Palestinian Advanced Physics School

Condensed Matter Physics

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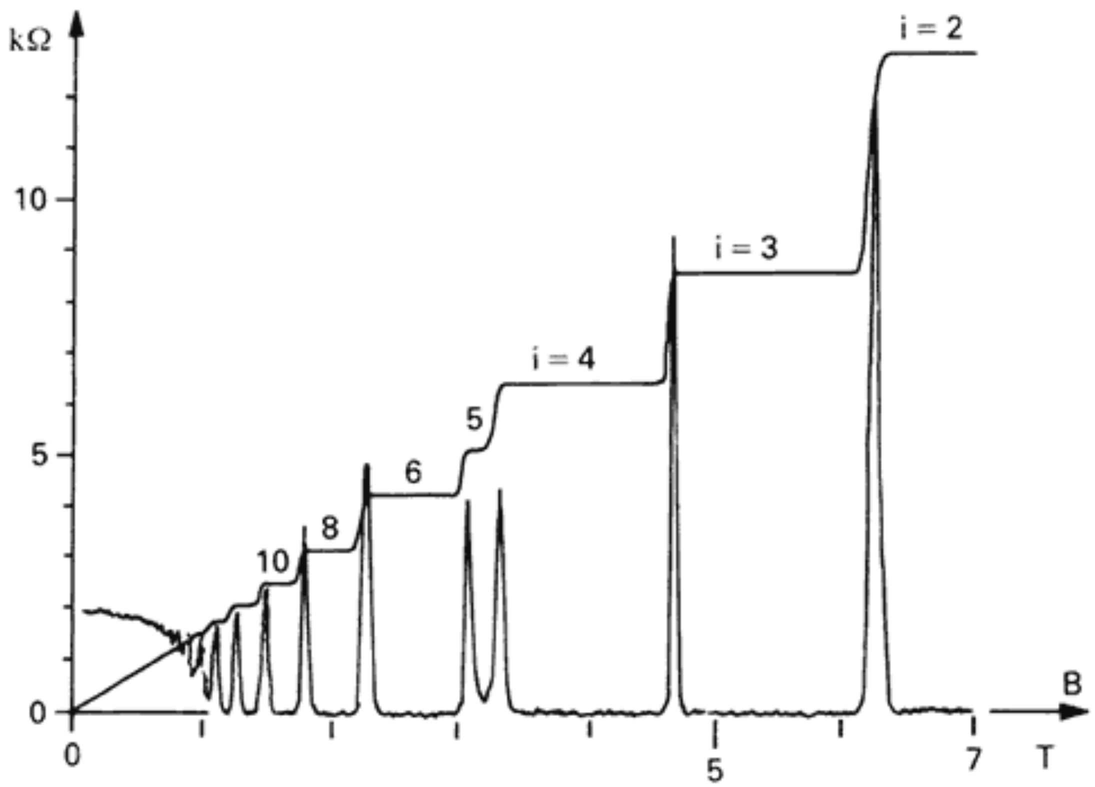
Lecture 4: The Fractional Quantum Hall Effect

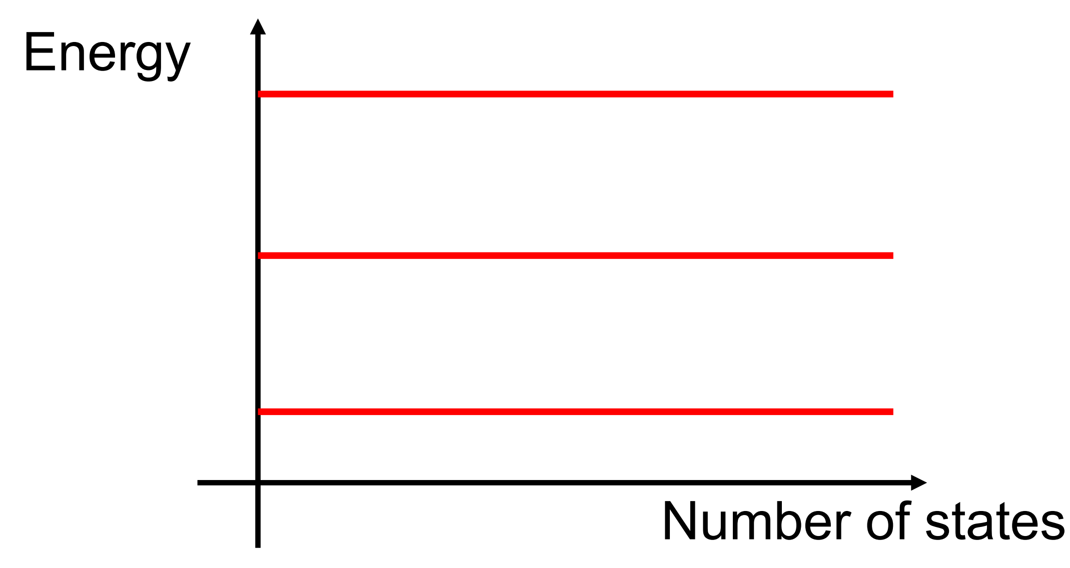
http://www.damtp.cam.ac.uk/user/tong/qhe.html

