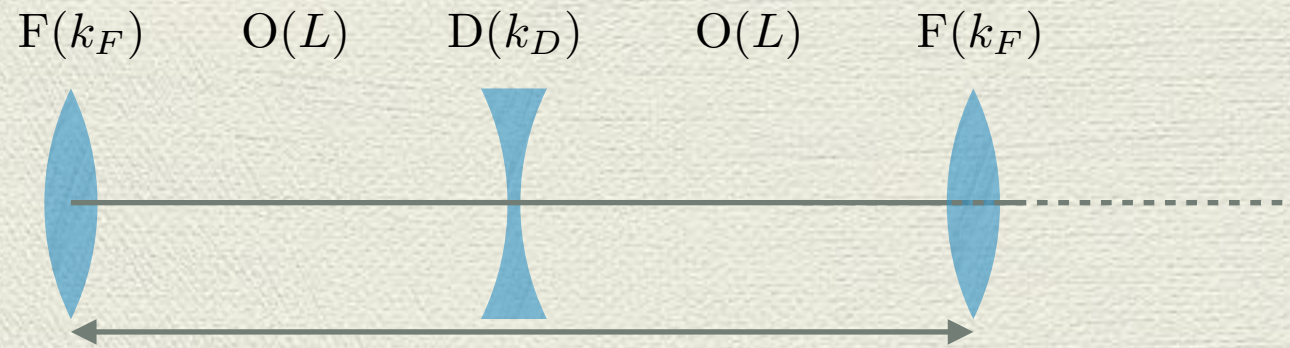
A "typical" collider consists of FODO cells, interaction region, dispersion suppressors, and RF.



Various arc lattices

lattice	characteristics	machines
FODO	<p>simplest structure best packing factor of dipoles</p>	<p>high energy machines (LEP, PEP, TRISTAN, etc.)</p>
Multi-bend achromat	<p>small emittance suitable for insertion devices</p>	<p>3G light sources</p>
Theoretical minimum emittance	<p>small emittance fast damping</p>	<p>damping rings (KEK-ATF)</p>
2.5π	<p>non-interleaved sextupole variable emittance/momentum compaction</p>	<p>KEKB/SuperKEKB</p>
Non- periodic	<p>All independent quadrupoles Maximum flexibility</p>	<p>CESR</p>

FODO cell (2)



Let us consider the on-axis, on-momentum transfer matrix M of a FODO cell, between the midpoints of QF. For the time being, we consider the 2 by 2 part for x and y . The $z\delta$ parts will be discussed later.

$$M_{\begin{smallmatrix} x \\ y \end{smallmatrix}} = \begin{pmatrix} 1 & 0 \\ \mp k_F/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm k_D & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp k_F/2 & 1 \end{pmatrix} \quad (42)$$

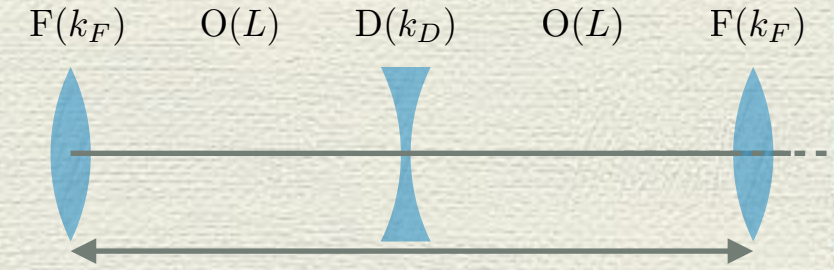
$$= \begin{pmatrix} 1 \pm (k_D - k_F)L - k_F k_D L^2 / 2 & L(2 \pm k_D L) \\ \pm(k_D - k_F) - k_F k_D L + k_F^2 L(1 \pm k_D L/2)/2 & 1 \pm (k_D - k_F)L - k_F k_D L^2 / 2 \end{pmatrix} \quad (43)$$

$$= \begin{pmatrix} 1 \pm (c_D - c_F) - \frac{c_D c_F}{2} & L(2 \pm c_D) \\ \frac{\pm(c_D - c_F) - c_F c_D + c_F^2 (1 \pm c_D/2)}{L} & 1 \pm (c_D - c_F) - \frac{c_F c_D}{2} \end{pmatrix}, \quad (44)$$

where L is the length between quads, and k_F, k_D are the focusing strengths of Q_F and Q_D , respectively.

We have used the thin-lens approximation for the quadrupoles, and introduced dimensionless parameters $c_{F,D} \equiv k_{F,D}L$. We assume $c_{F,D} > 0$.

FODO cell (3)



$$M_{x,y} = \begin{pmatrix} 1 \pm (c_D - c_F) - \frac{c_D c_F}{2} & L(2 \pm c_D) \\ \frac{\pm(c_D - c_F) - c_F c_D + c_F^2(1 \pm c_D/2)}{L} & 1 \pm (c_D - c_F) - \frac{c_F c_D}{2} \end{pmatrix}$$

If the beam optics is periodic, the transfer matrix above must be equal to

$$M_{x,y} = \begin{pmatrix} \cos \mu_{x,y} + \alpha_{x,y} \sin \mu_{x,y} & \beta_{x,y} \sin \mu_{x,y} \\ \frac{1 + \alpha_{x,y}^2}{\beta_{x,y}} \sin \mu_{x,y} & \cos \mu_{x,y} - \alpha_{x,y} \sin \mu_{x,y} \end{pmatrix}. \quad (45)$$

By comparing two expressions, we obtain the values of the Twiss parameters at the center of QF as

$$\alpha_{x,y} = 0, \quad (46)$$

$$\cos \mu_{x,y} = 1 + (c_{D,F} - c_{F,D}) - \frac{c_F c_D}{2}, \quad (47)$$

$$\beta_{x,y} = \frac{2 \pm c_D}{\sin \mu_{x,y}} L. \quad (48)$$

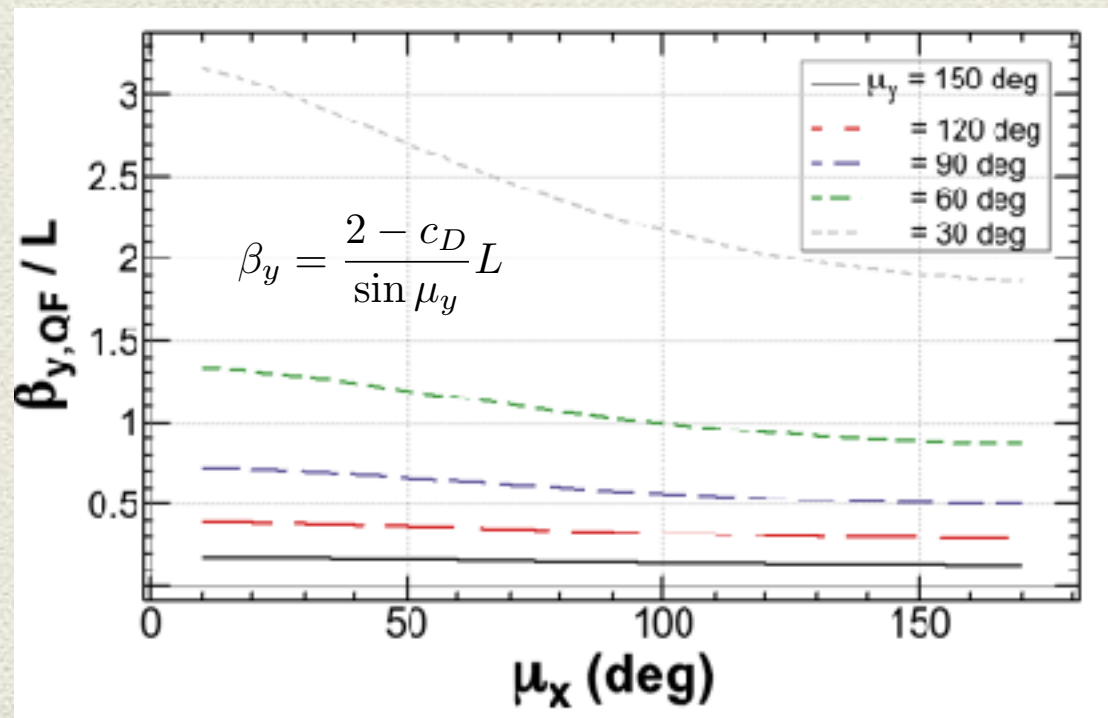
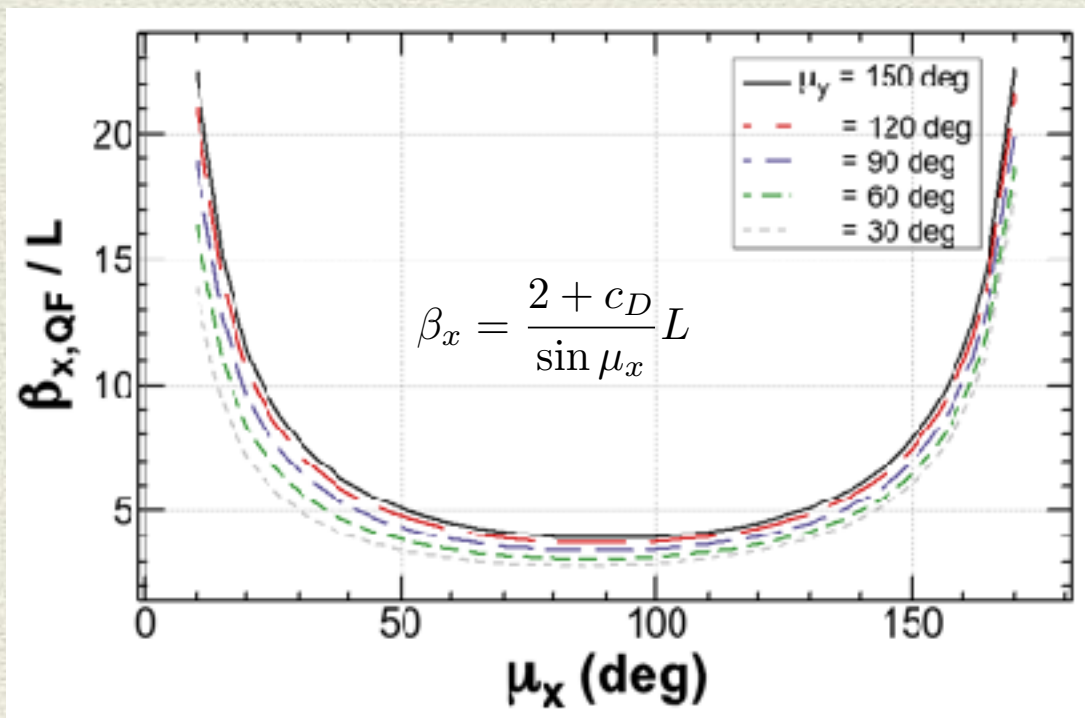
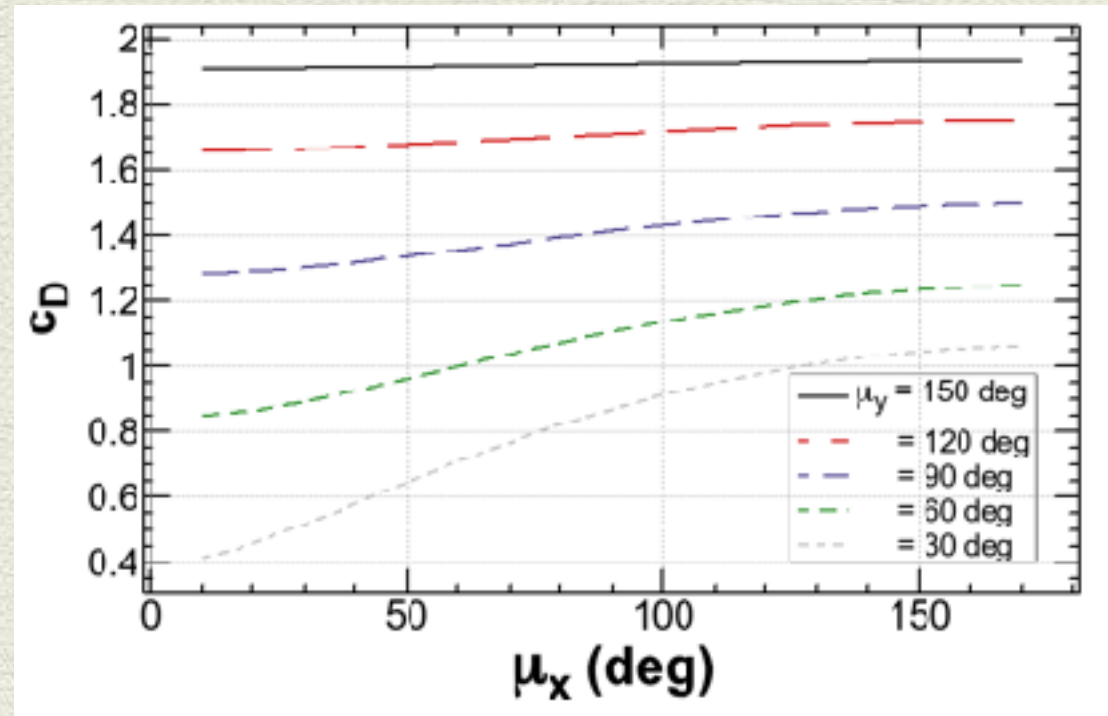
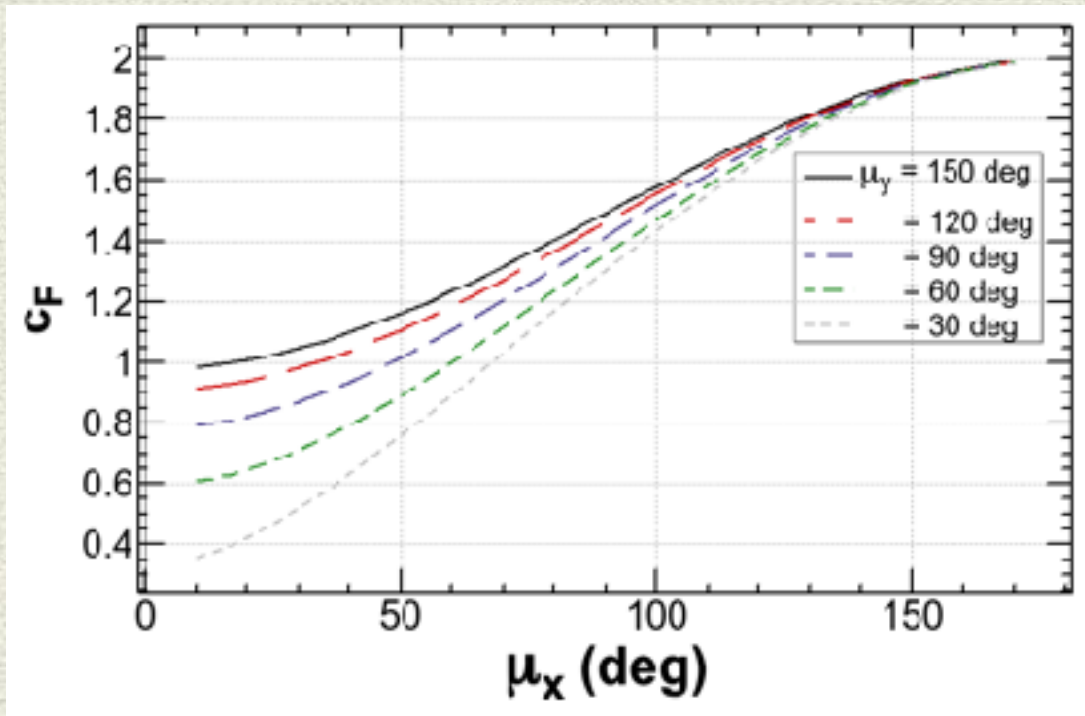
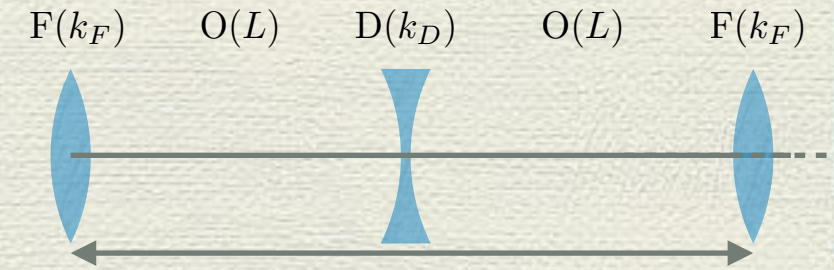
Then once the phase advances $\mu_{x,y}$ are given, parameters $c_{F,D}$ are determined as

