Palestinian Advanced Physics School

Condensed Matter Physics

Professor David Tong

جامعة بيرزيت
BIRZEIT UNIVERSITY

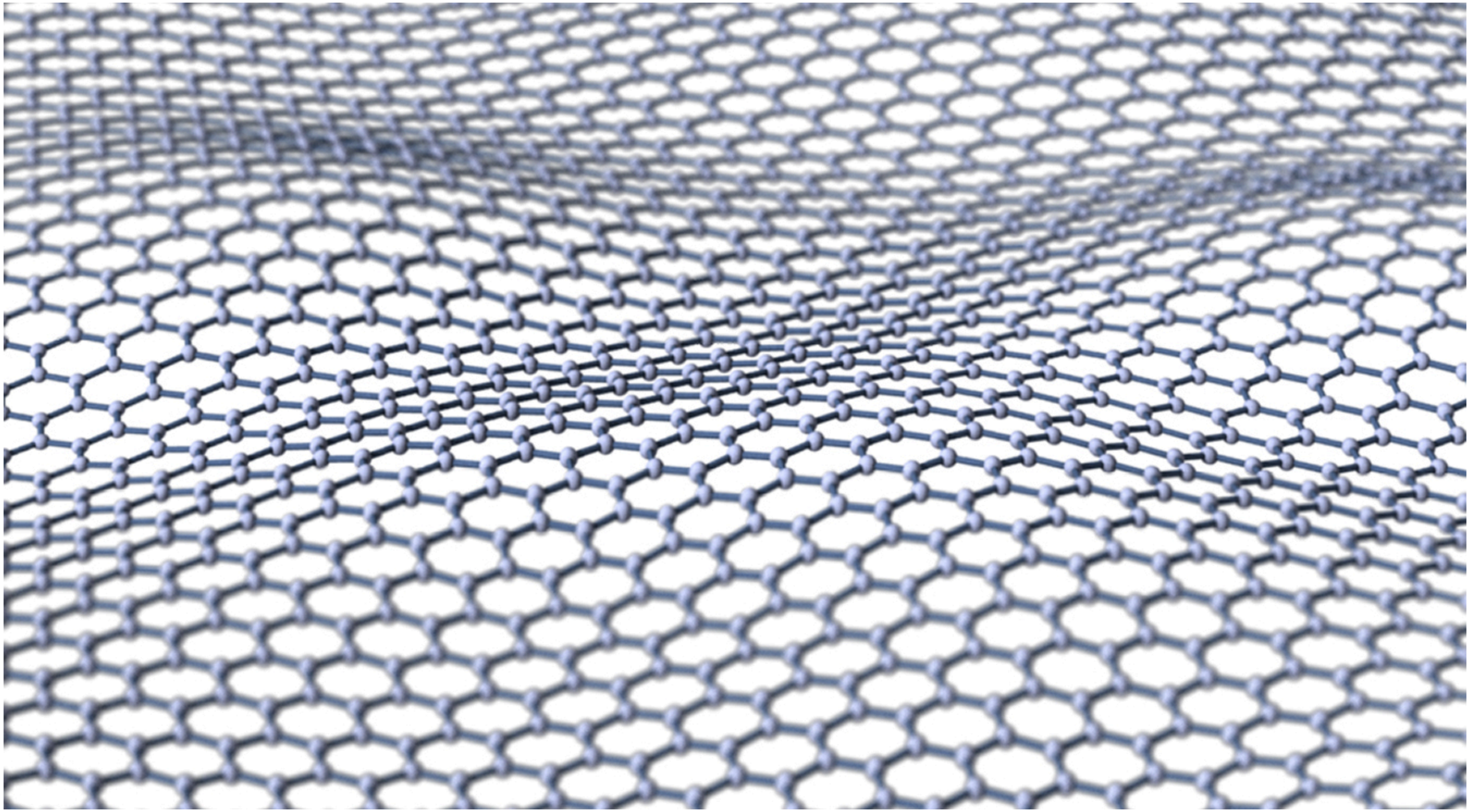
UNIVERSITY OF CAMBRIDGE

Lecture 3: Three Things

- Graphene
- Quantum Hall Effect
- Topological Insulators

Graphene

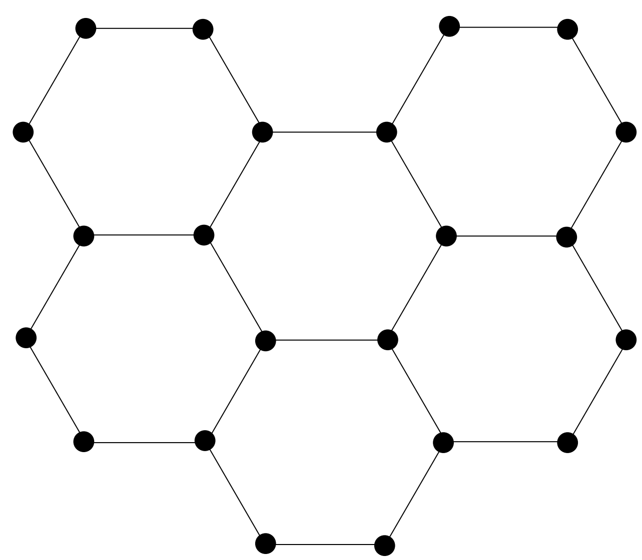
Graphene



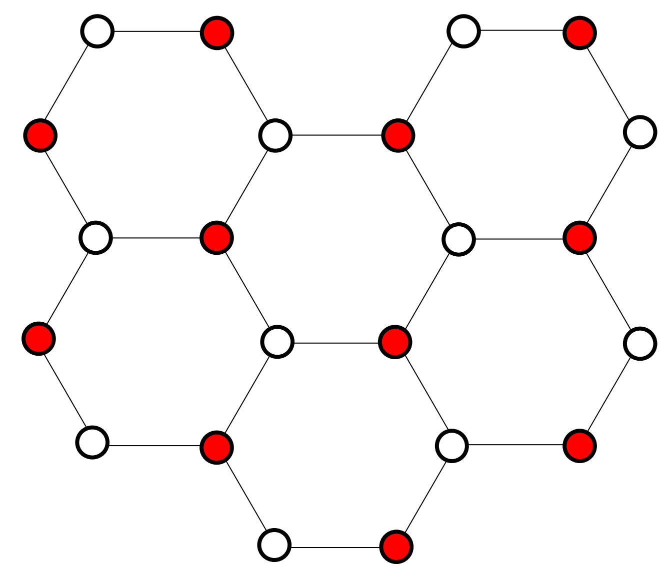
Geim and Novosolev 2004 (Nobel Prize 2010)

Graphene

Honeycomb lattice

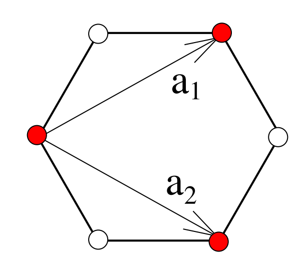


Best to think of this as two intersecting triangular lattices



Red dots = Lattice A: $\mathbf{r} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$