Recall From Lecture 3



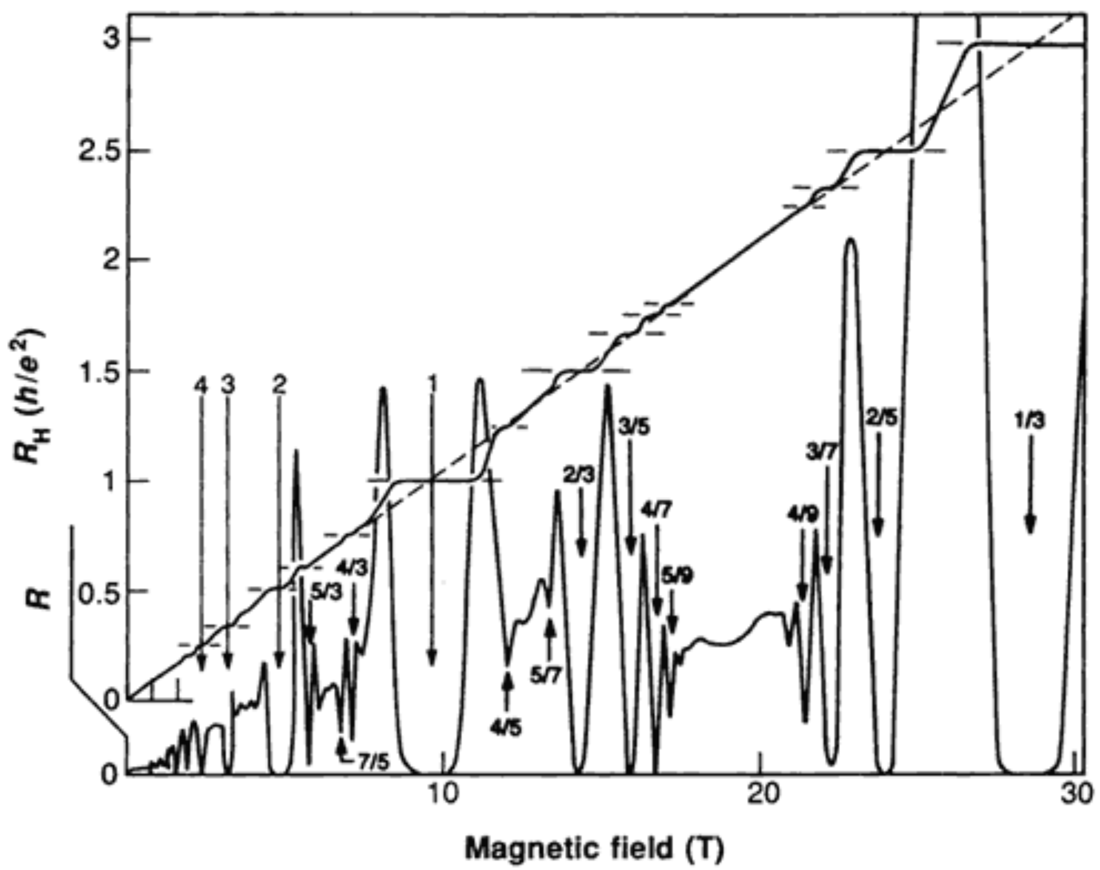
$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

$$\nu \in \mathbf{Z}$$



ν is counting the number of filled Landau levels

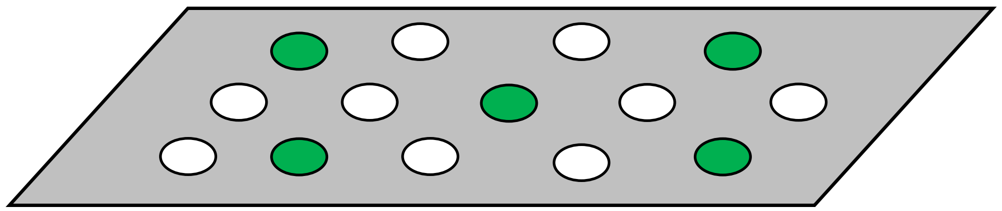
The Fractional Quantum Hall Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