$$c_F = \frac{1}{4} \left(\cos \mu_y - \cos \mu_x + \sqrt{32 - 16(\cos \mu_x + \cos \mu_y) + (\cos \mu_x - \cos \mu_y)^2} \right) \quad (49)$$

$$c_D = \frac{1}{4} \left(\cos \mu_x - \cos \mu_y + \sqrt{32 - 16(\cos \mu_x + \cos \mu_y) + (\cos \mu_x - \cos \mu_y)^2} \right), \quad (50)$$

The value of β -function at QF is obtained by Eq. (48). Note that $0 < c_{F,D} < 2$.

Dependence of FODO cell parameters on $\mu_{x,y}$



Twiss parameters along a beam line

Once we have the Twiss parameters α_1, β_1 at a particular location $s = s_1$ in a beam line, the 2 by 2 transfer matrix M from s_1 to s is expressed as

$$\begin{aligned}
 \mathbf{M} &= \left[\begin{pmatrix} 1 & 0 \\ \alpha(s) & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{\beta(s)} & 0 \\ 0 & \sqrt{\beta(s)} \end{pmatrix} \right]^{-1} \\
 &\times \begin{pmatrix} \cos \phi(s) & \sin \phi(s) \\ -\sin \phi(s) & \cos \phi(s) \end{pmatrix} \\
 &\times \left[\begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{\beta_1} & 0 \\ 0 & \sqrt{\beta_1} \end{pmatrix} \right] \\
 &= \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_1}} (\cos \phi + \alpha_1 \sin \phi) & \sqrt{\beta(s)\beta_1} \sin \phi \\ -\frac{(1 + \alpha(s)\alpha_1) \sin \phi + (\alpha(s) - \alpha_1) \cos \phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\frac{\beta_1}{\beta(s)}} (\cos \phi - \alpha(s) \sin \phi) \end{pmatrix}, \quad (51)
 \end{aligned}$$

Normal to physical conversion at exit

Rotation in the normal phase space

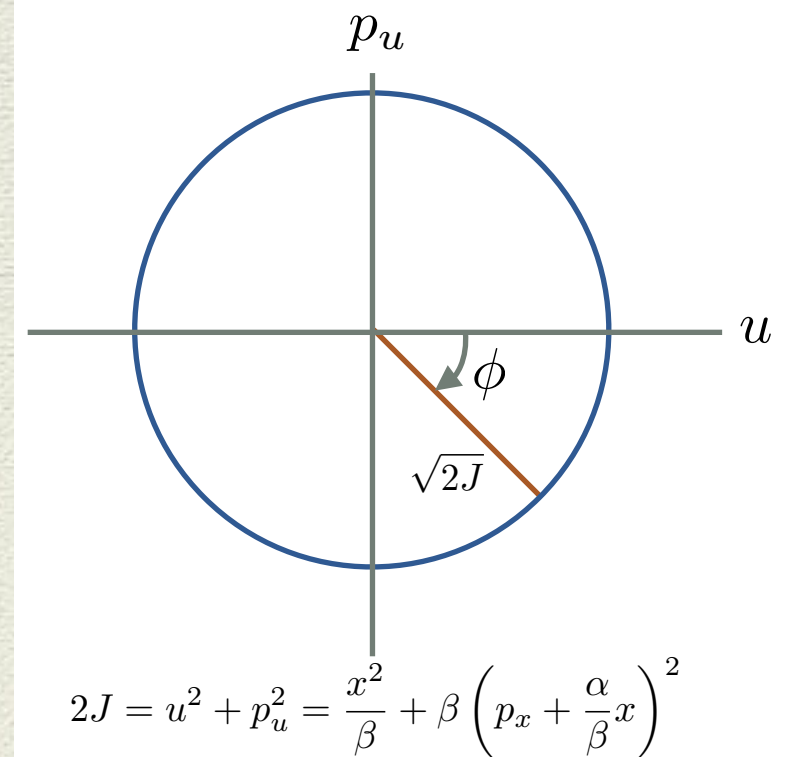
Physical to normal conversion at entrance

where $\alpha(s)$, $\beta(s)$, and $\phi = \phi(s)$ are the Twiss parameters at s and the phase advance from s_1 to s , respectively.

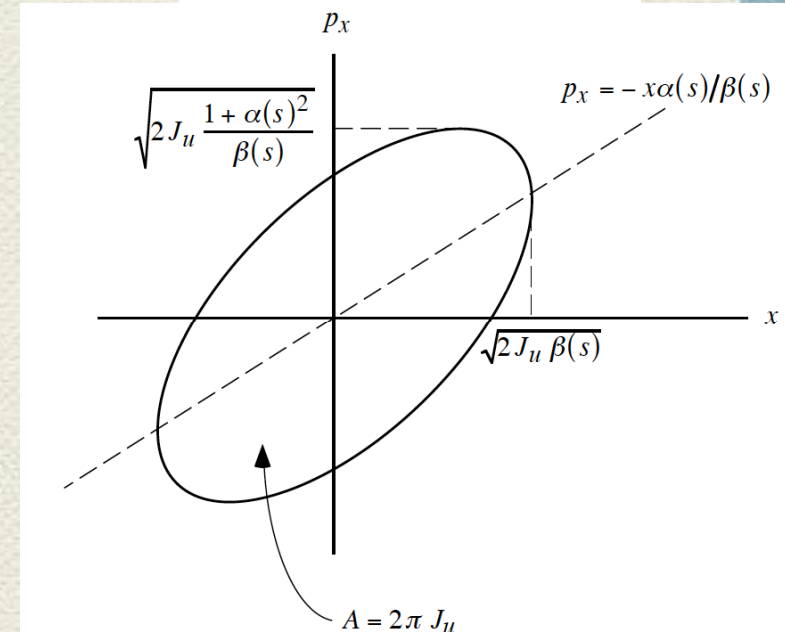
The above means that a symplectic motion is reduced to a *circular motion* in the normalized phase space:

$$(u, p_u) = \left(\frac{x}{\sqrt{\beta}}, p_x \sqrt{\beta} + x \frac{\alpha}{\sqrt{\beta}} \right). \quad (52)$$

normalized phase space



physical phase space



Twiss parameters along a beam line (2)

If we know the transfer matrix from s_1 to s

$$\mathbf{M}(s) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad (53)$$

we can calculate the Twiss parameters at s using Eq. (51) as

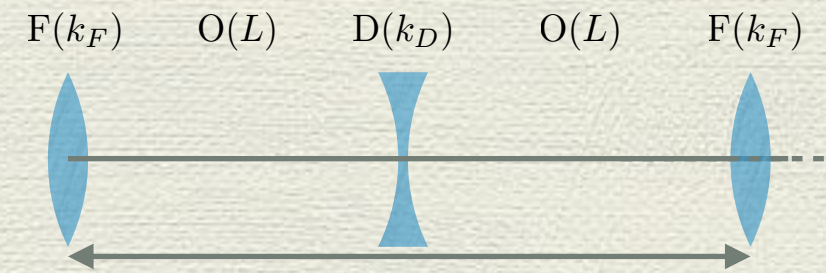
$$\alpha(s) = (M_{11}M_{22} + M_{12}M_{21})\alpha_1 - M_{11}M_{21}\beta_1 - M_{12}M_{22}\frac{1 + \alpha_1^2}{\beta_1} \quad (54)$$

$$\beta(s) = -2M_{11}M_{12}\alpha_1 + M_{11}^2\beta_1 + M_{12}^2\frac{1 + \alpha_1^2}{\beta_1} \quad (55)$$

