THREE NEUTRINO OSCILLATIONS IN MATTER

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- ...

the smaller of the lepton mixing angles is of similar magnitude to the larger of the quark mixing parameters, namely the Cabibbo angle [9]. One theoretical method often used to address this question involves the use of non-Abelian discrete subgroups of $SU(3)$ as flavor symmetries; the popularity of this method is due in part from the fact that these symmetries can give rise to the nearly *tri-bi-maximal*[2] structure of the PMNS matrix. Whether employing these flavor symmetries or other methods, any theoretical principle that attempts to describe the fundamental symmetries implied by the observed organization of quark and neutrino mixing — such as those proposed in unification models — leads to testable predictions such as sum rules between CKM and PMNS parameters [7, 8, 11, 12]. Data on the patterns of neutrino mixing are already proving crucial in the quest for a relationship between quarks and leptons and their seemingly arbitrary generation structure.

Clearly much work remains in order to complete the standard three-flavor mixing picture, particularly with regard to θ_{23} (is it less than, greater than, or equal to 45°?), mass hierarchy (normal or inverted?) and $\delta_{\rm CP}$. Additionally, there is great value in obtaining a set of measurements for multiple parameters *from a single experiment*, so that correlations and systematic uncertainties can be handled properly. Such an experiment would also be well positioned to extensively test the standard picture of three-flavor mixing. DUNE is designed to be this experiment.

3.2 Expected Event Rate and Sensitivity Calculations

The oscillation probability of $\nu_\mu \to \nu_e$ through matter in a constant density approximation is, to first order [13]:

$$\begin{aligned} P(\nu_\mu \to \nu_e) \;\simeq\; & \sin^2\theta_{23}\sin^2 2\theta_{13}\frac{\sin^2(\Delta_{31}-aL)}{(\Delta_{31}-aL)^2}\Delta_{31}^2 \\ & + \sin 2\theta_{23}\sin 2\theta_{13}\sin 2\theta_{12}\frac{\sin(\Delta_{31}-aL)}{(\Delta_{31}-aL)}\Delta_{31}\frac{\sin(aL)}{(aL)}\Delta_{21}\cos(\Delta_{31}+\delta_{\rm CP}) \\ & + \cos^2\theta_{23}\sin^2 2\theta_{12}\frac{\sin^2(aL)}{(aL)^2}\Delta_{21}^2, \end{aligned} \tag{3.5}$$

where $\Delta_{ij} = \Delta m^2_{ij}L/4E_\nu$, $a = G_F N_e/\sqrt{2}$, G_F is the Fermi constant, N_e is the number density of electrons in the Earth, L is the baseline in km, and E_ν is the neutrino energy in GeV. In the equation above, both $\delta_{\rm CP}$ and a switch signs in going from the $\nu_\mu \to \nu_e$ to the $\bar\nu_\mu \to \bar\nu_e$ channel; i.e., a neutrino-antineutrino asymmetry is introduced both by CP violation ($\delta_{\rm CP}$) and the matter effect (a). The origin of the matter effect asymmetry is simply the presence of electrons and absence

We use the approximate see-saw structure of the full Hamiltonian. This way one can diagonalize the 3x3 matrix by two successive diagonalizations of 2x2 matrices. The starting point is the Schroedinger equation

$$i\frac{d}{dx}\nu = \mathcal{H}\nu \tag{1}$$

$$\mathcal{H} = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{\odot}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_a^2}{2E} \end{pmatrix} U^\dagger + \begin{pmatrix} V(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

$$S_{\alpha\beta} = T \ e^{-i\int_{x_0}^{x_f} \mathcal{H}(x)dx} \tag{3}$$

$$S_{\alpha\beta} = e^{-iU_m\mathcal{H}_m U_m^\dagger (x_f - x_0)} = U_m e^{-i\mathcal{H}_m L} U_m^\dagger \tag{4}$$