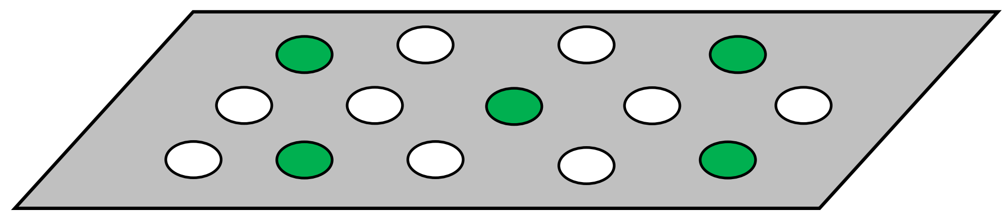
$$\nu \in \mathbf{Q}$$

Now ν is telling us that the lowest Landau level is only fractionally filled



Why? And what picks the filling fractions?

The Fractional Quantum Hall Effect



$$H = \frac{1}{2m}\sum_{i=1}^{N}(\mathbf{p}_i + e\mathbf{A})^2 + \sum_{i\neq j}\frac{e^2}{4\pi\epsilon_0|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{i=1}^{N}\omega^2\mathbf{r}_i^2$$

All electrons feel the same background field

Repulsive Coulomb force

Trap

What is the ground state? Solving problems like this with many interacting electrons is hard*

*hard = no one knows how to do it!

The Laughlin Wavefunction

Here is an ansatz for the ground state of this system

$$\psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

$$l_B = \sqrt{\frac{\hbar}{eB}}$$

$$z = x + iy$$

This is not the correct answer but, numerically, it is very close for a small number of electrons

Some properties of this wavefunction:

- The electrons are spread out homogeneously in a disc of radius

$$R \approx \sqrt{2mN}l_B$$

- This describes a “liquid” of electrons (strictly speaking, a new phase of matter)
- The filling fraction is $\nu = \dfrac{1}{m}$
- But electrons are fermions, so only *m* odd is allowed. This wavefunction describes the 1/3 and 1/5 observed Hall plateaux.

The Laughlin Wavefunction

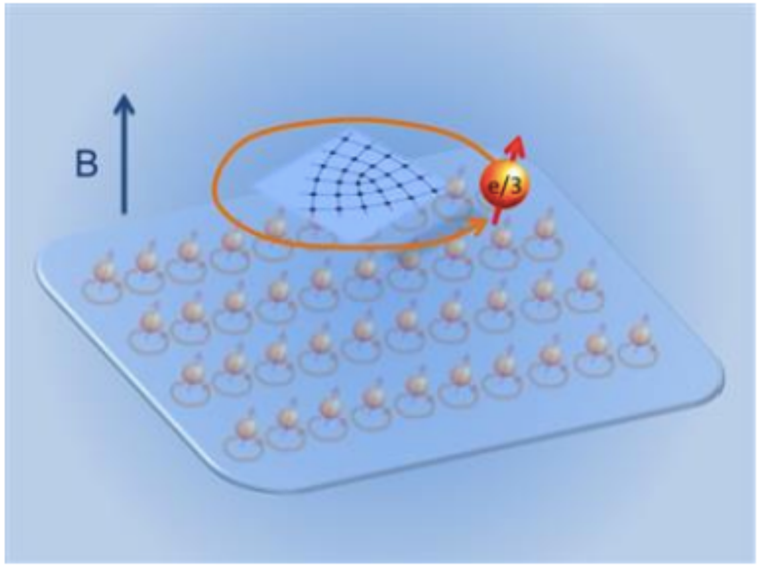
We can also write down excited states

$$\psi_{\rm hole}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)\prod_{k<l}(z_k - z_l)^m\, e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

position of some excitation, known as a *quasihole*

Some properties of this wavefunction:

- If we place *m* quasiholes in the same place, then it looks like an electron
- A quasihole carries fractional charge e/m
- The indivisible electron has miraculously split into, for example, 3 pieces!
- Detected experimentally



Anyons

The quasi-holes don't just have fractional charge. They have fractional statistics!

Recall why we get bosons and fermions.