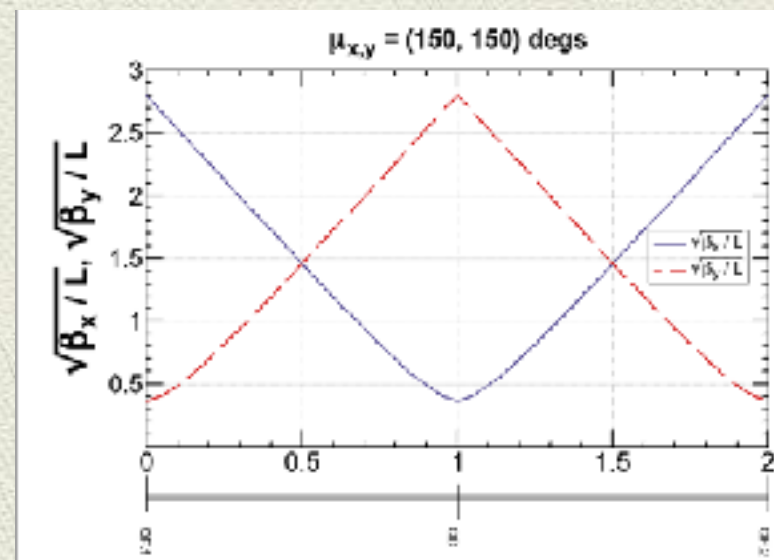
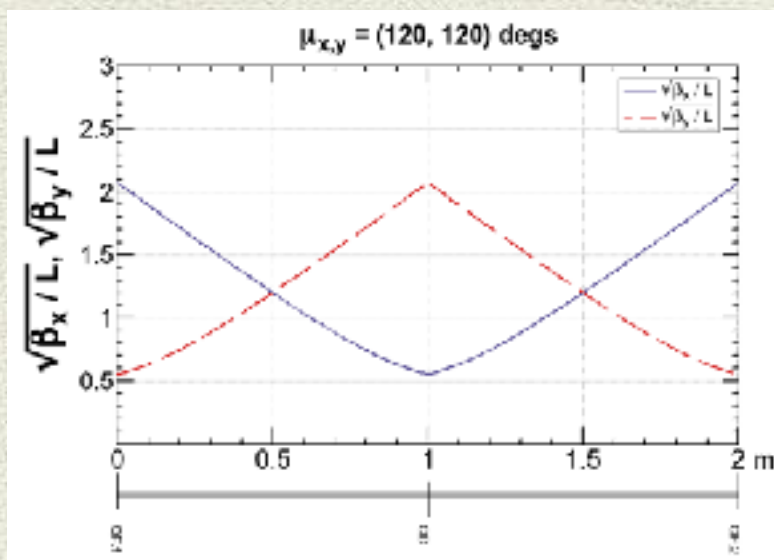
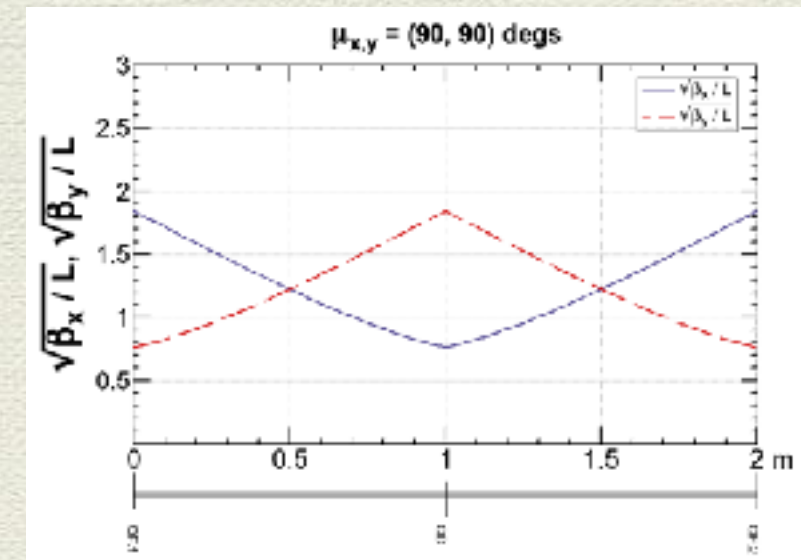
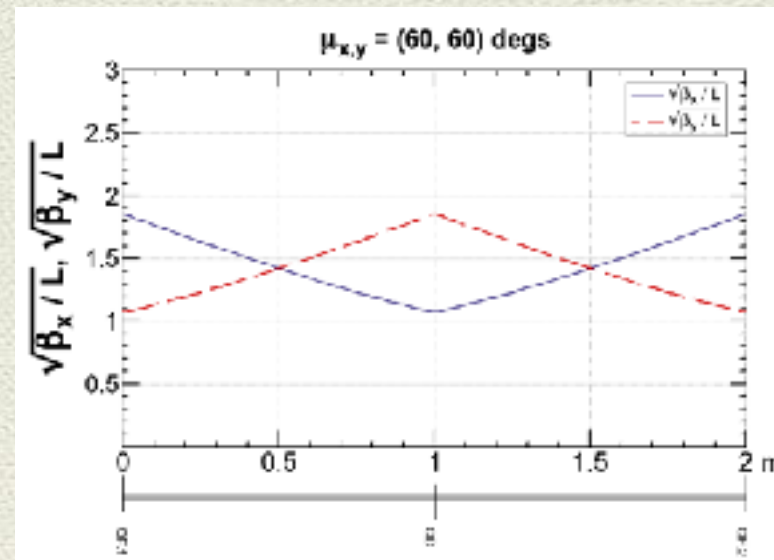
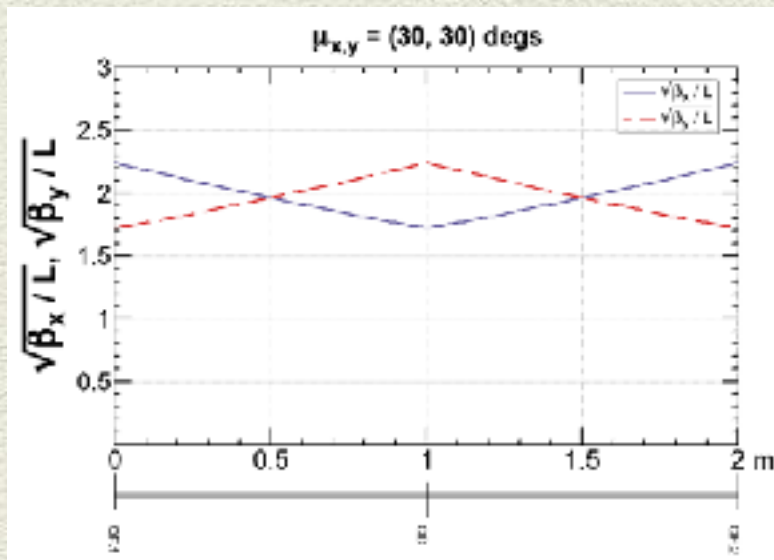
$$\phi(s) = \arg(-M_{12}\alpha_1 + M_{11}\beta_1 + iM_{12}) \quad (56)$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_1}}(\cos \phi + \alpha_1 \sin \phi) & \sqrt{\beta(s)\beta_1} \sin \phi \\ -\frac{(1 + \alpha(s)\alpha_1) \sin \phi + (\alpha(s) - \alpha_1) \cos \phi}{\sqrt{\beta(s)\beta_1}} & \sqrt{\frac{\beta_1}{\beta(s)}}(\cos \phi - \alpha(s) \sin \phi) \end{pmatrix}, \quad (51)$$

FODO cell (4)



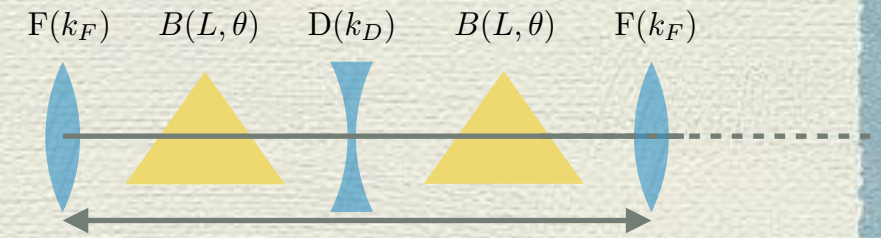
β -functions along a FODO cell with various phase advances ($\mu_x = \mu_y$)



$$\beta_{\max} \approx \beta_{\min} \rightarrow \infty (\mu \rightarrow 0)$$

$$\beta_{\max} \gg \beta_{\min} (\mu \rightarrow \pi)$$

Dispersion



In a FODO cell, now let us consider a horizontal dipole with length L and the bending angle θ in place of the drift space. The on-axis, on-momentum transfer matrix between the center of QFs in (x, p_x, z, δ) phase space is written as

$$M = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ -k_F/2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \cdot & L\theta/2 \\ \cdot & 1 & \cdot & \theta \\ -\theta & -L\theta/2 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ k_D & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & L & \cdot & L\theta/2 \\ \cdot & 1 & \cdot & \theta \\ -\theta & -L\theta/2 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ -k_F/2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cdot & \cdot & \frac{(c_D + 4)L\theta}{2} & \cdot \\ \cdot & M_x & \cdot & \cdot \\ \frac{(c_D + 4)(c_F - 2)\theta}{4} & \cdot & \cdot & \cdot \\ \cdot & \frac{(c_D + 4)L\theta}{2} & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \frac{(c_D + 4)L\theta^2}{4} \end{pmatrix}, \quad (57)$$

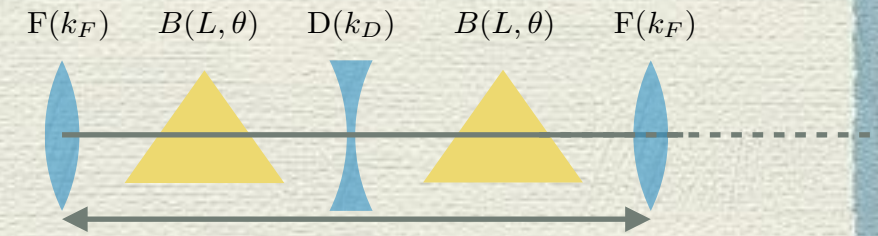
where M_x is the 2 by 2 matrix of FODO given by Eq.(44).

dipole magnet

$$\begin{pmatrix} 1 & \frac{L}{1+\delta} & \cdot & \cdot & \cdot & \frac{L\theta}{2(1+\delta)^2} \\ \cdot & 1 & \cdot & \cdot & \cdot & \theta \\ \cdot & \cdot & 1 & \frac{L}{1+\delta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & \cdot & \cdot & 1 & \frac{v-v_d}{v_d}L \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + O(\theta)^2$$

$$M_x = \begin{pmatrix} 1 \pm (c_D - c_F) - \frac{c_D c_F}{2} & L(2 \pm c_D) \\ \frac{\pm(c_D - c_F) - c_F c_D + c_F^2(1 \pm c_D/2)}{L} & 1 \pm (c_D - c_F) - \frac{c_F c_D}{2} \end{pmatrix}, \quad (44)$$

Dispersion (3)



Consider a FODO with dipoles having the cell transfer matrix for x, z planes:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdot & M_{16} \\ M_{21} & M_{22} & \cdot & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad (60)$$

The matrix M is semi-diagonalized by another matrix

$$D = \begin{pmatrix} 1 & \cdot & \cdot & \eta_x \\ \cdot & 1 & \cdot & \eta_{px} \\ -\eta_{px} & \eta_x & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \text{ as } D^{-1}MD = \begin{pmatrix} M_{11} & M_{12} & \cdot & \cdot \\ M_{21} & M_{22} & \cdot & \cdot \\ \cdot & \cdot & 1 & M_{D56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (61)$$

with

physical to normal

normal to physical

$$\eta_x = \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4 \sin^2(\mu_x/2)},$$

$$\eta_{px} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4 \sin^2(\mu_x/2)},$$

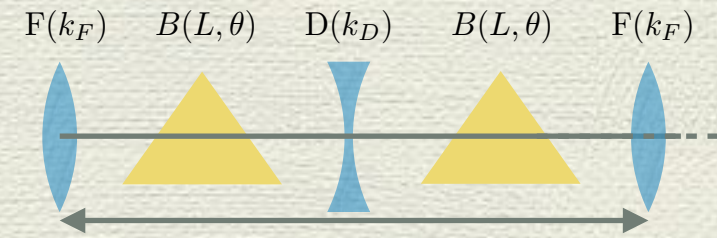
$$D^{-1} = D / \{ \eta_x \rightarrow -\eta_x, \eta_{px} \rightarrow -\eta_{px} \}$$