and the U_m is the neutrino mixing matrix in matter. Defining $\phi_{21} = (\mathcal{H}_2 - \mathcal{H}_1)L$ and $\phi_{31} = (\mathcal{H}_3 - \mathcal{H}_1)L$, we can write

$$S_{\alpha\beta} = \left[U_m \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\phi_{21}} & 0 \\ 0 & 0 & e^{-i\phi_{31}} \end{pmatrix} U_m^\dagger \right] \tag{5}$$

$$\mathcal{H} = U_m \mathcal{H}_m U_m^\dagger \tag{6}$$

It is convenient to do it in two steps, first calculating the hamiltonian in a certain auxiliary basis. This way, to an excellent approximation, we can diagonalize the 3x3 matrix by two successive diagonalizations of the 2x2 matrices.
The auxiliary basis is defined by the following equation

$$\mathcal{H}' = U^{aux\dagger} \mathcal{H} U^{aux} \quad and \quad S = U^{aux} e^{(-i\mathcal{H}'L)} U^{aux\dagger} \tag{7}$$

where

$$U^{aux} = \mathcal{O}_{23} U^\delta \mathcal{O}_{13} \tag{8}$$

and the rotations $\mathcal{O}_{ij}$ are defined by the decomposition of the mixing matrix U in the vacuum (see eq. 2) as follows:

$$\begin{aligned} U &= \mathcal{O}_{23} U^\delta \mathcal{O}_{13} U^{\delta *} \mathcal{O}_{12} \\ &= \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \end{aligned} \tag{9}$$

($c_{12} \equiv \cos\theta_{12}$, $s_{12} \equiv \sin\theta_{12}$ etc).

The matrices $\mathcal{O}_{ij}$ are orthogonal matrices. It is more convenient to rewrite the matrix U in another form

$$U \to U \cdot U^{\delta} = \mathcal{O}_{23} U^{\delta} \mathcal{O}_{13} \mathcal{O}_{12}$$
$$= \begin{pmatrix} c_{13} c_{12} & c_{13} s_{12} & s_{13} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & c_{13} s_{23} e^{i\delta} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{13} c_{23} e^{i\delta} \end{pmatrix} \quad (10)$$

Using eqs. (2,7) we obtain

$$\begin{aligned} \mathcal{H}' &= \mathcal{O}_{13}^T U^{\delta *} \mathcal{O}_{23}^T \; \mathcal{H} \; \mathcal{O}_{23} U^{\delta} \mathcal{O}_{13} \\ &= \begin{pmatrix} V c_{13}^2 & s_{12} c_{12} \frac{\Delta m_{\odot}^2}{2E} & s_{13} c_{13} \; V \\ s_{12} c_{12} \frac{\Delta m_{\odot}^2}{2E} & (c_{12}^2 - s_{12}^2) \frac{\Delta m_{\odot}^2}{2E} & 0 \\ s_{13} c_{13} \; V & 0 & \frac{\Delta m_{ee}^2}{2E} + V s_{13}^2 \end{pmatrix}, \quad (11) \\ \Delta m_{ee}^2 &= c_{12}^2 \Delta m_a^2 + s_{12}^2 (\Delta m_a^2 - \Delta m_{\odot}^2) \quad (12) \end{aligned}$$

This matrix has a see-saw structure, with the $(13),(31)$ elements much smaller than the (33) element and can be put in an almost diagonal form by two rotations

$$\mathcal{O}_{12}^{m\ T}\mathcal{O}_{13}'^{T}\mathcal{H}'\ \mathcal{O}_{13}'\mathcal{O}_{12}^{m} = \begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix} \quad (13)$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_{31}^2}{2E} \end{pmatrix} + \mathcal{H}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14)$$