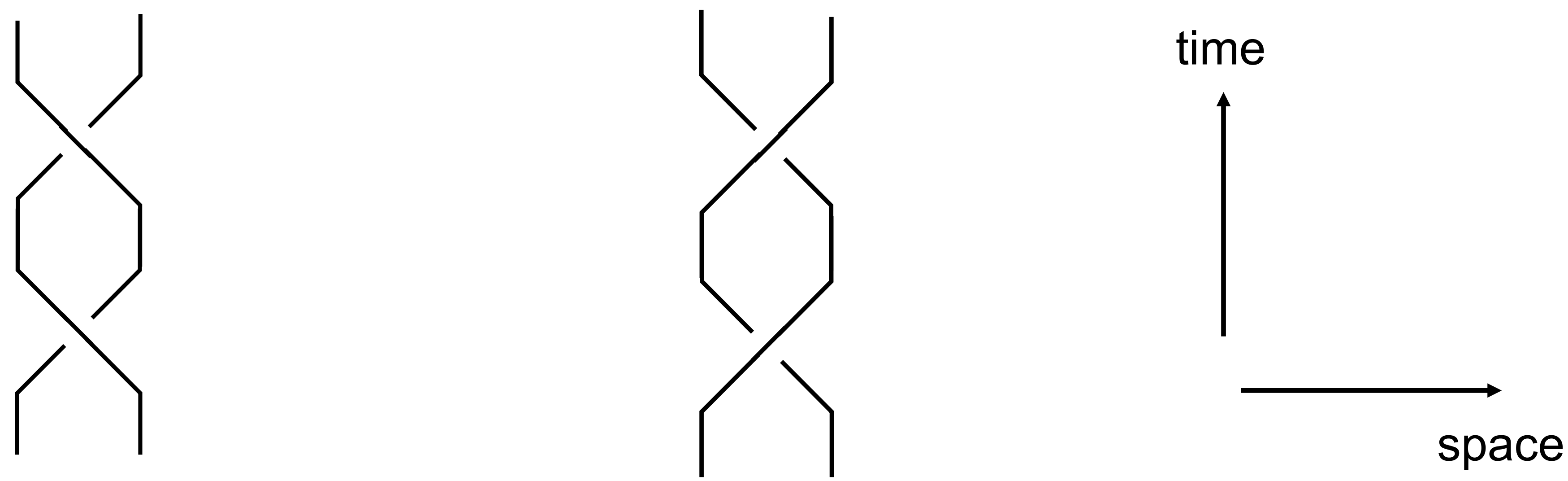
- Exchange two particles to get a phase $\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\pi\alpha}\psi(\mathbf{r}_2, \mathbf{r}_1)$
- Now exchange again to get back to where we started

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{2i\pi\alpha}\psi(\mathbf{r}_1, \mathbf{r}_2) \Longrightarrow e^{2\pi i\alpha} = 1$$

- There are two solutions to this:
  - $\alpha=0$ This is a *boson*
  - $\alpha=1$ This is a *fermion*
- But there's a loophole to this argument in 2 spatial dimensions!

Anyons

Consider the worldline of 2d particles moving in spacetime



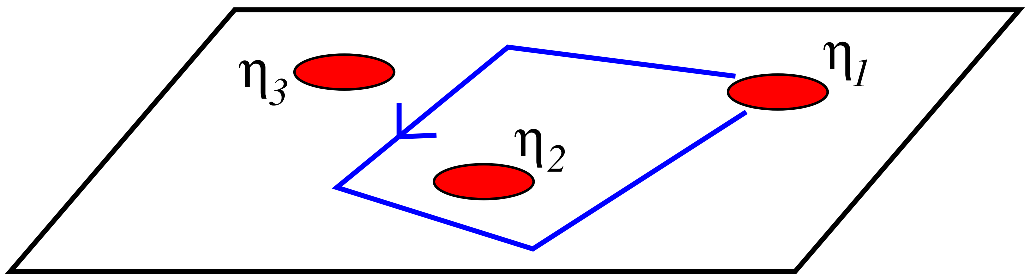
exchange anti-clockwise exchange clockwise

- In d=2+1, these two worldline configurations cannot be continuously deformed into each other.
- In d=2+1, it is therefore possible to have $\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\pi\alpha}\psi(\mathbf{r}_2, \mathbf{r}_1)$ with any α
- These are called *anyons.*

Quasiholes are Anyons

Claim: The quasi-hole in the quantum Hall effect is an anyon with

$$\alpha = \frac{1}{m}$$



Proof: Start with the wavefunction for N electrons and M quasi-holes

$$\psi_{\text{quasi-holes}}(z;\eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m \, e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