(62)

where we have used $M_{11} + M_{22} = 2 \cos \mu_x$ from Eq. (45).

$$M_{x,y} = \begin{pmatrix} \cos \mu_{x,y} + \alpha_{x,y} \sin \mu_{x,y} & \beta_{x,y} \sin \mu_{x,y} \\ \frac{1 + \alpha_{x,y}^2}{\beta_{x,y}} \sin \mu_{x,y} & \cos \mu_{x,y} - \alpha_{x,y} \sin \mu_{x,y} \end{pmatrix}. \quad (45)$$

Dispersion (4)



In the case of the FODO transfer matrix Eq. (57), the solution Eq. (62) is

$$= \begin{pmatrix} M_x & \cdot & \frac{(c_D + 4)L\theta}{2} & \cdot \\ \frac{(c_D + 4)(c_F - 2)\theta}{4} & -\frac{(c_D + 4)L\theta}{2} & \cdot & -\frac{(c_D + 4)(c_F - 2)\theta}{4} \\ \cdot & \cdot & 1 & -\frac{(4 + c_D)L\theta^2}{4} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (57)$$

$$\eta_x = \frac{(4 + c_D)L\theta}{2(c_F - c_D) + c_F c_D} \quad (63)$$

$$= \frac{2L + \beta_x \sin \mu_x}{4 \sin^2(\mu_x/2)} \theta, \quad (64)$$

$$\eta_{px} = 0, \quad (65)$$

then the matrix M is semi-diagonalized as

$$D^{-1}MD = \begin{pmatrix} M_x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{(c_F - 4)\eta_x \theta}{2} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (66)$$

We will discuss the nature of M_{D56} in the matrix Eq. (61) later.

$$\eta_x = \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4 \sin^2(\mu_x/2)}, \quad (62)$$

$$\eta_{px} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4 \sin^2(\mu_x/2)},$$

$$D^{-1}MD = \begin{pmatrix} M_{11} & M_{12} & \cdot & \cdot \\ M_{21} & M_{22} & \cdot & \cdot \\ \cdot & \cdot & 1 & M_{D56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (61)$$

Dispersion (5)

In this case, the betatron coordinates by Eq. (59) are written as

$$\begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta_\beta \end{pmatrix} = D^{-1} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \quad (59)$$

$$x_\beta = x - \eta_x \delta, \quad (67)$$

$$p_{x\beta} = p_x - \eta_{px} \delta, \quad (68)$$

$$z_\beta = z + \eta_{pz} x - \eta_x p_x, \quad (69)$$

and other variables are unchanged.

In this special case, the resulting 2 by 2 transfer matrix M_x in Eq. (66) did not change, but in the z -plane, the 2 by 2 matrix or $M_{z\delta}$ in Eq. (66) has changed.

Note that such a simple decoupling by dispersions as Eqs. (67–69) is only possible when there is no acceleration in the beam line. If there is an RF cavity in a non-dispersive location in a beam line, the x - z coupling requires more parameters to decouple. The resulting 2 by 2 matrices will not preserve the forms of Eq. (66), in such a general case.

$$D^{-1}MD = \begin{pmatrix} M_x & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \frac{(c_F - 4)\eta_x \theta}{2} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (66)$$

Dispersion along a beam line

After obtained the dispersions at a particular location $s = s_1$ of a beam line, it is possible to transfer the dispersion at s_1 to another location s in the beam line.

Let us denote the transfer matrix from s_1 to s by $M(s)$.

The decoupling matrix $D(s)$ should be chosen to semi-diagonalize $M(s)$ to $M_D(s)$ as:

$$D(s)^{-1}M(s)D = M_D(s) = \begin{pmatrix} M_x(s) & \cdot & \cdot \\ \cdot & M_y(s) & \cdot \\ \cdot & \cdot & M_z(s) \end{pmatrix}, \quad (70)$$

physical to normal at s

normal to physical at s_1

where $M_{x,y,z}(s)$ are the 2 by 2 transfer matrices from s_1 to s .

Dispersion along a beam line (2)

In the case of a simple dispersion in x - z plane, $M(s)$ is expressed as

$$M(s) = \begin{pmatrix} M_{11} & M_{12} & \cdot & M_{16} \\ M_{21} & M_{22} & \cdot & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad (71)$$

Then the matrix $D(s)$ is written as

$$D(s) = \begin{pmatrix} 1 & \cdot & \cdot & \eta_x(s) \\ \cdot & 1 & \cdot & \eta_{px}(s) \\ -\eta_{px}(s) & \eta_x(s) & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$D(s)^{-1}M(s)D = M_D(s) = \begin{pmatrix} M_x(s) & \cdot & \cdot \\ \cdot & M_y(s) & \cdot \\ \cdot & \cdot & M_z(s) \end{pmatrix}$$

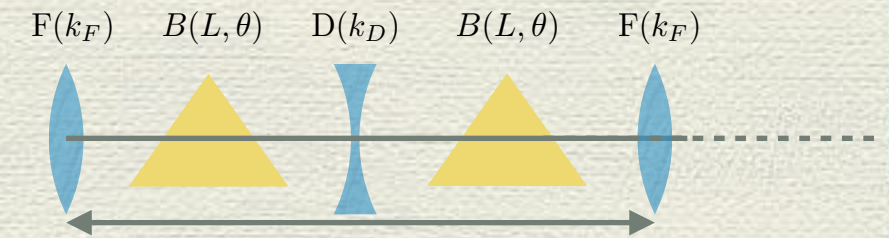
(72)

with dispersions at s :

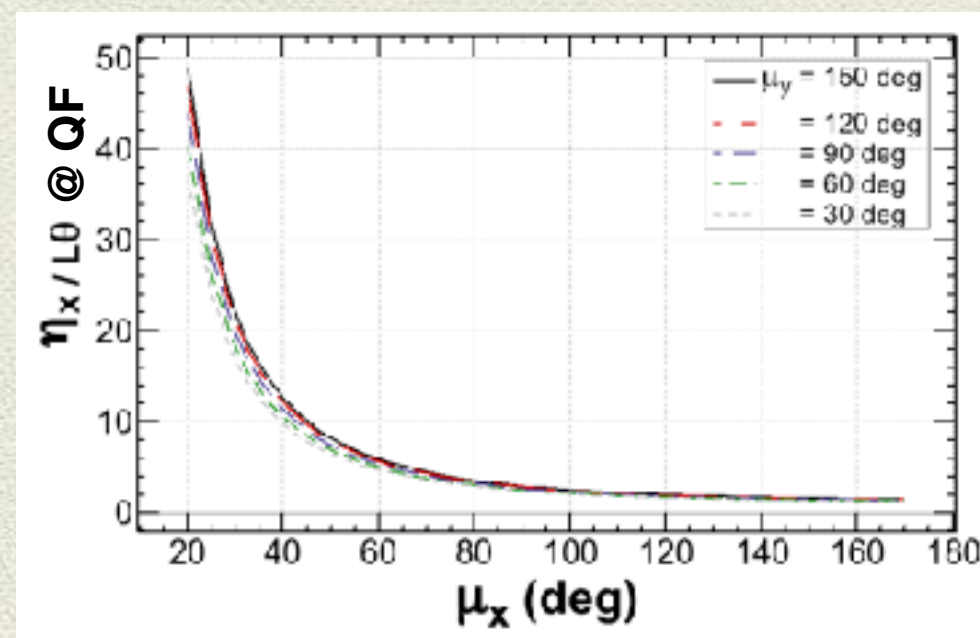
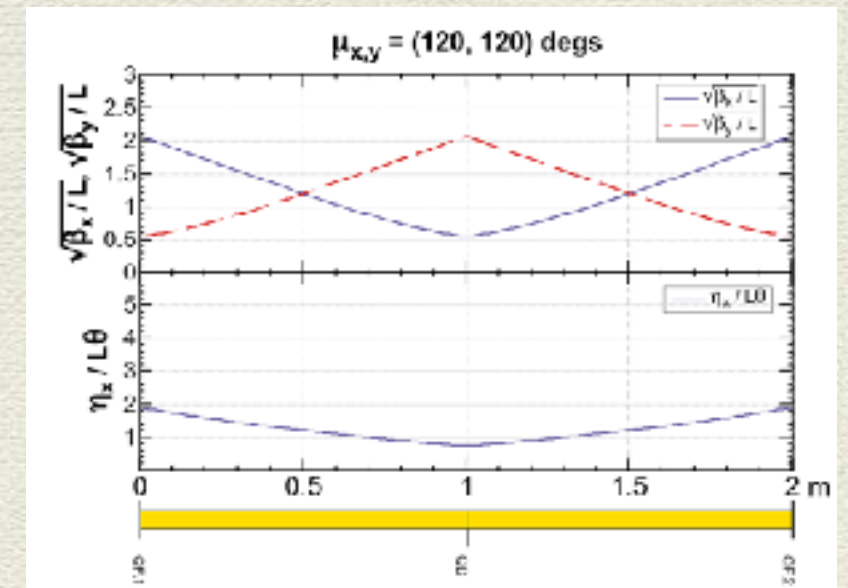
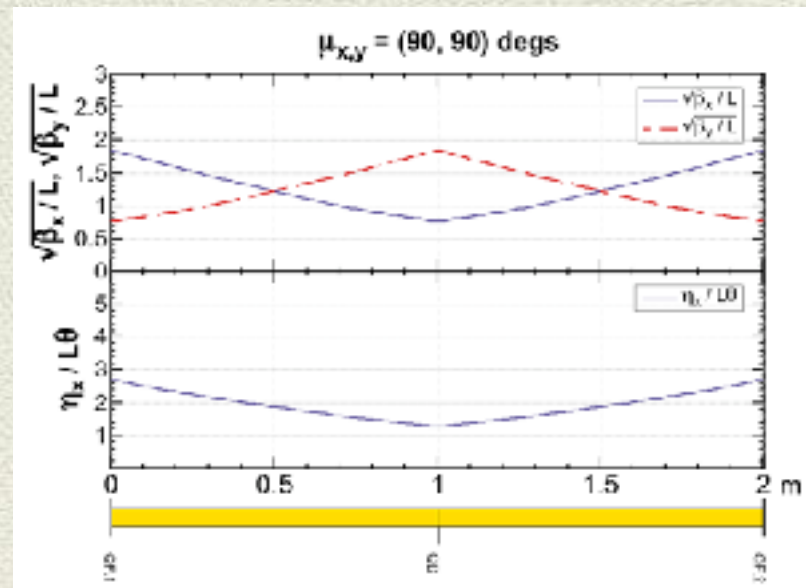
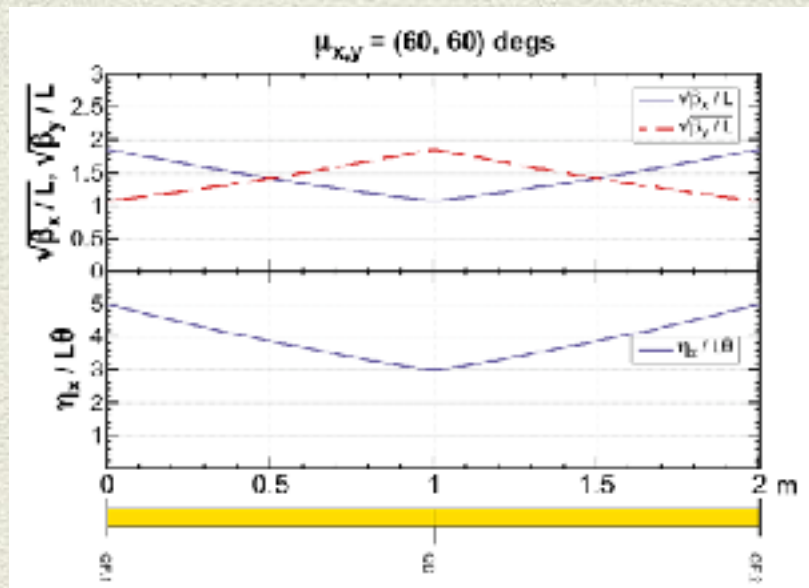
$$\begin{aligned} \eta_x(s) &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16}, \\ \eta_{px}(s) &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26}. \end{aligned} \quad (73)$$

Above means that the the dispersions $(\eta_x(s), \eta_{px}(s))$ behave in the same way as variables (x, p_x) when the inhomogeneous terms M_{16} and M_{26} are zero.