White dots = Lattice B: $\mathbf{r} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + \mathbf{d}$



$$\mathbf{a}_1 = \frac{\sqrt{3}a}{2}(\sqrt{3}, 1)$$

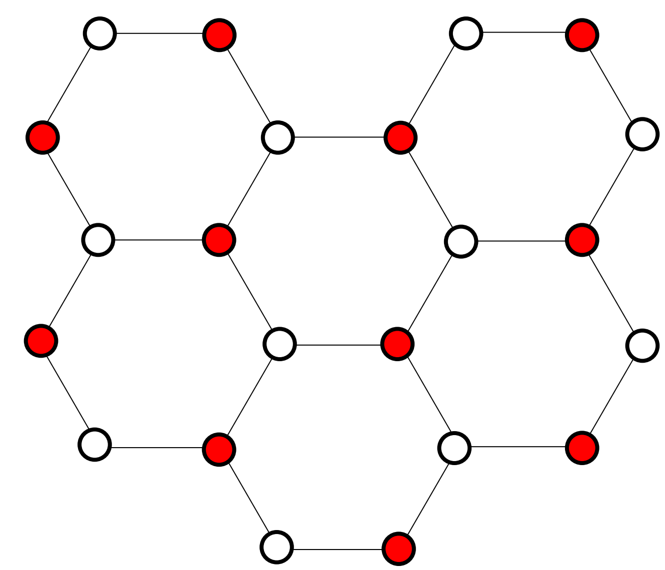
$$\mathbf{a}_2 = \frac{\sqrt{3}a}{2}(\sqrt{3}, -1)$$

$$n_i \in \mathbf{Z} \qquad \mathbf{d} = (-a, 0)$$

Tight Binding for Graphene

$$H = -t \sum_{\mathbf{r} \in \mathbf{\Lambda}} \Big[|\mathbf{r}; A\rangle\langle \mathbf{r}; B| + |\mathbf{r}; A\rangle\langle \mathbf{r} + \mathbf{a}_1; B| + |\mathbf{r}; A\rangle\langle \mathbf{r} + \mathbf{a}_2; B| + \text{h.c.} \Big]$$

This Hamiltonian hops from the red dots to the white dots, and back again



$$|\mathbf{r}; A\rangle = |\mathbf{r}\rangle$$

$$|\mathbf{r}; B\rangle = |\mathbf{r} + \mathbf{d}\rangle$$

To solve the Schrodinger equation, we make the ansatz;

$$|\psi(\mathbf{k})\rangle = \frac{1}{\sqrt{2N}} \sum_{\mathbf{r} \in \Lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \Big(c_A |\mathbf{r}; A\rangle + c_B |\mathbf{r}; B\rangle \Big)$$

Then we solve $H|\psi\rangle = E|\psi\rangle$

$$\begin{pmatrix} 0 & \gamma(\mathbf{k}) \\ \gamma^\star(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} c_A \\ c_B \end{pmatrix} = E(\mathbf{k}) \begin{pmatrix} c_A \\ c_B \end{pmatrix} \quad \text{with} \quad \gamma(\mathbf{k}) = -t\Big(1 + e^{i\mathbf{k}\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2}\Big)$$

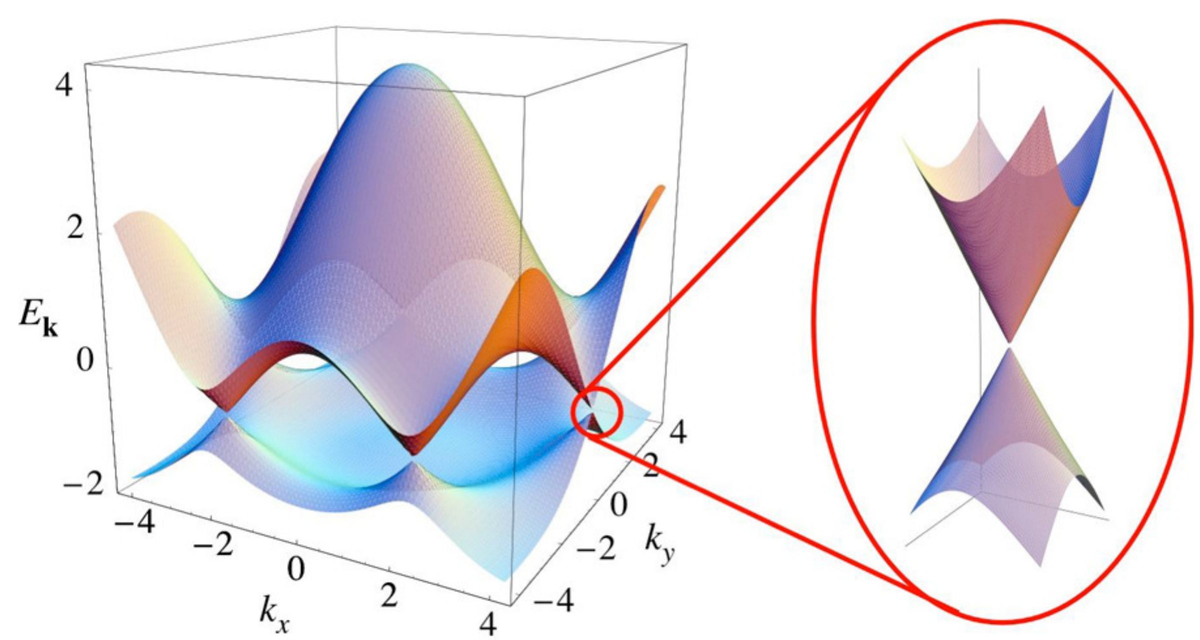
Tight Binding for Graphene

$$\begin{pmatrix} 0 & \gamma(\mathbf{k}) \\ \gamma^\star(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} c_A \\ c_B \end{pmatrix} = E(\mathbf{k}) \begin{pmatrix} c_A \\ c_B \end{pmatrix} \quad \text{with} \quad \gamma(\mathbf{k}) = -t\left(1 + e^{i\mathbf{k}\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2}\right)$$