Move one quasi-hole around the other and compute the *Berry phase*

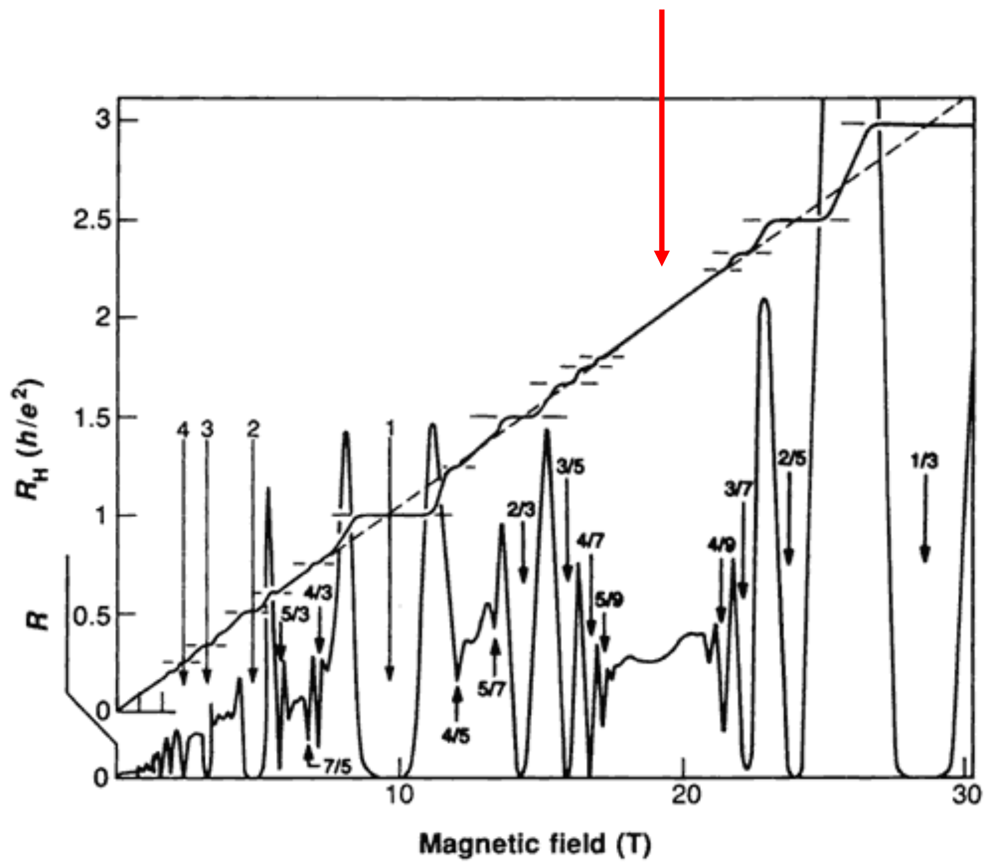
$$e^{i\alpha} = \exp\left(-i\oint_C \mathcal{A}_\eta d\eta + \mathcal{A}_{\bar\eta} d\bar\eta\right) \quad \text{with} \quad \mathcal{A}_\eta(\eta,\bar\eta) = -i\langle\psi|\frac{\partial}{\partial\eta}|\psi\rangle$$

More Advanced Topics: Half-Filling

http://www.damtp.cam.ac.uk/user/tong/qhe.html

The Fractional Quantum Hall Effect

At half-filling, it looks as if nothing strange is going on!



Closer examination shows that this is one of the most surprising regions of the diagram.

This is a metal!

Recent suggestion that there are relativistic Dirac fermions arising here!