Dispersion in a FODO cell



β -functions and dispersions with various phase advances ($\mu_x = \mu_y$)



Dispersion in a ring

If we put $s = C + s_1$ in Eq. (73), then $M(s)$ becomes the one-turn transfer matrix M in Eq. (60), and

$$\begin{aligned}\eta_x(s) &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16}, \\ \eta_{px}(s) &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26}.\end{aligned}\quad (73)$$

$$\begin{aligned}\eta_x &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16}, \\ \eta_{px} &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26},\end{aligned}\quad (74)$$

by the periodic condition $(\eta_x(s), \eta_{px}(s)) = (\eta_x, \eta_{px})$.

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdot & M_{16} \\ M_{21} & M_{22} & \cdot & M_{26} \\ -M_{11}M_{26} + M_{21}M_{16} & -M_{12}M_{26} + M_{22}M_{16} & 1 & M_{56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad (60)$$

The solution of Eqs. (74) is identical to the solution of semi-diagonalization, Eqs. (62).

$$\begin{aligned}\eta_x &= \frac{(1 - M_{22})M_{16} + M_{12}M_{26}}{4 \sin^2(\mu_x/2)}, \\ \eta_{px} &= \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4 \sin^2(\mu_x/2)},\end{aligned}\quad (62)$$

Momentum compaction

Then the resulting semi-diagonal matrix $M_D(s)$ becomes

$$M_D(s) = \begin{pmatrix} M_{11} & M_{12} & \cdot & \cdot \\ M_{21} & M_{22} & \cdot & \cdot \\ \cdot & \cdot & 1 & M_{D56} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (75)$$

$$D(s)^{-1}M(s)D = M_D(s)$$

$$\begin{aligned} M_{D56} &= (M_{21}\eta_x + M_{22}\eta_{px})M_{16} - (M_{11}\eta_x + M_{12}\eta_{px})M_{26} + M_{56}, \\ &= \eta_{px}(s)M_{16} - \eta_x(s)M_{26} + M_{56}, \end{aligned} \quad (76)$$

where we have used Eq. (73).

$$\begin{aligned} \eta_x(s) &= M_{11}\eta_x + M_{12}\eta_{px} + M_{16}, \\ \eta_{px}(s) &= M_{21}\eta_x + M_{22}\eta_{px} + M_{26}. \end{aligned} \quad (73)$$

Let us consider a small portion of a beam line whose transfer matrix is given by ΔM . According to Eq. (76), the change of M_{D56} is written as

$$\Delta M_{D56} = \eta_{px}(s)\Delta M_{16} - \eta_x(s)\Delta M_{26} + \Delta M_{56}. \quad (77)$$

Thus the component M_{D56} changes only when there are non-zero ΔM_{16} , ΔM_{26} , ΔM_{56} . Thus M_{D56} does not change in drift spaces nor multipole magnets at least for an on-momentum & on-axis orbit, where $M_{*6} = 0$.

Momentum compaction (2)

In the case of a dipole with a small length $L = ds$ and a bending angle $\theta = ds/\rho$, the transfer matrix Eq. (29) becomes

$$M = \begin{pmatrix} 1 & \frac{L}{1+\delta} & \cdot & \cdot & \cdot & \frac{L\theta}{2(1+\delta)^2} \\ \cdot & 1 & \cdot & \cdot & \cdot & \frac{\theta}{\rho} \\ \cdot & \cdot & 1 & \frac{L}{1+\delta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ -\theta & -\frac{L\theta(1+2\delta)}{2(1+\delta)^2} & \cdot & \cdot & 1 & \frac{v-v_d}{v_d}L \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + O(\theta)^2, \quad (29)$$

$$\Delta M = \begin{pmatrix} 1 & ds & \cdot & \cdot \\ \cdot & 1 & \cdot & \frac{ds}{\rho} \\ -\frac{ds}{\rho} & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + O(ds)^2. \quad (78)$$

Then if we plug in above into Eq. (77), the change of ΔM_{D56} is written as

$$\Delta M_{D56} = \eta_{px}(s)\Delta M_{16} - \eta_x(s)\Delta M_{26} + \Delta M_{56}. \quad (77)$$

$$dM_{D56} = -\eta_{xs} \frac{ds}{\rho}. \quad (79)$$

By integrating above, we obtain an expression for M_{D56} :

$$M_{D56} = - \int \frac{\eta_x(s)}{\rho(s)} ds. \quad (80)$$

For a storage ring, the quantity M_{56} is sometimes expressed in terms of the ratio to the circumference C as

$$\alpha_p \equiv -\frac{M_{D56}}{C} = \frac{1}{C} \oint \frac{\eta_x(s)}{\rho(s)} ds, \quad (81)$$

where the ratio α_p is called *momentum compaction factor*.

Synchrotron motion

Let us consider a ring with an RF cavity with the voltage V_c and the phase ϕ_{RF} at a dispersion-free location. Using Eq. (40), the one-turn, on-axis transfer matrix M_z in z -plane from the center of the cavity is written as:

$$M_z = \begin{pmatrix} 1 & \cdot \\ k_z/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha_p C \\ \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ k_z/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\alpha_p C k_z}{2} & -\alpha_p C \\ k_z - \frac{\alpha_p C k_z^2}{4} & 1 - \frac{\alpha_p C k_z}{2} \end{pmatrix}, \quad (82)$$

$$\text{with } k_z = \frac{\omega_{\text{RF}}}{v^2} \frac{eV_c}{cP_d} \cos \phi_{\text{RF}}. \quad (83)$$

By equating M_z with the representation by Twiss parameters

$$M_z = \begin{pmatrix} \cos \mu_z + \alpha_z \sin \mu_z & \beta_z \sin \mu_z \\ \frac{1 + \alpha_z^2}{\beta_z} \sin \mu_z & \cos \mu_z - \alpha_z \sin \mu_z \end{pmatrix}, \quad (84)$$

we obtain at the middle of the RF cavity:

$$\begin{aligned} \alpha_z &= 0, \\ \cos \mu_z &= 1 - \frac{\alpha_p C k_z}{2}, \\ \beta_z &= -\frac{\alpha_p C}{\sin \mu_z}. \end{aligned} \quad (85)$$

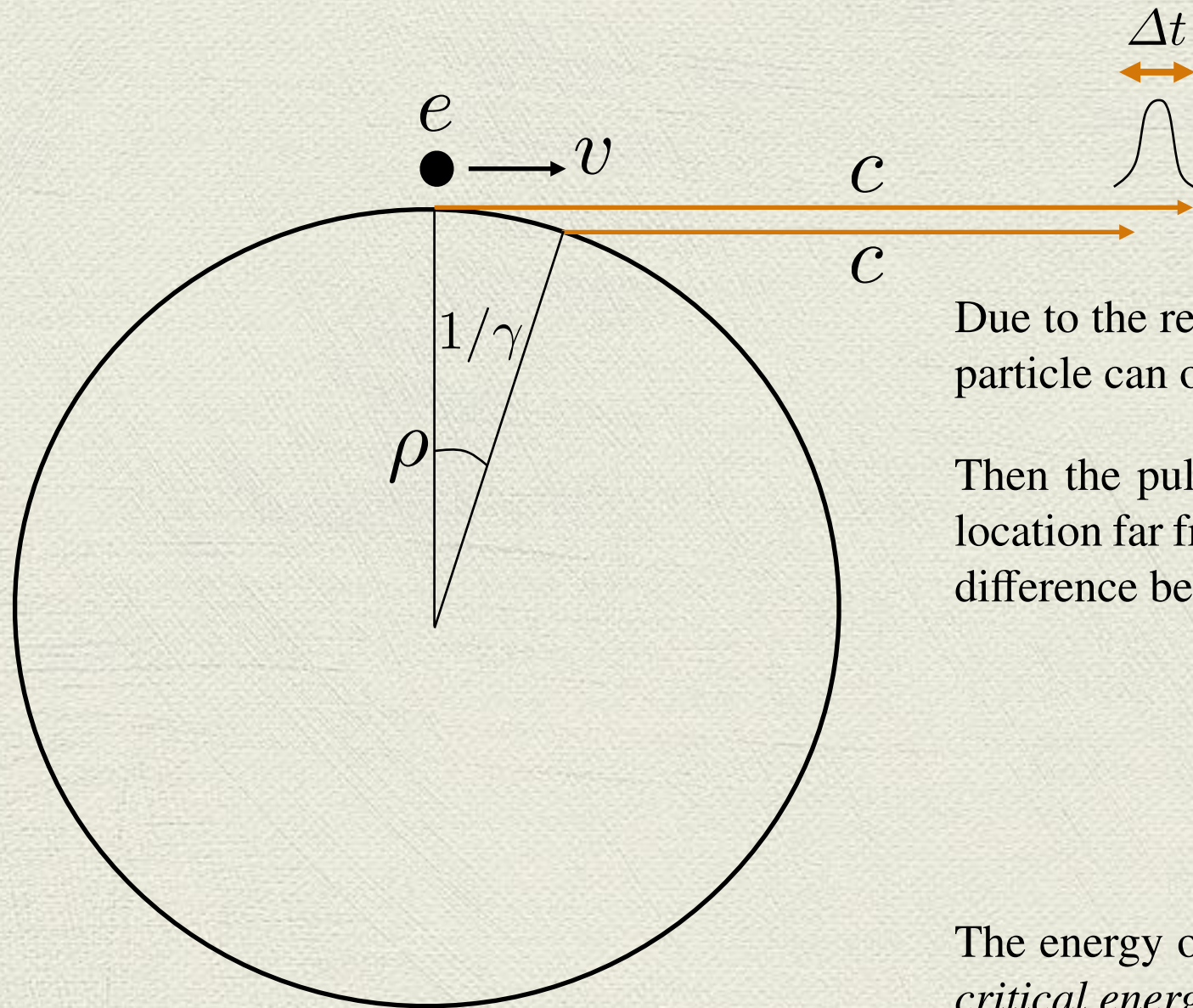
Note that $\alpha_p k_z > 0$ for the stability, and $\mu_z < 0$ for $\alpha_p > 0$.

In the case of $|\mu_z| \ll 1$, α_z and β_z are nearly constant over the ring, and:

$$\mu_z^2 \approx \alpha_p C k_z, \quad \beta_z \approx -\frac{\alpha_p}{\mu_z} C. \quad (86)$$

$$E = E_0 + \frac{eV_c}{cP_d} \sin\left(-\omega_{\text{RF}} \frac{z}{v} + \phi_{\text{RF}}\right), \quad (40)$$

Overview of synchrotron radiation



Due to the relativistic kinematics, an ultra-relativistic particle can only radiate within an angle $1/\gamma$.

Then the pulse length of the radiation observed at a location far from the ring can be estimated by the time difference between the particle and the light:

$$\Delta t \approx \frac{\rho/\gamma}{v} - \frac{\rho \sin 1/\gamma}{c} \quad (87)$$

$$\approx \frac{2\rho}{3\gamma^3 c} \quad (88)$$

The energy of the photon should be characterized by *critical energy* using the uncertainty principle:

$$u_c = \frac{\hbar}{\Delta t} = \frac{3}{2}\gamma^3 \frac{\hbar c}{\rho} = \frac{3}{2}\gamma^3 mc^2 \left(\frac{r_e}{\alpha \rho} \right), \quad (89)$$

where α and r_e are the fine structure constant and the classical electron radius, respectively.