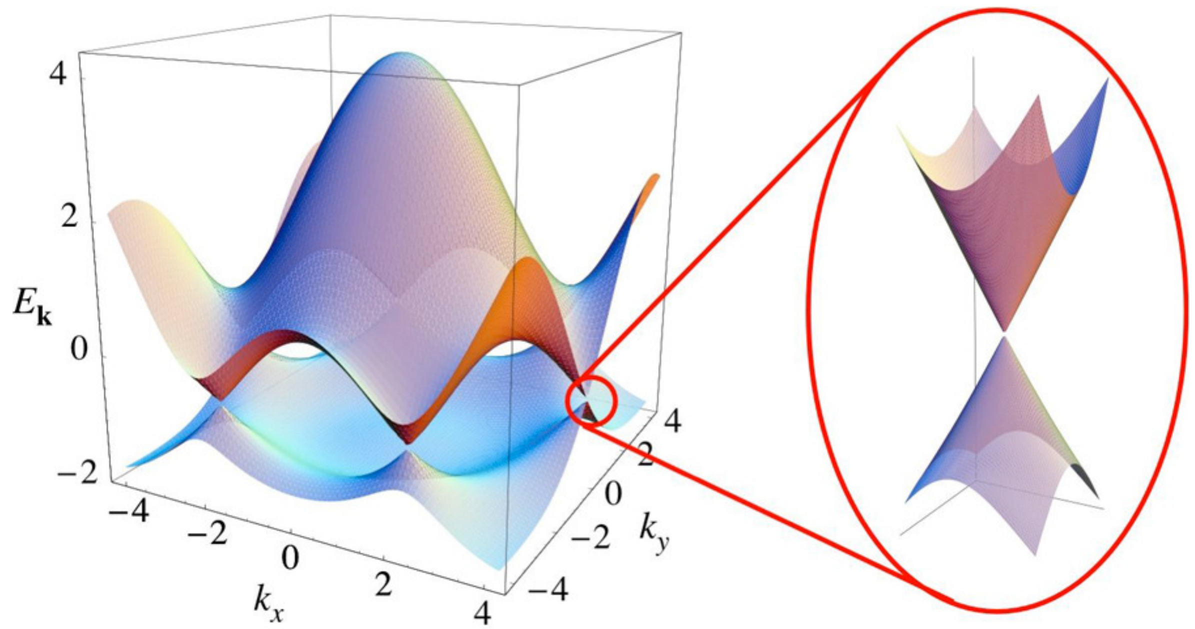
The eigenvalues are $E(\mathbf{k}) = \pm|\gamma(\mathbf{k})|$

$$\Rightarrow \quad E(\mathbf{k}) = \pm t\sqrt{1 + 4\cos\left(\frac{3k_x a}{2}\right)\cos\left(\frac{\sqrt{3}k_y a}{2}\right) + 4\cos^2\left(\frac{\sqrt{3}k_y a}{2}\right)}$$

- Carbon atoms have valency *Z=1.*
- This means that the lower band is filled. The upper band is empty
- The Fermi surface is simply points. These are *Dirac points*



Tight Binding for Graphene



We can also expand the Hamiltonian near the *Dirac point.* We find the Dirac equation for a massless, relativistic particle!

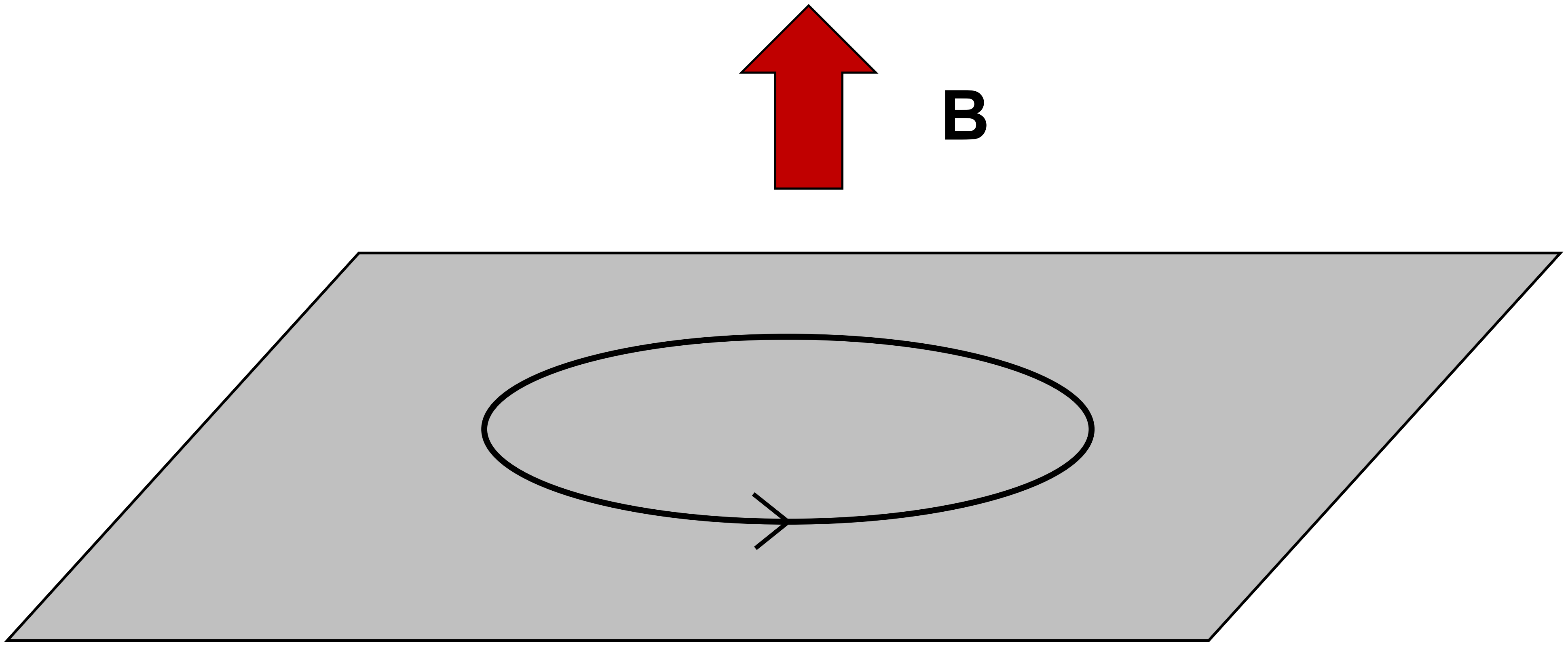
$$H = -v_F \hbar (q_x \sigma^1 + q_y \sigma^2)$$

- *q* is the momentum near the Dirac point
- σ are Pauli matrices
- The electron always travels with speed $v_F = \dfrac{3ta}{2\hbar}$. This is 300 times smaller than *c*.

- In graphene, the two components of the wavefunction tell us what sublattice the electron sits on; in particle physics it is what we call spin

The Quantum Hall Effect

Electron in a Magnetic Field



$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B} \quad \Longrightarrow \quad x = \frac{v}{\omega}\cos\omega t, \qquad y = -\frac{v}{\omega}\sin\omega t \qquad \omega = \frac{eB}{m}$$