More Advanced Topics: Non-Abelian Anyons

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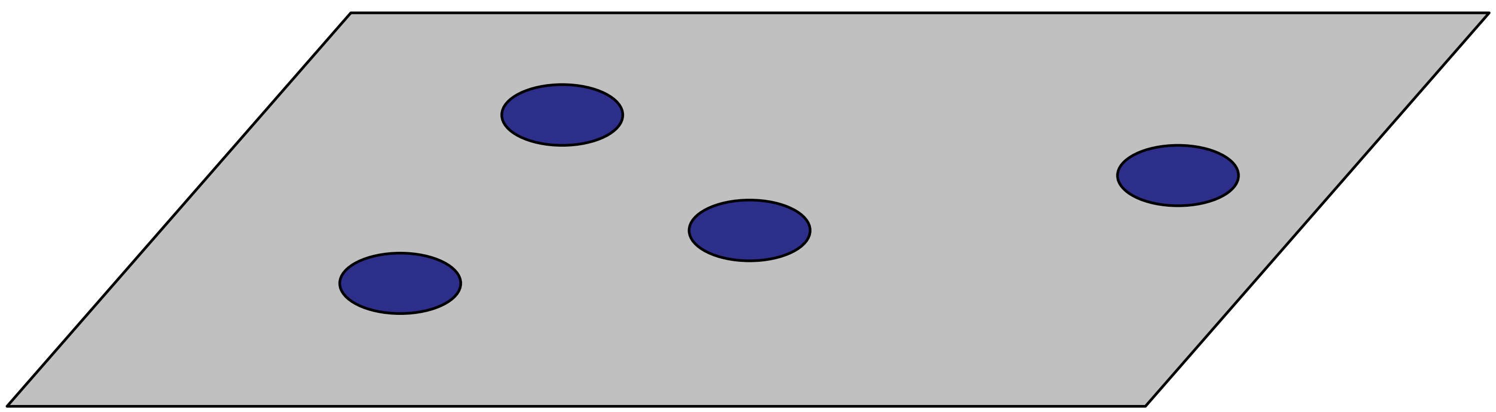
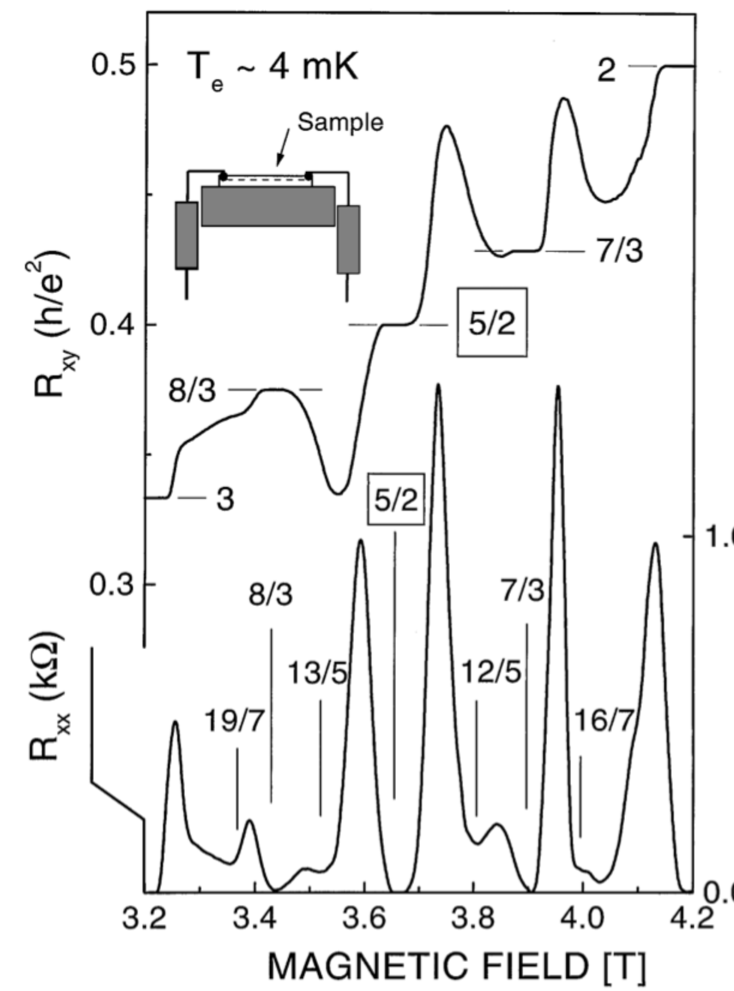
Non-Abelian Anyons