Overview of synchrotron radiation (2)

Most basic characteristics of the radiation, concerning to the beam, are represented by the following formulas: The expected number of photons in a bending angle ϕ :

$$\langle N \rangle = \frac{5}{2\sqrt{3}} \alpha \gamma \phi . \quad (90)$$

The expected value of photon energy and the square of photon energy:

$$\langle u \rangle = \frac{8}{15\sqrt{3}} u_c , \quad (91)$$

$$\langle u^2 \rangle = \frac{11}{27} u_c^2 . \quad (92)$$

The expected energy loss per an angle ϕ :

$$\langle \Delta E \rangle = -\langle N \rangle \langle u \rangle = -\frac{2}{3} \gamma^4 m c^2 \left(\frac{r_e}{\rho} \right) \phi . \quad (93)$$

The expected energy spread per an angle ϕ :

$$\langle \Delta E^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = \langle N \rangle \langle u^2 \rangle = \frac{55}{24\sqrt{3}} \frac{\gamma^7}{\alpha} m^2 c^4 \left(\frac{r_e}{\rho} \right)^2 \phi . \quad (94)$$

Overview of synchrotron radiation (3)

Let us denote the amount of the momentum loss, normalized by the design momentum P_d , in a small section of a beam line by $-dW(x, y, \delta; s)$. As we have seen, dW is expressed by

$$dW = \frac{\langle N \rangle \langle u \rangle}{cP_d} . \quad (95)$$

We obtain the energy loss per revolution by integrating dW at the standard orbit:

$$U_0 = cP_d \oint dW_d = \frac{C_\gamma}{2\pi} E_d^4 \oint \frac{1}{\rho^2} ds , \quad (96)$$

where $E_d \approx cP_d$ and

$$C_\gamma \equiv \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} . \quad (97)$$

If the bending radius is constant, $\rho = \rho_0$, along the ring,

$$U_0 = C_\gamma \frac{E_d^4}{\rho_0} = 88.5[\text{keV}] \times \frac{E_d[\text{GeV}]^4}{\rho_0[\text{m}]} . \quad (98)$$

Table 1: One-turn energy loss U_0 for FCC-ee.

Beam Energy E_d	[GeV]	45.6	182.5
Bending radius ρ_0	[m]	10760	
Energy loss / turn U_0	[MeV]	35.6	9124

Justification of $\langle E^2 \rangle - \langle E \rangle^2 = \langle N \rangle \langle u^2 \rangle$

If the system emits k uncorrelated photons randomly, its probability is given by

$$\lambda_k = \frac{\langle N \rangle^k}{k!} \exp(-\langle N \rangle). \quad (99)$$

Note that

$$\sum_{k=0}^{\infty} \lambda_k = 1, \quad \sum_{k=0}^{\infty} k \lambda_k = \langle N \rangle. \quad (100)$$

The beam energy changes as

$$E_k = E_0 - \sum_i^k u_i, \quad (101)$$

$$\begin{aligned} \langle E_k^2 \rangle &= E_0^2 - 2E_0 \left\langle \sum_{i=1}^k u_i \right\rangle + \left\langle \sum_{i=1}^k \sum_{j=1}^k u_i u_j \right\rangle \\ &= E_0^2 - 2kE_0 \langle u \rangle + k(k-1) \langle u \rangle^2 + k \langle u^2 \rangle. \end{aligned} \quad (102)$$

Then

$$\begin{aligned} \langle E^2 \rangle &= \sum_{k=0}^{\infty} \lambda_k \langle E_k^2 \rangle \\ &= E_0^2 - 2E_0 \langle N \rangle \langle u \rangle + \sum_{k=0}^{\infty} k(k-1) \frac{\langle N \rangle^k}{k!} \exp(-\langle N \rangle) \langle u^2 \rangle + \langle N \rangle \langle u^2 \rangle \\ &= E_0^2 - 2E_0 \langle N \rangle \langle u \rangle + \langle N \rangle^2 \langle u^2 \rangle + \langle N \rangle \langle u^2 \rangle \\ &= \langle E \rangle^2 + \langle N \rangle \langle u^2 \rangle. \end{aligned}$$

$$\langle E \rangle = E_0 - \langle N \rangle \langle u \rangle$$

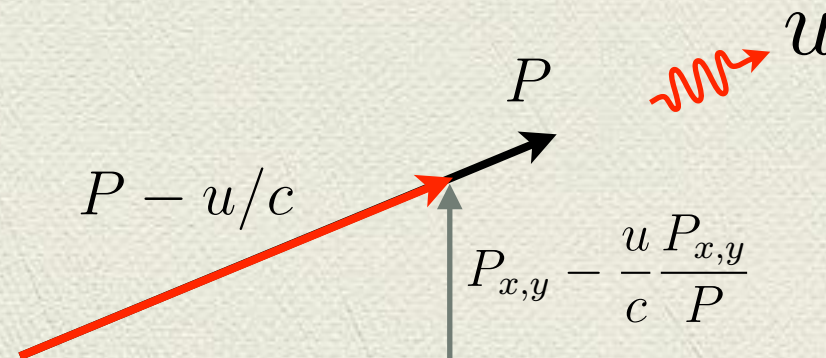
(103)

Effects of radiation on the beam motion

Hereafter we assume the particle is ultra-relativistic, ie., $\gamma \gg 1$. The effects of radiation or emission of photons on the particle motion are characterized as:

- The position of a particle does not change due to radiation.
- The radiation is almost in the same direction of the particle momenta, within an angular deviation $\lesssim 1/\gamma$, The angular divergence will be omitted below for the time being.
- Thus the particle almost does not change the direction of motion. The changes of the momenta are expressed as

$$P \rightarrow P - u/c, \quad P_{x,y} \rightarrow P_{x,y} - \frac{u P_{x,y}}{c P}. \quad (104)$$



- The emission of photons is a totally random process. It is not possible to tell when and where a particle emits a photon. Each amount of photon is also stochastic under a given probability distribution.

Local radiation damping

Now let us consider a particle having small deviations $(x, p_x, y, p_y, z, \delta)$ from the *standard orbit*, where the radiation is given by dW_d .

Then the change of momenta relative to the standard orbit due to radiation are written as

$$\begin{aligned} d\delta &= -(dW - dW_d) \\ &= -\frac{\partial dW}{\partial \delta} \delta - \frac{\partial dW}{\partial x} x - \frac{\partial dW}{\partial y} y, \end{aligned} \quad (105)$$

$$dp_{x,y} = -(dW p_{x,y} - dW_d p_{x,y;d}) \quad (106)$$

$$= -dW p_{x,y}, \quad (107)$$

where we have assumed the momentum components at the standard orbit $p_{x,y;d}$ are zero.

In a simple case with $\frac{\partial dW}{\partial x} = \frac{\partial dW}{\partial y} = 0$, the equations above become

$$d\delta = -\frac{\partial dW}{\partial \delta} \delta, \quad dp_{x,y} = -dW p_{x,y}, \quad (108)$$

which means a *local damping* of the momenta around the standard orbit.

Local radiation damping (2)

In the case of a uniform (transverse) magnetic field B , the momentum loss in a length ds is expressed as

$$dW = -\frac{\langle N \rangle \langle u \rangle}{cP_d} \quad (109)$$

$$-\langle N \rangle \langle u \rangle = -\frac{2}{3}\gamma^4 mc^2 \left(\frac{r_e}{\rho}\right) \phi \quad (93)$$

$$\propto P^2 B^2 ds' \propto (1 + \delta)^2 \quad (110)$$

using Eq.(93) with $\phi = ds'/\rho$ and $P = (1 + \delta)P_d = eB\rho$. Here we have introduced the orbit length ds' , which will be described later.

Then the momentum damping coefficient in Eq. (108) is written as

$$\frac{\partial dW}{\partial \delta} = 2dW, \quad (111)$$

which means the longitudinal damping rate is twice faster than the transverse ones in this simple case.

$$d\delta = -\frac{\partial dW}{\partial \delta} \delta, \quad dp_{x,y} = -dW p_{x,y}, \quad (103)$$

Effects of dispersion on damping

If there are dispersions, the motion should be considered in the betatron coordinates instead of physical coordinates:

$$\begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \end{pmatrix} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} - \begin{pmatrix} \eta_x \\ \eta_{p_x} \\ \eta_y \\ \eta_{p_y} \end{pmatrix} \delta \quad (112)$$

as well as

$$z_\beta = z + \eta_{p_x} x - \eta_x p_x + \eta_{p_y} y - \eta_y p_y . \quad (113)$$

Since we know the physical position (x, y, z) does not change due to radiation:

$$dx = dy = dz = 0 , \quad (114)$$

we obtain

$$\begin{aligned} d\delta &= -\frac{\partial dW}{\partial x}(x_\beta + \eta_x \delta) - \frac{\partial dW}{\partial y}(y_\beta + \eta_y \delta) - \frac{\partial dW}{\partial \delta} \delta , \\ dx_\beta &= -\eta_x d\delta , \\ dp_{x\beta} &= -dW(p_{x\beta} + \eta_{p_x} \delta) - \eta_{p_x} d\delta , \\ dy_\beta &= -\eta_y d\delta , \\ dp_{y\beta} &= -dW(p_{y\beta} + \eta_{p_y} \delta) - \eta_{p_y} d\delta , \\ dz_\beta &= \eta_{p_x} dx_\beta - \eta_x dp_{x\beta} + \eta_{p_y} dy_\beta - \eta_y dp_{y\beta} . \end{aligned} \quad (115)$$

Effects of dispersion on damping (2)

The effect of radiation in the previous equation can be expressed by a matrix:

$$\begin{pmatrix} dx_\beta \\ dp_{x\beta} \\ dy_\beta \\ dp_{y\beta} \\ dz_\beta \\ d\delta \end{pmatrix} = dR_D \begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial dW}{\partial x} \eta_x & \cdot & \frac{\partial dW}{\partial y} \eta_x & \cdot & \cdot & -dR_{D66} \eta_x \\ \frac{\partial dW}{\partial x} \eta_{p_x} & -dW & \frac{\partial dW}{\partial y} \eta_{p_x} & \cdot & \cdot & -(dR_{D66} + dW) \eta_{p_x} \\ \frac{\partial dW}{\partial x} \eta_y & \cdot & \frac{\partial dW}{\partial y} \eta_y & \cdot & \cdot & -dR_{D66} \eta_y \\ \frac{\partial dW}{\partial x} \eta_{p_y} & \cdot & \frac{\partial dW}{\partial y} \eta_{p_y} & -dW & \cdot & -(dR_{D66} + dW) \eta_{p_y} \\ \cdot & dW \eta_x & \cdot & dW \eta_y & \cdot & dW(\eta_x \eta_{p_x} + \eta_y \eta_{p_y}) \\ -\frac{\partial dW}{\partial x} & \cdot & -\frac{\partial dW}{\partial y} & \cdot & \cdot & dR_{D66} \end{pmatrix} \begin{pmatrix} x_\beta \\ p_{x\beta} \\ y_\beta \\ p_{y\beta} \\ z_\beta \\ \delta \end{pmatrix}, \quad (116)$$

$$dR_{D66} = -\frac{\partial dW}{\partial x} \eta_x - \frac{\partial dW}{\partial y} \eta_y - \frac{\partial dW}{\partial \delta}. \quad (117)$$

Then the local damping rates $d\kappa_{x,y,z}$ are calculated by the diagonal parts of dR_D :

$$\begin{aligned} d\kappa_x &= -\frac{dR_{D11} + dR_{D22}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial x} \eta_x \right), \\ d\kappa_y &= -\frac{dR_{D33} + dR_{D44}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial y} \eta_y \right), \\ d\kappa_z &= -\frac{dR_{D55} + dR_{D66}}{2} = \frac{1}{2} \left(\frac{\partial dW}{\partial x} \eta_x + \frac{\partial dW}{\partial y} \eta_y + 2dW \right), \end{aligned} \quad (118)$$

where we have used Eq. (111).

$$\frac{\partial dW}{\partial \delta} = 2dW, \quad (111)$$

Note that $d\kappa_x + d\kappa_y + d\kappa_z = 2dW$.

Radiation damping per revolution

We can integrate the local damping decrements Eq. (118) all over the ring to obtain the damping rate per revolution. We also assume there is no x - y coupling nor more complicated betatron-synchrotron couplings beyond that expressed by the dispersions.