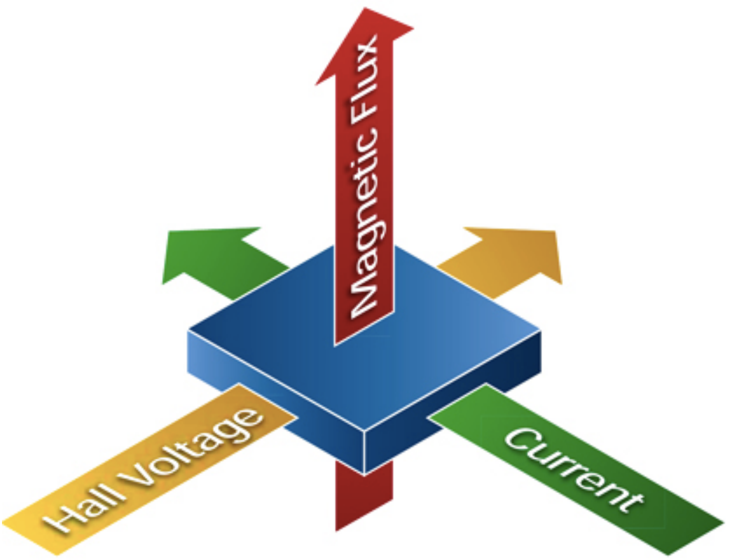
The Classical Hall Effect

friction=resistance

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau}$$

Electric field in x-direction

magnetic field in z-direction



In equilibrium, we solve for the velocity **v**. The solution takes the form

$$\mathbf{v} = \sigma\mathbf{E}$$

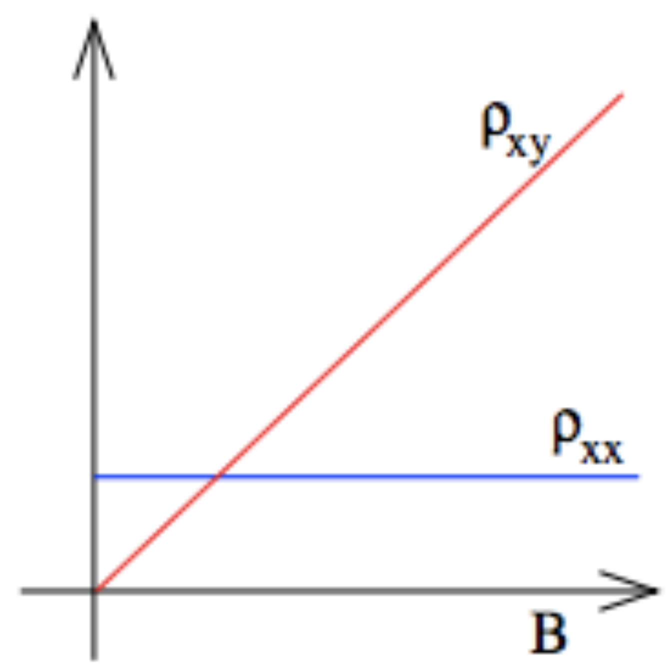
with σ a 2x2 matrix called the *conductivity*

The Classical Hall Effect

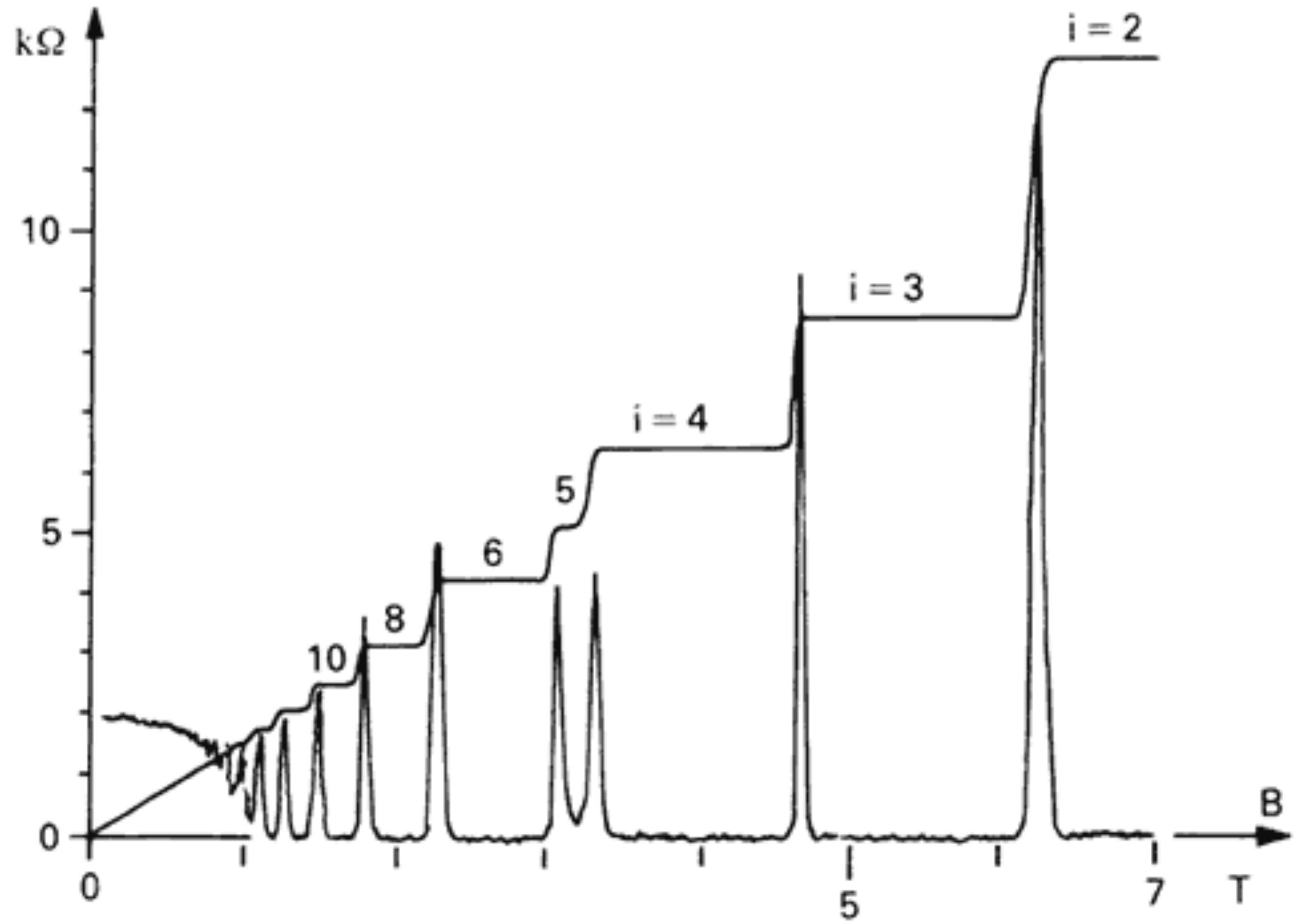
We usually plot the *resistivity* matrix

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$

The classical calculation above tells us how the components should change with **B**