There is one more level in the quantum Hall story

$$\nu = \frac{5}{2}$$





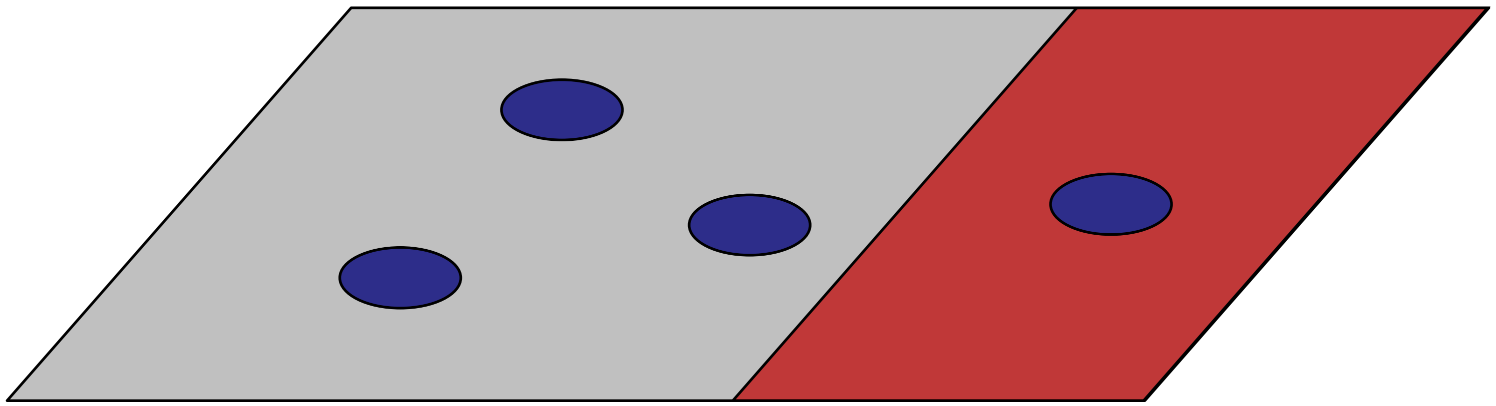
Look at the excitations above the ν=5/2 state

Suppose that we have *n* excitations. These excitations do not have a unique ground state. Instead, the number of ground states is:

$$2^{n/2}$$

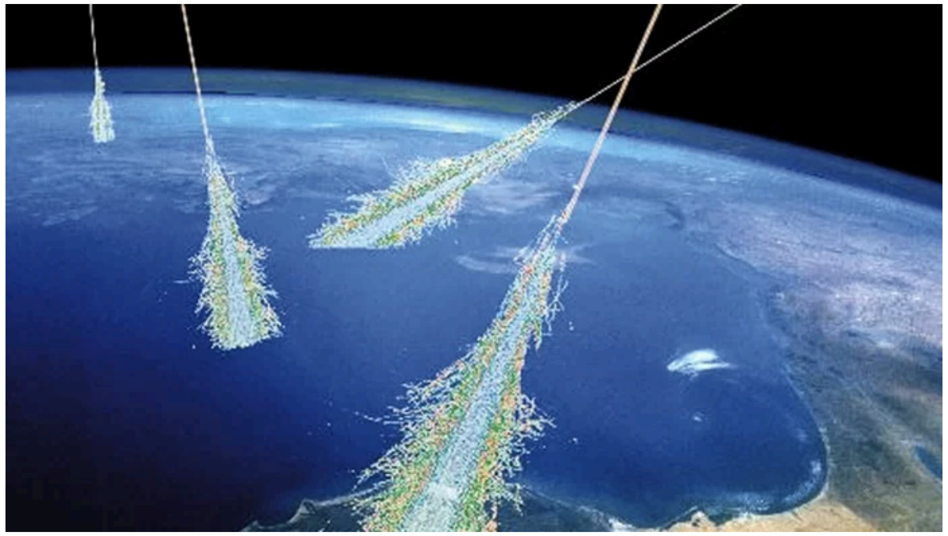
Non-Abelian Anyons

- $2^{n.2}$ is a strange number
- If each particle had 2 different states (e.g. spin up or down) we would get 2^n



The state is a global property of the system. If we only have access to a subset of the system, there's no way of telling which state we're in.