The local momentum loss dW is expressed as

$$-\langle N \rangle \langle u \rangle = -\frac{2}{3} \gamma^4 m c^2 \left(\frac{r_e}{\rho} \right) \phi \quad (93)$$

$$\begin{aligned} dW &= \frac{\langle N \rangle \langle u \rangle}{c P_d} \\ &= \frac{2}{3} \gamma_0^3 r_e e^2 (1 + \delta)^2 B^2 ds' , \end{aligned} \quad (119)$$

with the orbit length ds'

$$ds' = \left(1 + \frac{x}{\rho_x} + \frac{y}{\rho_y} \right) ds , \quad (120)$$

where $\rho_x = -P_d/eB_y$ and $\rho_y = P_d/eB_x$ are the bending radius in each plane, and $B^2 = B_x^2 + B_y^2$.

$$\begin{aligned} d\kappa_x &= -\frac{dR_{D11} + dR_{D22}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial x} \eta_x \right) , \\ d\kappa_y &= -\frac{dR_{D33} + dR_{D44}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial y} \eta_y \right) , \\ d\kappa_z &= -\frac{dR_{D55} + dR_{D66}}{2} = \frac{1}{2} \left(\frac{\partial dW}{\partial x} \eta_x + \frac{\partial dW}{\partial y} \eta_y + 2dW \right) , \end{aligned} \quad (118)$$

Radiation damping per revolution (2)

We obtain the derivatives of dW in Eq. (118), on the standard orbit where $(x, p_x, y, p_y, z, \delta) = 0$. Using Eqs. (119,120), they are written as:

$$\begin{aligned} d\kappa_x &= -\frac{dR_{D11} + dR_{D22}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial x} \eta_x \right), \\ d\kappa_y &= -\frac{dR_{D33} + dR_{D44}}{2} = \frac{1}{2} \left(dW - \frac{\partial dW}{\partial y} \eta_y \right), \end{aligned} \quad (118)$$

$$d\kappa_z = -\frac{dR_{D55} + dR_{D66}}{2} = \frac{1}{2} \left(\frac{\partial dW}{\partial x} \eta_x + \frac{\partial dW}{\partial y} \eta_y + 2dW \right),$$

$$dW = \frac{2}{3} \gamma_0^3 r_e e^2 (1 + \delta)^2 B^2 ds', \quad (119)$$

$$ds' = \left(1 + \frac{x}{\rho_x} + \frac{y}{\rho_y} \right) ds \quad (120)$$

$$\frac{1}{dW} \frac{\partial dW}{\partial x} = \frac{1}{\rho_x} + \frac{2B_y}{B^2} \frac{\partial B_y}{\partial x}, \quad (121)$$

$$\frac{1}{dW} \frac{\partial dW}{\partial y} = \frac{1}{\rho_y} + \frac{2B_x}{B^2} \frac{\partial B_x}{\partial y}, \quad (122)$$

$$\frac{1}{dW} \frac{\partial dW}{\partial \delta} = 2. \quad (123)$$

We have omitted skew-focusing components $\frac{\partial B_x}{\partial x}$ and $\frac{\partial B_y}{\partial y}$, which can cause x - y coupling.

Thus the damping decrement per revolution at the standard orbit becomes

$$\kappa_x = \oint d\kappa_x = \oint \left(1 - \frac{\eta_x}{\rho_x} - 2\eta_x \frac{B_y}{B^2} \frac{\partial B_y}{\partial x} \right) \frac{dW_d}{2}, \quad (124)$$

$$\kappa_y = \oint d\kappa_y = \oint \left(1 - \frac{\eta_y}{\rho_y} - 2\eta_y \frac{B_x}{B^2} \frac{\partial B_x}{\partial y} \right) \frac{dW_d}{2}, \quad (125)$$

$$\kappa_z = \oint d\kappa_z = \oint \left(2 + \frac{\eta_x}{\rho_x} + \frac{\eta_y}{\rho_y} + 2\eta_x \frac{B_y}{B^2} \frac{\partial B_y}{\partial x} + 2\eta_y \frac{B_x}{B^2} \frac{\partial B_x}{\partial y} \right) \frac{dW_d}{2}. \quad (126)$$

where $dW_d = \frac{C_\gamma}{2\pi c} E_d^3 \frac{1}{\rho^2} ds$, as shown in Eq. (96).

$$U_0 = cP_d \oint dW_d = \frac{C_\gamma}{2\pi} E_d^4 \oint \frac{1}{\rho^2} ds, \quad (91)$$

Radiation damping per revolution (3)

These damping decrements are expressed in terms of *damping partitions* $J_{x,y,z}$ as:

$$\kappa_{x,y,z} = \frac{J_{x,y,z}}{2} \oint dW_d = \frac{J_{x,y,z}}{2} \frac{U_0}{cP_d}. \quad (127)$$

Thus

$$J_x + J_y + J_z = 4. \quad (128)$$

In the expressions in Eqs. (124–126), the terms with field gradients $\frac{2B_y}{B^2} \frac{\partial B_y}{\partial x}$ and $\frac{2B_x}{B^2} \frac{\partial B_x}{\partial y}$ are often zero, in the case of a ring consisting of flat dipoles ($\because B' = 0$) and separated quadrupoles ($\because B = 0$ at the design orbit). Also the ratio η_x/ρ is usually small for a large machine. Then in such a case we can approximate as:

$$J_x \approx J_y \approx 1, \quad J_z \approx 2. \quad (129)$$

More terms in the damping partitions can arise from the edge angle of a dipole. Also a non-zero closed orbit will change them.

$$\kappa_x = \oint d\kappa_x = \oint \left(1 - \frac{\eta_x}{\rho_x} - 2\eta_x \frac{B_y}{B^2} \frac{\partial B_y}{\partial x} \right) \frac{dW_d}{2}, \quad (124)$$

$$\kappa_y = \oint d\kappa_y = \oint \left(1 - \frac{\eta_y}{\rho_y} - 2\eta_y \frac{B_x}{B^2} \frac{\partial B_x}{\partial y} \right) \frac{dW_d}{2}, \quad (125)$$

$$\kappa_z = \oint d\kappa_z = \oint \left(2 + \frac{\eta_x}{\rho_x} + \frac{\eta_y}{\rho_y} + 2\eta_x \frac{B_y}{B^2} \frac{\partial B_y}{\partial x} + 2\eta_y \frac{B_x}{B^2} \frac{\partial B_x}{\partial y} \right) \frac{dW_d}{2}. \quad (126)$$

Momentum spread

The stochastic nature of the synchrotron radiation generates beam emittance, which is the spread of particles in the phase space.

First let us look at the z -plane. From Eq. (94), we can write the increment of the square of momentum spread in a small section ds as

$$\langle N \rangle \langle u^2 \rangle = \frac{55}{24\sqrt{3}} \frac{\gamma^7}{\alpha} m^2 c^4 \left(\frac{r_e}{\rho} \right)^2 \phi. \quad (94)$$

$$dA_z = \frac{\langle N \rangle \langle u^2 \rangle}{c^2 P_d^2} = \frac{55}{24\sqrt{3}} \gamma^5 \frac{r_e^2}{\alpha} \frac{ds}{|\rho|^3}. \quad (130)$$

The momentum spread of the beam follows an equation of damping and excitation:

$$\frac{d\sigma_\delta^2}{dn} = -2\kappa_z \sigma_\delta^2 + \frac{1}{2} \oint dA_z, \quad (131)$$

where n is the number of turns.

The factor 1/2 in front of the integral above comes from the synchrotron oscillation, where the average of δ^2 is equal to $\delta_{\text{peak}}^2/2$. We have also assumed the synchrotron motion is slow ($\beta_z \approx \text{const.}$, $\alpha_z \approx 0$).

Using Eq. (127), at the equilibrium the momentum spread becomes

$$\sigma_\delta^2 = \frac{1}{4\kappa_z} \oint dA_0 = \frac{1}{2J_z} \frac{cP_d}{U_0} \oint dA_z \quad (132)$$

$$= \frac{C_q}{J_z} \gamma_0^2 \frac{\oint 1/|\rho|^3 ds}{\oint 1/\rho^2 ds}, \quad (133)$$

where $C_q \equiv \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 0.9923\lambda_e$.

$$\kappa_{x,y,z} = \frac{J_{x,y,z}}{2} \oint dW_d = \frac{J_{x,y,z}}{2} \frac{U_0}{cP_d}. \quad (122)$$

Transverse emittance

The transverse equilibrium emittance can be estimated in a similar way. In the normalized coordinate

$$(u_x, p_{ux}) = \left(\frac{x}{\sqrt{\beta_x}}, \sqrt{\beta_x} p_x + \frac{\alpha_x}{\sqrt{\beta_x}} x \right), \quad (134)$$

the emittance is equal to the expected value $\langle u_x^2 \rangle = \langle p_{ux}^2 \rangle$.

Then the excitation term dA_x , corresponding to dA_z for z -plane is

$$dA_x = \langle du_x^2 \rangle + \langle dp_{ux}^2 \rangle \quad (135)$$

$$= \gamma_x \langle dx_\beta^2 \rangle + \alpha_x \langle dx_\beta dp_{x\beta} \rangle + \beta_x \langle dp_{x\beta}^2 \rangle, \quad (136)$$

$$= (\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2) \langle d\delta^2 \rangle, \quad (137)$$

$$= \mathcal{H}_x \langle d\delta^2 \rangle, \quad (138)$$

where $\gamma_x = (1 + \alpha_x^2)/\beta_x$, and we have used $dx_\beta = -\eta_x d\delta$ and $dp_{x\beta} = -\eta_{px} d\delta$. Then in a similar way as the z -plane, the equilibrium emittance can be obtained as

$$\begin{aligned} \varepsilon_{x,y} &= \frac{1}{4\kappa_{x,y}} \oint dA_{x,y} = \frac{1}{2J_{x,y}} \frac{cP_d}{U_0} \oint dA_{x,y} \\ &= \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{\oint \mathcal{H}_{x,y} / |\rho|^3 ds}{\oint 1/\rho^2 ds}, \end{aligned} \quad (139)$$

where

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y} \eta_{x,y}^2 + 2\alpha_{x,y} \eta_{x,y} \eta_{px,py} + \beta_{x,y} \eta_{px,py}^2. \quad (140)$$

Longitudinal emittance

The result for the transverse equilibrium emittance Eq. (139) is extendable to z -plane as:

$$\varepsilon_z = \frac{C_q}{J_z} \gamma_0^2 \frac{\oint \mathcal{H}_z / |\rho|^3 ds}{\oint 1/\rho^2 ds}, \quad (141)$$

with

$$\mathcal{H}_z \equiv \gamma_z \eta_z^2 + 2\alpha_{x,y} \eta_z \eta_\delta + \beta_z \eta_\delta^2 \quad (142)$$

$$\approx \beta_z, \quad (143)$$

where we have used $\eta_\delta = 1$ and $\eta_z \approx 0$.

From the emittance, the resulting bunch length and the energy spread are written as

$$\sigma_z^2 = \beta_z \varepsilon_z, \quad (144)$$

$$\sigma_\delta^2 = \gamma_z \varepsilon_z = \frac{1 + \alpha_z^2}{\beta_z} \varepsilon_z. \quad (145)$$

In the case of a slow synchrotron motion, above agree with previous results by setting $\beta_z \approx \text{const.} = -\frac{\alpha_p}{\mu_z} C$ as in Eq. (86).

x - y coupling

- The expression for the vertical equilibrium emittance Eq. (139) is only true when the x - y coupling is zero. It is no longer correct under x - y coupling.
- For instance, if we put a skew quadrupole in a dispersion-free region of a perfect ring with zero vertical dispersion, the vertical dispersion still remains zero.
- On the other hand, the vertical emittance can be increased depending on the magnitude of the skew quadrupole.
- Thus the expression \mathcal{H}_\dagger should not refer the dispersions in the physical coordinate, but the dispersions in the betatron coordinate.