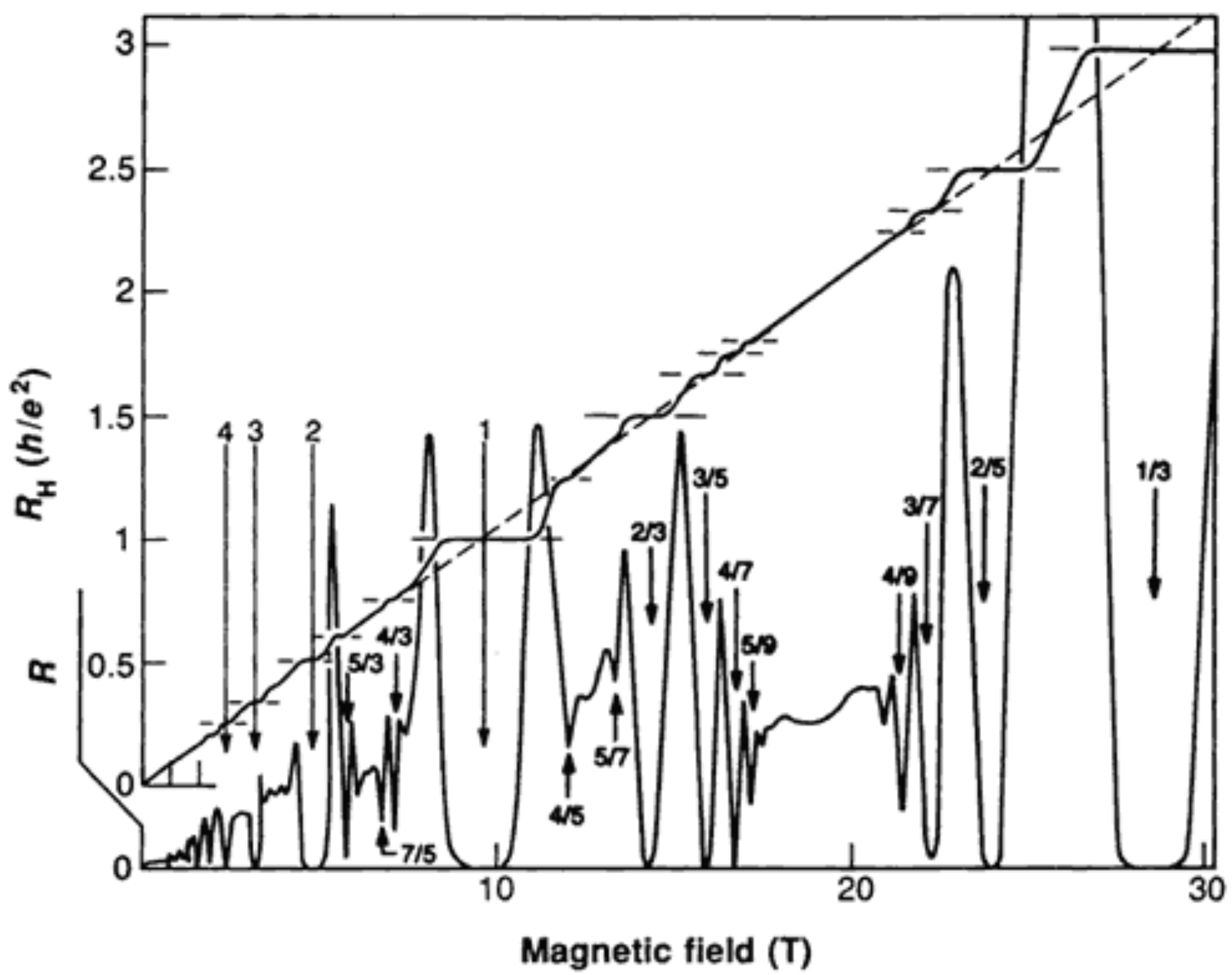
The Integer Quantum Hall Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \qquad \nu \in \mathbf{Z}$$

von Klitzing, Dorda and Pepper, 1981 (Nobel Prize 1985)

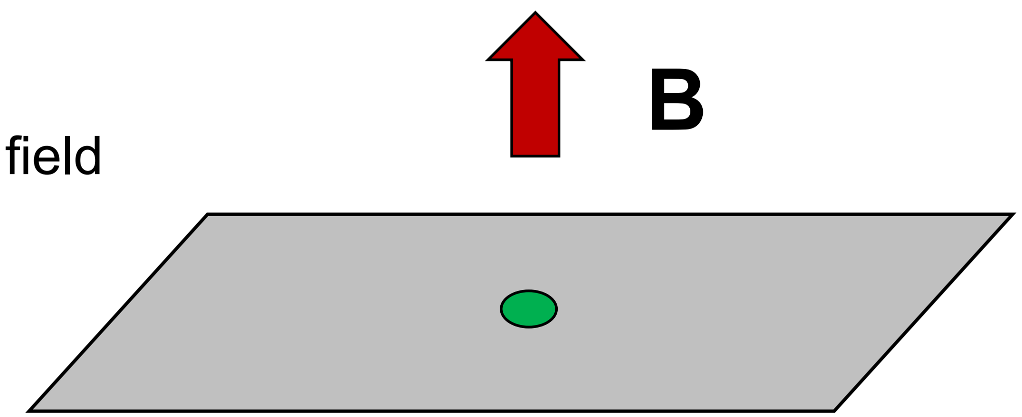
The Fractional Quantum Hall Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \qquad \nu \in \mathbf{Q}$$

Tsui, Stormer and Goddard, 1982 (Nobel Prize with Laughlin 1998)

Landau Levels



Consider quantum mechanics of particle moving in a magnetic field

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2$$

Gauge potential for magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Many choices; we work with $\mathbf{A} = xB\hat{\mathbf{y}}$

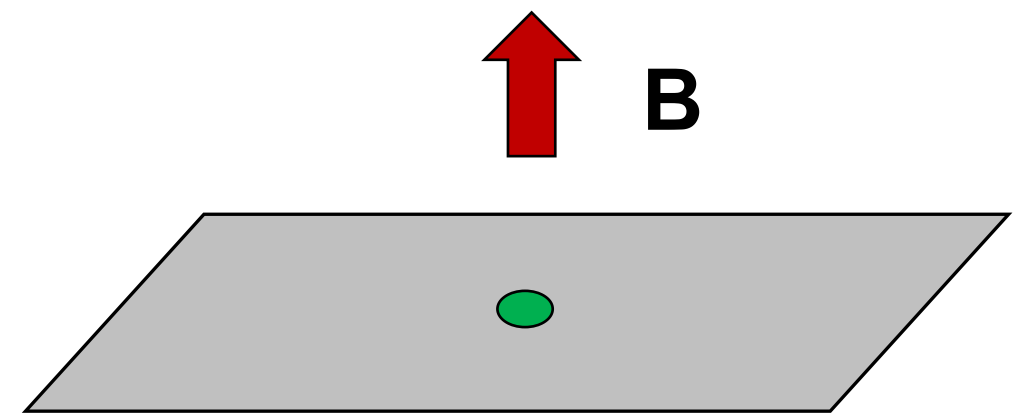
$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right)$$

Try the ansatz $\psi_k(x,y) = e^{iky} f_k(x)$. Then the Schrodinger equation becomes

$$\left(\frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2\right) f_k(x) = E f_k(x) \qquad l_B = \sqrt{\frac{\hbar}{eB}}$$

But this is the harmonic oscillator, displaced from the origin by *k!*

Landau Levels



A particle in a magnetic field is a harmonic oscillator

$$\left(\frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2\right) f_k(x) = E f_k(x)$$