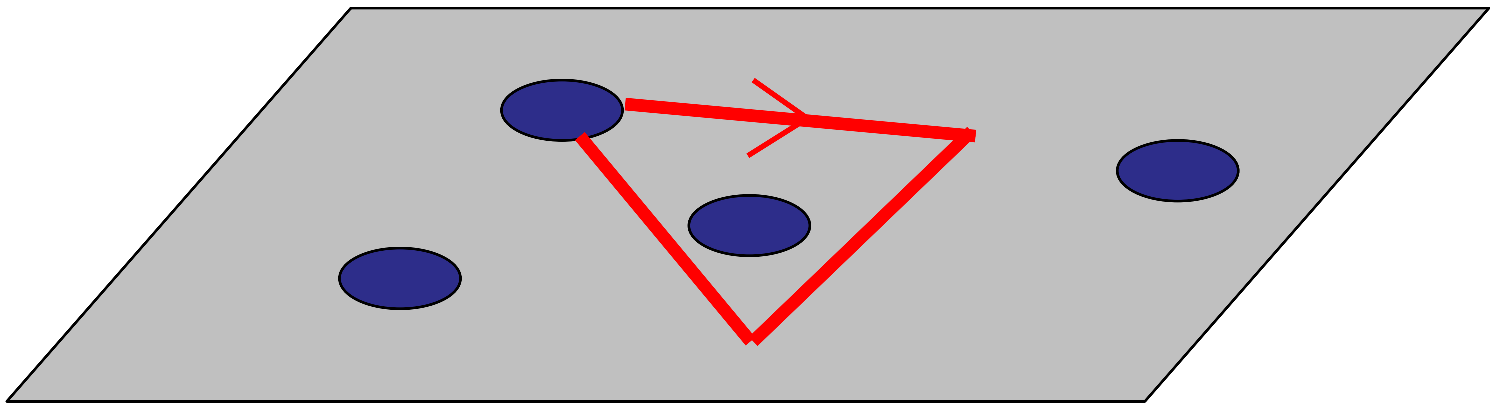
This makes these quantum states robust…



Topological Quantum Computing

We describe a state by a $2^{n/2}$ dimensional vector ψ

Now move one particle around on a path

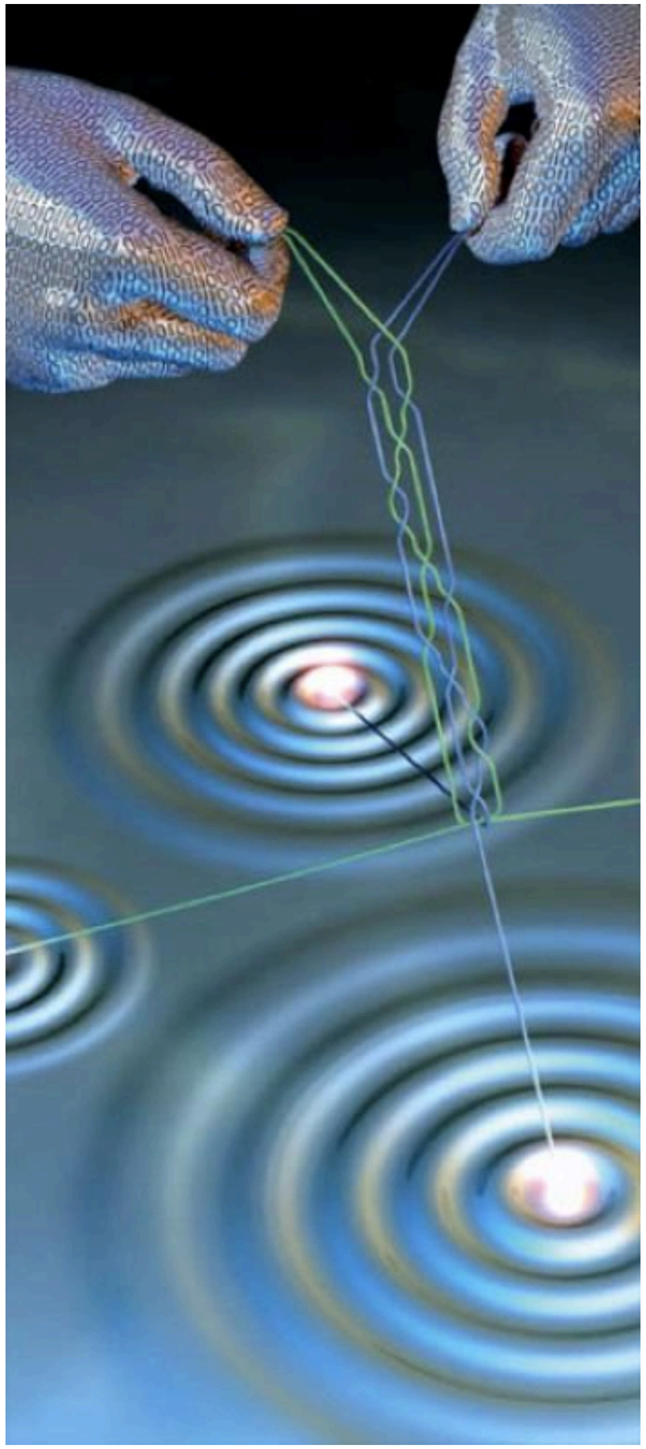


$$\psi \to U_{\text{path}}\psi$$

with U_{path} a unitary matrix that depends on the path taken.

These particles are called *non-Abelian anyons*.

This allows us to do quantum computation without error!



Summary

There is a *lot* of interesting physics hiding in this diagram