Optional Exercises-Set 2
(Ann Nelson's Cosmology lectures)

[5] The Axion is a hypothetical elementary particle, which was proposed as a solution to the strong CP problem, which is the problem of why the QCD $\overline{\theta}$ parameter is apparently so small, less than 10^{-10}. This solution makes $\overline{\theta}$ a dynamical field, $\theta\ a/f_a$, where $a{=}a(\vec{x},t)$. The energy density as a function of $\overline{\theta}$ may be estimated to be of order $m_\pi^2 f_\pi^2(1-\cos\overline{\theta})$, so is minimized at the CP conserving value $\overline{\theta}=0$. The axion mass, and all the axion couplings, are inversely proportional to f_a. Using dimensional analysis, and the fact that for a scalar field the square of the mass is given by the second derivative of the potential energy density at the minimum, **very roughly estimate** the axion mass and the axion lifetime, for $f_a = 10^{11}$ GeV. For the lifetime you can use the fact that the rate of axion decay has to be proportional to its coupling squared , and that besides f_a, the only other relevant dimensionful parameters in the problem are the axion mass, $\hbar$ and c.

[6] Assuming the Universe to be filled with homogeneous and isotropic matter, Einstein's theory of general relativity relates the time varying expansion rate of the Universe (called $H(t) = \dot{a}/a$) to the energy density ϵ and the pressure p of the matter. In order to solve for these three functions (H, ϵ and p), one needs three equations. From general relativity, assuming flat space, the first two equations are (in units with $c = 1$

$$H(t)^2 = \tfrac{8\pi}{3} G\epsilon(t) \ ,$$

$$\tfrac{d\epsilon(t)}{dt} = -3H(t)\left(\epsilon(t) + p(t)\right) \ .$$

where $G = G_N$ is Newton's gravitational constant. The third equation is the equation of state relating $p(t)$ to $\epsilon(t)$, and it depends on the type of energy filling the universe.

$$p = w\epsilon$$

Given these three equations, sssuming the equation of state parameter w is constant, solve for the expansion rate $H(t)$ and the scale factor $a(t)$. You should find that for w greater than some critical value, the acceleration of the scale factor $\ddot{a}$ is negative, (expansion rate is slowing down), while for w less than than this critical value, the acceleration of the scale factor $\ddot{a}$ is positive.

[7] The causal horizon size $d_h(t)$ is defined to be the farthest proper distance that a signal we receive at time t which travels at the speed of light could have travelled during the lifetime of the universe. Taking Ω=1 (flat universe) and $H_0 = (\dot{a}/a)|_{today} = 71$ Km/s/Mpc, and using the Robertson-Walker metric for flat space, compute the age of the universe and $d_h(t)$ in the scenario where there is only 1 kind of stuff in the universe, with equation of state:

$$p = w\epsilon$$

where p is the pressure, ϵ is the energy density, and w is a constant (which for any kind of stuff with nonnegative kinetic energy and speed of sound less than the speed of light lies between -1 and 1). You can use the results of the previous problem for $a(t)$. You can also use the fact that light follows a geodesic with $ds = 0$ so that $dt/a(t) = dr$ and the total coming distance travelled between times t_1 and t_2 (in units with $c = 1$) is $\int_{t_1}^{t_2} \frac{dt}{a(t)}$ while the total physical distance travelled is $a(t_2)\int_{t_1}^{t_2} \frac{dt}{a(t)}$. You should find that for $-1 > w > w_{\text{critical}}$ your answer for the causal horizon diverges as t_1 is taken to 0. What happens for $w = -1$? Explain why, provided that during the early universe w is less than the critical value, your results can explain the observation that the cosmic microwave background indicates that the entire observable universe appears to have once been in causal contact.