We know the energy immediately

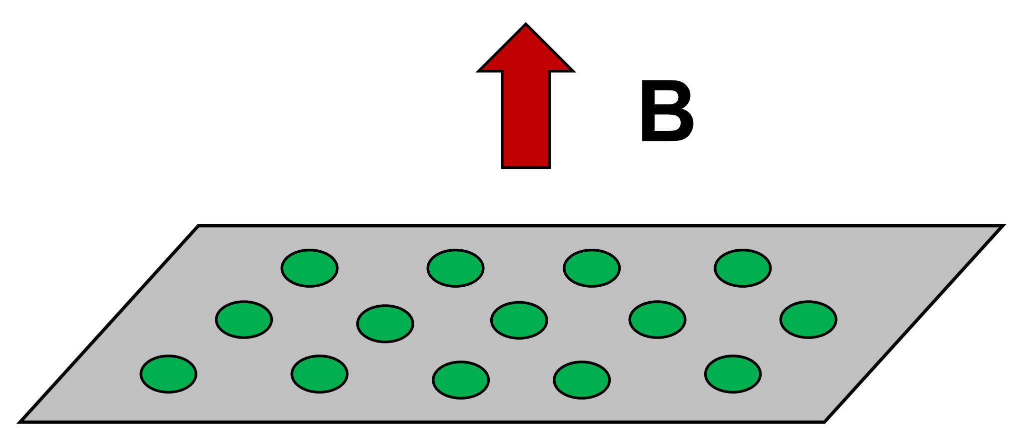
$$E = \hbar\omega\left(n + \frac{1}{2}\right)$$

with $\omega = \dfrac{eB}{m}$

Also we have the same energy for every *k* in the wavefunction $\psi_k(x, y) = e^{iky} f_k(x)$ .We have lots of states with the same energy.

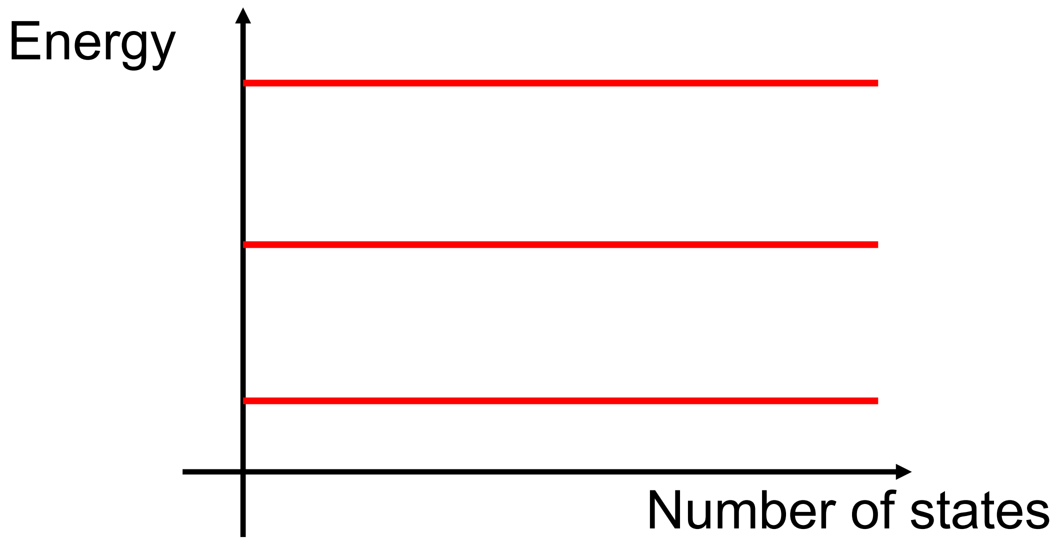
The degeneracy in an area *A* is

$$\mathcal{N} = \frac{e}{2\pi\hbar}AB$$



Landau Levels

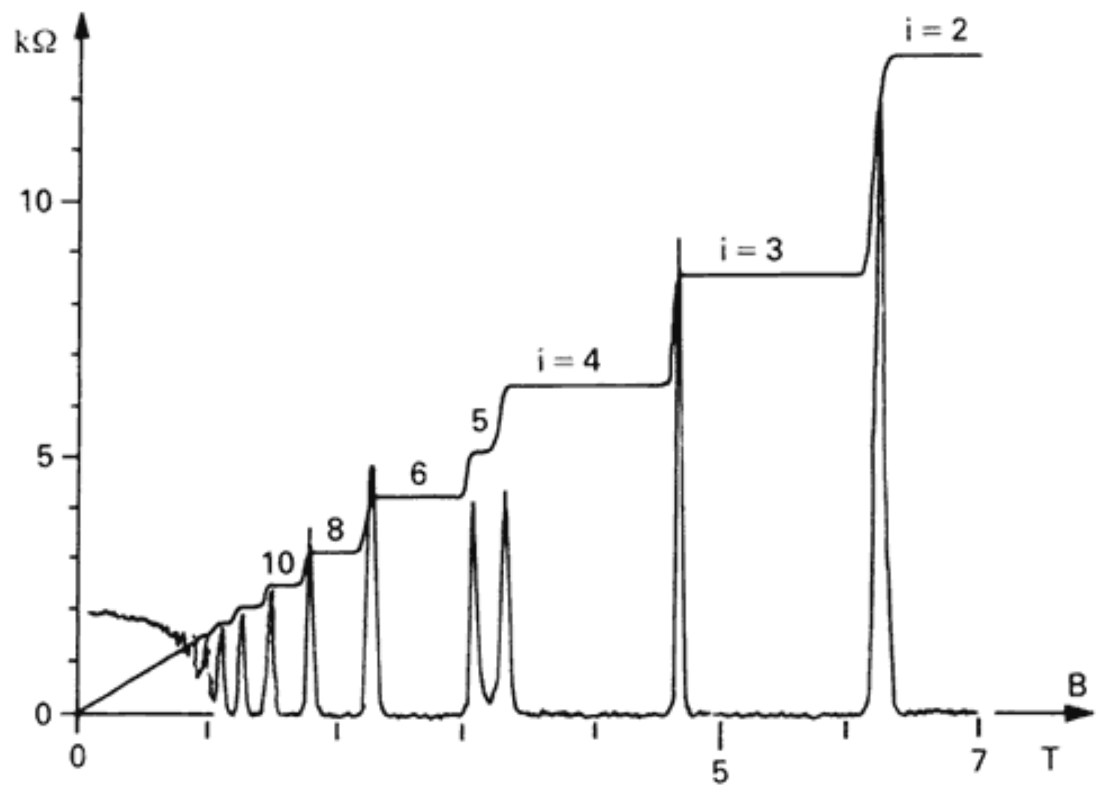
This is what the degeneracy of states looks like