There are several ways to express such betatron coordinates (u, p_u, v, p_v) . One way is

$$\begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = \mathbf{R} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} \mu & \cdot & -r_4 & r_2 \\ \cdot & \mu & r_3 & -r_1 \\ r_1 & r_2 & \mu & \cdot \\ r_3 & r_4 & \cdot & \mu \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}, \quad (146)$$

where $r_{1,2,3,4}$ are the coupling coefficients at each location s and $\mu^2 + (r_1 r_4 - r_2 r_3) = 1$.

$$\varepsilon_{x,y} = \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{\oint \mathcal{H}_{x,y} / |\rho|^3 ds}{\oint 1/\rho^2 ds}, \quad (139)$$

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y} \eta_{x,y}^2 + 2\alpha_{x,y} \eta_{x,y} \eta_{px,py} + \beta_{x,y} \eta_{px,py}^2$$

x - y coupling (2)

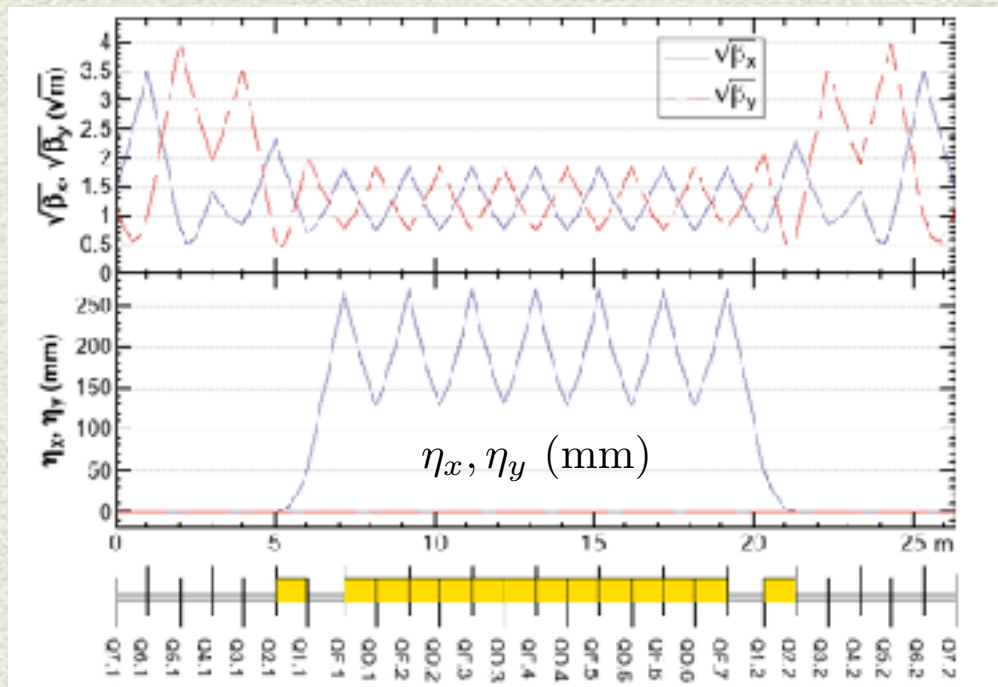
Then for the function \mathcal{H} we should use dispersions for (u, p_u, v, p_v) ., it ie.,

$$\begin{pmatrix} \eta_u \\ \eta_{pu} \\ \eta_v \\ \eta_{pv} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \end{pmatrix} = \begin{pmatrix} \mu & \cdot & -r_4 & r_2 \\ \cdot & \mu & r_3 & -r_1 \\ r_1 & r_2 & \mu & \cdot \\ r_3 & r_4 & \cdot & \mu \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \end{pmatrix}, \quad (147)$$

Thus even with $\eta_y = \eta_{py} = 0$, the dispersions in the normal coordinates can be nonzero due to the x - y coupling.

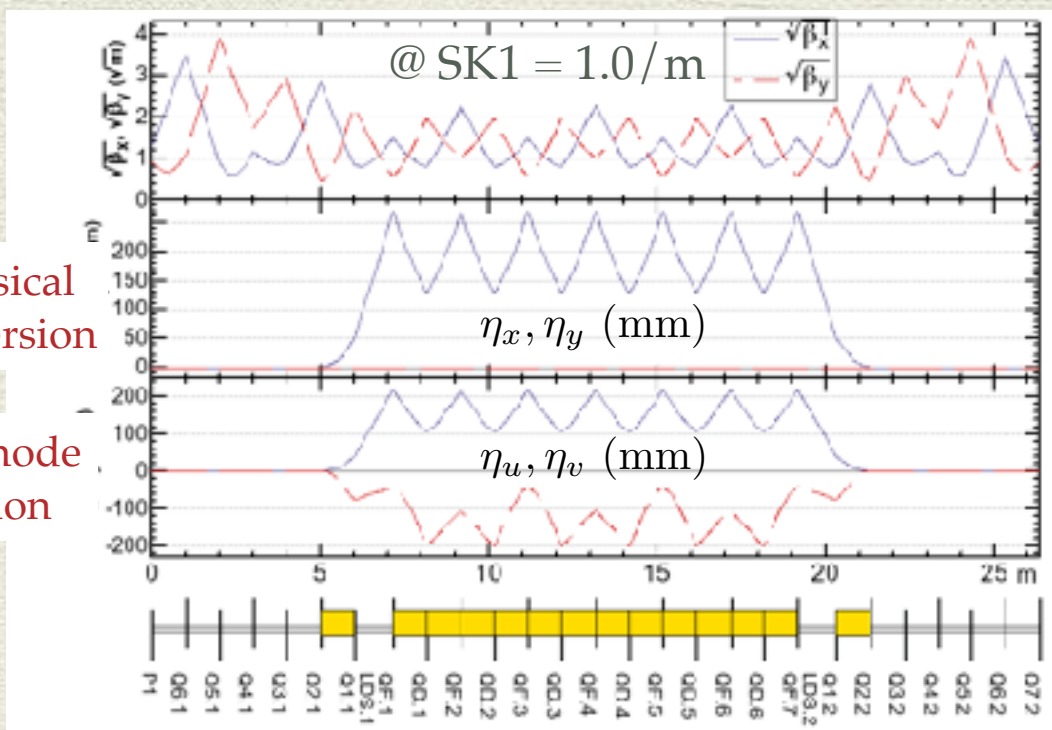
- Practically such an x - y coupling can arise from the vertical offset of sextupoles, which generates skew quadrupole components.
- Usually a sextupole is placed at a location with non-zero horizontal dispersion for the chromaticity correction, then such a vertical offset also produces the vertical dispersion.
- Thus to reduce the vertical emittance, correcting the vertical dispersion is not enough and x - y coupling correction through the ring is necessary.

x - y coupling (3)

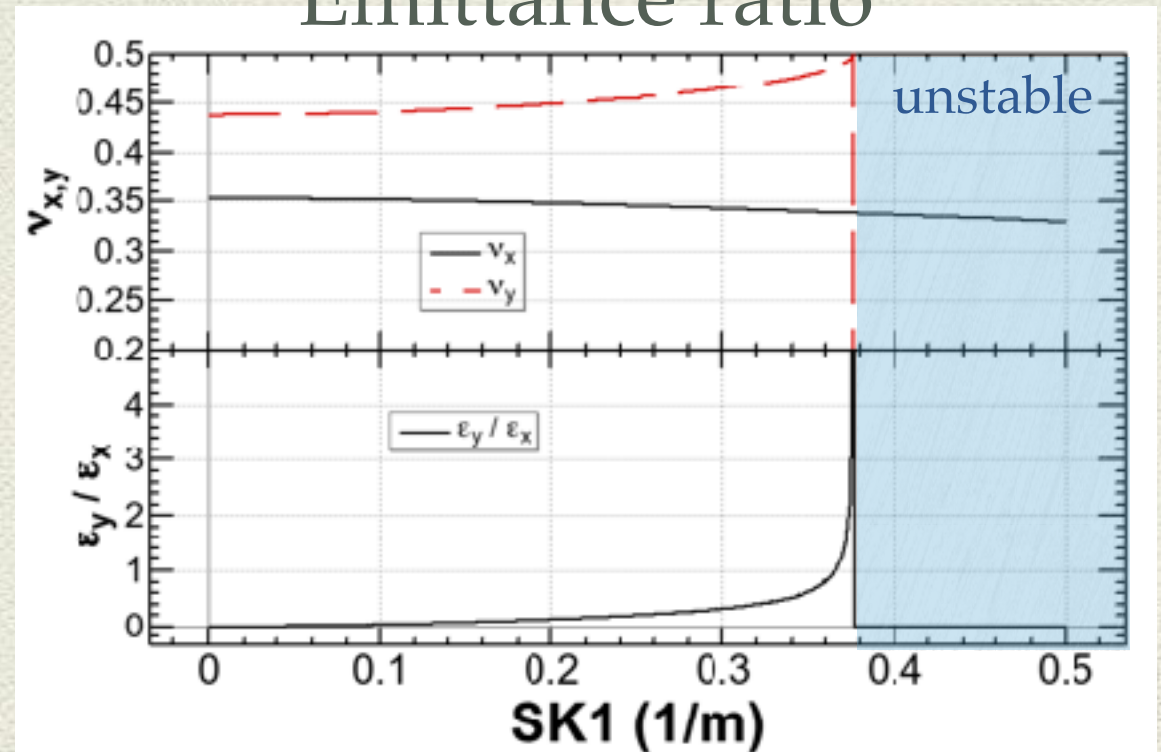


Under the presence of x - y coupling, the vertical emittance can increase even with zero (physical) vertical dispersion.

Skew quadrupole (SK1)



Emittance ratio



Emittance by the opening angle of photons

Usually for a vertical direction in a flat storage ring, the design vertical dispersion is zero and there is no x - y coupling in the ring. Then the vertical emittance given by Eq. (139) becomes zero.

In such a case, the vertical emittance due to the angular fluctuation of photons becomes the ultimate limit of the vertical emittance. The change of action $2dS_{y0}$ is given as in Eq. (136):

$$dA_{y,1/\gamma} = \beta_y \langle d\delta^2 / \gamma^2 \rangle, \quad (148)$$

assuming the emitted photons have angular divergence $\sim 1/\gamma$.

Then the equilibrium vertical emittance is

$$\varepsilon_{y,1/\gamma} = \frac{C_q}{J_y} \frac{\oint \beta_y / |\rho|^3 ds}{\oint 1/\rho^2 ds}. \quad (149)$$

In the case of uniform bending radius $\rho = \rho_0$,

$$\varepsilon_{y,1/\gamma} = \frac{C_q}{J_y} \frac{\langle \beta_y \rangle}{\rho_0} \approx \frac{\lambda_e}{J_y} \frac{\langle \beta_y \rangle}{\rho_0}. \quad (150)$$

Interestingly, this limit on the vertical emittance does not explicitly depend on the beam energy. Anyway the value is more important for a small ring.

$$dA_x = \gamma_x \langle dx_\beta^2 \rangle + \alpha_x \langle dx_\beta dp_{x\beta} \rangle + \beta_x \langle dp_{x\beta}^2 \rangle, \quad (136)$$

$$\varepsilon_{x,y} = \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{\oint \mathcal{H}_{x,y} / |\rho|^3 ds}{\oint 1/\rho^2 ds}, \quad (139)$$

$$\mathcal{H}_{x,y} \equiv \gamma_{x,y} \eta_{x,y}^2 + 2\alpha_{x,y} \eta_{x,y} \eta_{px,py} + \beta_{x,y} \eta_{px,py}^2$$

Synchrotron radiation integrals

We can summarize these formulas related to synchrotron radiation in terms of *radiation integrals* defined by:

$$\begin{aligned}
 I_1 &\equiv \oint \left(\frac{\eta_x}{\rho_x} + \frac{\eta_y}{\rho_y} \right) ds, & I_2 &\equiv \oint \frac{1}{\rho^2} ds, & I_3 &\equiv \oint \frac{1}{|\rho|^3} ds, \\
 I_{4x} &\equiv \oint \eta_x \left(\frac{1}{\rho_x} + 2 \frac{B_y}{B^2} \frac{\partial B_y}{\partial x} \right) ds, & I_{4y} &\equiv \oint \eta_y \left(\frac{1}{\rho_y} + 2 \frac{B_x}{B^2} \frac{\partial B_x}{\partial y} \right) ds, & I_{5x,y} &\equiv \oint \frac{\mathcal{H}_{x,y}}{|\rho|^3} ds.
 \end{aligned}
 \tag{144}$$

Then we can write as:

$$\alpha_p = \frac{I_1}{C}, \quad U_0 = \frac{C_\gamma}{2\pi} E_d^4 I_2, \tag{145}$$

$$J_{x,y} = 1 - \frac{I_{4x,y}}{I_2}, \quad J_z = 4 - J_x - J_y = 2 + \frac{I_{4x} + I_{4y}}{I_2}, \tag{146}$$

$$\kappa_{x,y,z} = \frac{J_{x,y,z}}{2} \frac{U_0}{cP_d}, \tag{147}$$

$$\sigma_\delta^2 = \frac{C_q}{J_z} \gamma_0^2 \frac{I_3}{I_2}, \quad \varepsilon_{x,y} = \frac{C_q}{J_{x,y}} \gamma_0^2 \frac{I_{5x,y}}{I_2}. \tag{148}$$

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