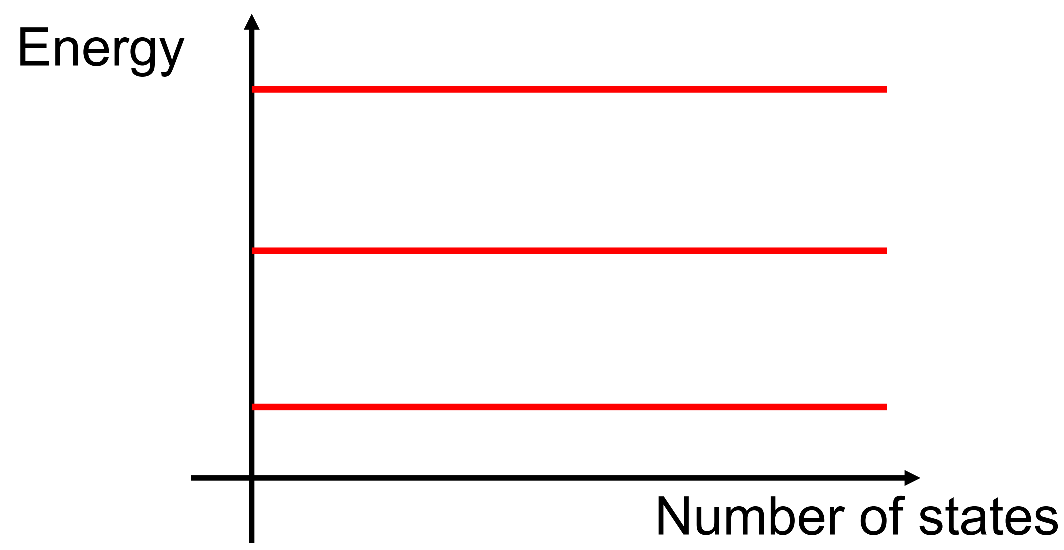


$$\mathcal{N} = \frac{e}{2\pi\hbar} AB$$

These are Landau levels

Understanding the Integer Quantum Hall Effect

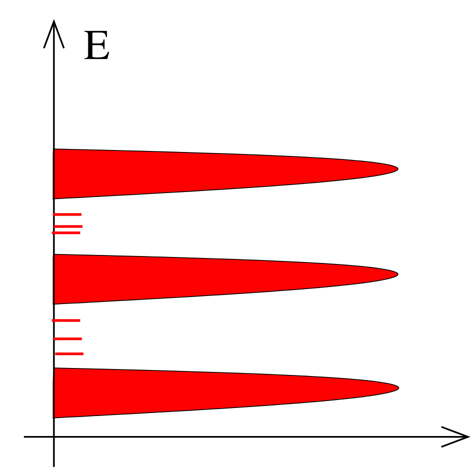




$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

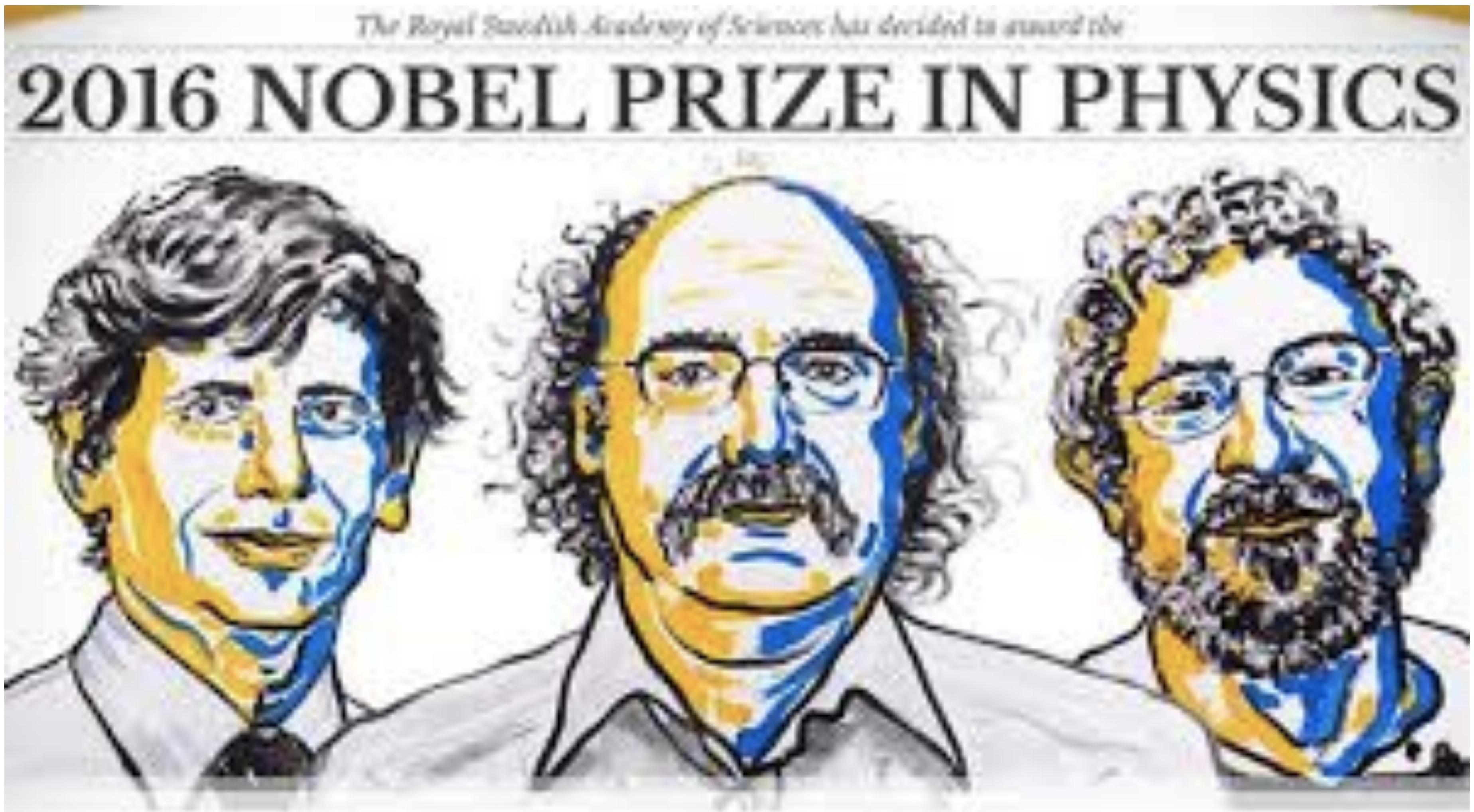
ν is counting the number of filled Landau levels

(A full understanding of why plateau exist needs disorder)



Topological Insulators

An Aside: Topological Insulators



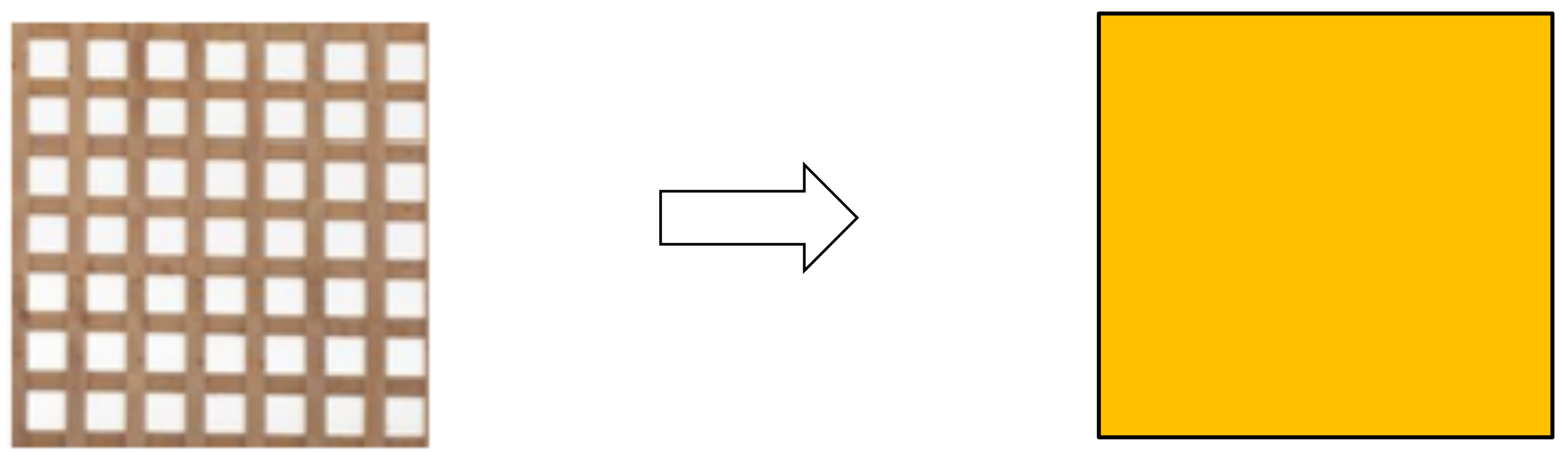
David Thouless

Michael Kosterlitz

Duncan Haldane

Topological Insulators in 2d

As we saw in the first lecture, if space is discrete then the momentum sits in the Brillouin zone



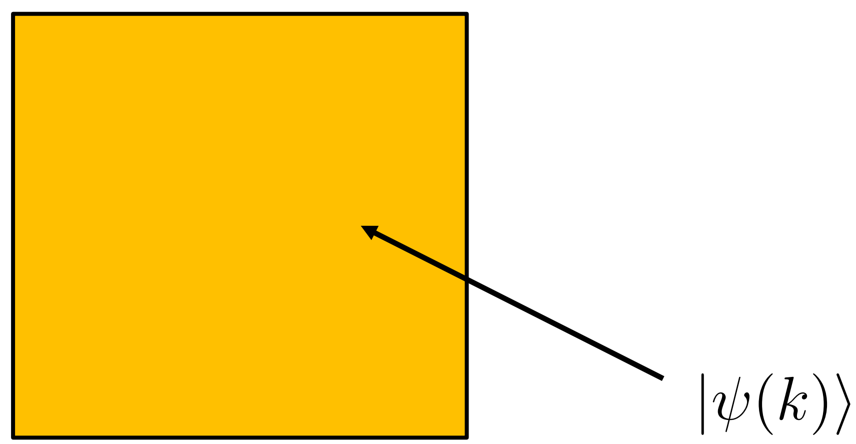
For an insulator, this Brillouin zone is completely filled

At each point ***k*** of the Brillouin zone there is a quantum state $|\psi(k)\rangle$

The idea of topological insulators is that the phase of the wavefunction can wind as we move around in the Brillouin zone.

Topological Insulators in 2d

The Brillouin zone



Define the *Berry connection*

$$\mathcal{A}_i = -i\langle\psi(k)|\frac{\partial}{\partial k^i}|\psi(k)\rangle$$

and the *Berry curvature*

$$\mathcal{F}_{xy} = \frac{\partial\mathcal{A}_x}{\partial k^y} - \frac{\partial\mathcal{A}_y}{\partial k^x}$$

We can then define the Chern number: $C = -\frac{1}{2\pi}\int_{BZ} d^2k\ \mathcal{F}_{xy}$. This is an integer

The famous TKNN formula is:

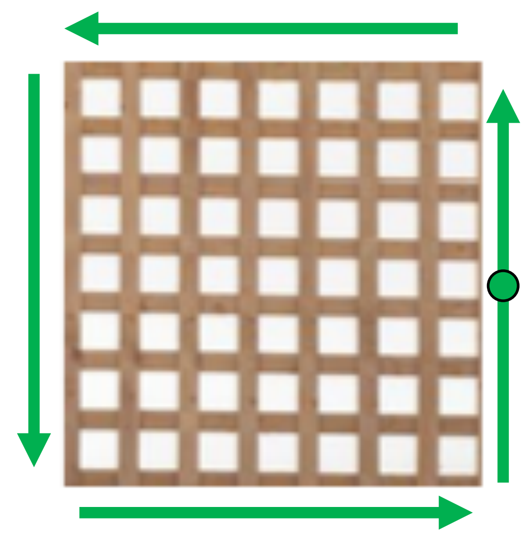
$$\boxed{\sigma_{xy} = \frac{e^2}{2\pi\hbar}C}$$

No magnetic fields in sight!

Thouless et al '82
Haldane, '88

Edge Modes

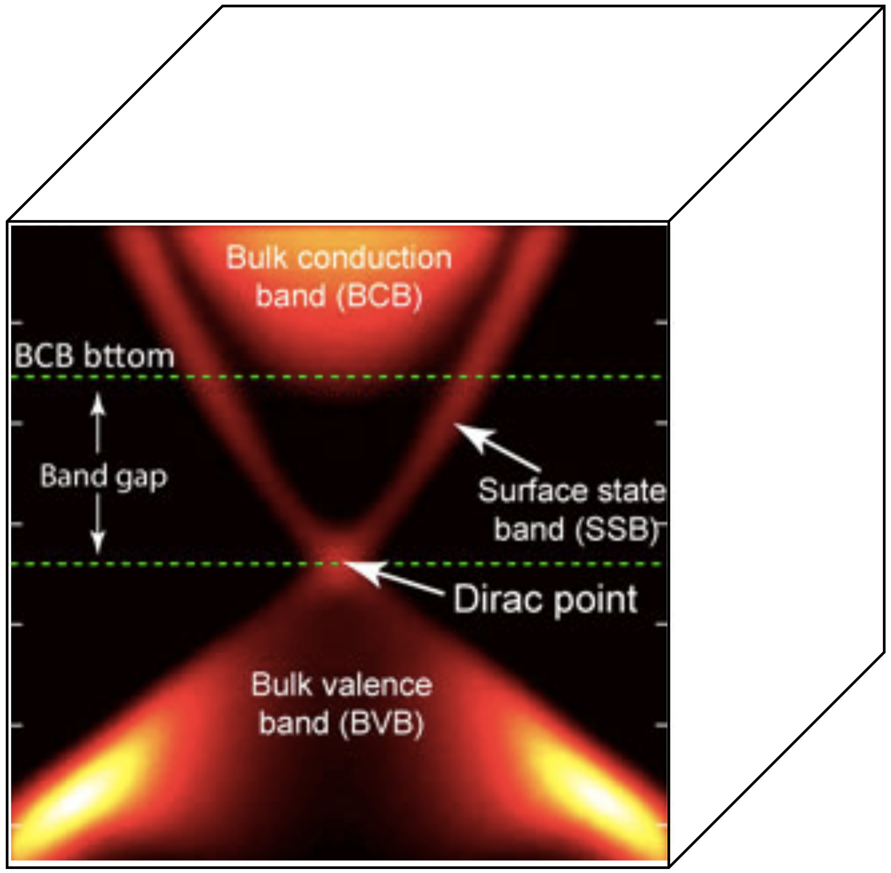
Topological insulators have interesting things happening on the edge



- For a 2d topological insulator, there are *chiral* edge modes.
- This move in just one direction

Topological Insulators in 3d

Predicted by Kane and Mele in 2005. Discovered soon after.



Again, interesting things happen on the edge. Now we a single relativistic Dirac fermions