After the first rotation we have for $\mathcal{O}_{13}'^{T}\mathcal{H}'\mathcal{O}_{13}' =$

$$\begin{pmatrix} \sin^2\theta_{13}'\frac{\Delta m_{ee}^2}{2E} + \cos^2(\theta_{13}+\theta_{13}')V & \cos\theta_{13}'\ s_{12}c_{12}\frac{\Delta m_{\odot}^2}{2E} & 0 \\ \cos\theta_{13}'\ s_{12}c_{12}\frac{\Delta m_{\odot}^2}{2E} & (c_{12}^2 - s_{12}^2)\frac{\Delta m_{\odot}^2}{2E} & \sin\theta_{13}'\ s_{12}c_{12}\frac{\Delta m_{\odot}^2}{2E} \\ 0 & \sin\theta_{13}'\ s_{12}c_{12}\frac{\Delta m_{\odot}^2}{2E} & \cos^2\theta_{13}'\frac{\Delta m_{ee}^2}{2E} + V\sin^2(\theta_{13}+\theta_{13}' \end{pmatrix} \quad (15)$$

$$\sin 2\theta_{13}' = \frac{\epsilon_a \sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \quad \epsilon_a = \frac{2EV}{\Delta m_{ee}^2} \quad (16)$$

We can safely neglect the $(23), (32)$ elements which are generated after the first rotation (see Appendix A).
Now we can diagonalize the latter with the second rotation

$$\sin 2\theta_{12}^{m} = \frac{\cos\theta'_{13}\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12}-\epsilon_{\odot})^2+\cos^2\theta'_{13}\sin^2 2\theta_{12}}}. \tag{17}$$

The eigenvalues of $\mathcal{H}$ are

$$\mathcal{H}_2-\mathcal{H}_1 \equiv \frac{\Delta m_{21}^2}{2E} = \frac{\Delta m_{\odot}^2}{2E}\sqrt{(\cos 2\theta_{12}-\epsilon_{\odot})^2+\cos^2\theta'_{13}\sin^2 2\theta_{12}}, \tag{18}$$

$$\begin{aligned}\mathcal{H}_3-\mathcal{H}_1 \equiv \frac{\Delta m_{31}^2}{2E} &= \frac{\Delta m_{ee}^2}{2E}\sqrt{(\cos 2\theta_{13}-\epsilon_a)^2+\sin^2 2\theta_{13}}\\ &-\frac{1}{4}\frac{\Delta m_{ee}^2}{2E}\sqrt{(\cos 2\theta_{13}-\epsilon_a)^2+\sin^2 2\theta_{13}}\\ &+\frac{1}{4}[\frac{\Delta m_{ee}^2}{2E}+V]+\frac{1}{4E}(\Delta m_{21}^2-\Delta m_{\odot}^2\cos 2\theta_{12})\end{aligned} \tag{19}$$

Finally, for the mixing matrix U_m in matter we obtain

$$U_m = U^{aux}\mathcal{O}'_{13}\mathcal{O}^m_{12} = \mathcal{O}_{23}U^\delta\mathcal{O}_{13}\mathcal{O}'_{13}\mathcal{O}^m_{12} = \mathcal{O}_{23}U^\delta\mathcal{O}^m_{13}\mathcal{O}^m_{12}, \qquad (20)$$

so that

$$U_m = U_{23}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}\begin{pmatrix} \cos\theta^m_{13} & 0 & \sin\theta^m_{13} \\ 0 & 1 & 0 \\ -\sin\theta^m_{13} & 0 & \cos\theta^m_{13} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix}\begin{pmatrix} \cos\theta^m_{12} & \sin\theta^m_{12} & 0 \\ -\sin\theta^m_{12} & \cos\theta^m_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\theta^m_{13} = \theta_{13} + \theta'_{13} \qquad (22)$$

$$\sin 2\theta^m_{13} = \frac{\sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \qquad (23)$$

and

$$\sin 2\theta^m_{12} = \frac{\cos\theta'_{13}\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2\theta'_{13}\sin^2 2\theta_{12}}}, \qquad (24)$$

$$\cos 2\theta_{13}^m = \frac{\cos 2\theta_{13} - \epsilon_a}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}} \quad (25)$$

$$\cos 2\theta_{12}^m = \frac{\cos 2\theta_{12} - \epsilon_\odot}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta'_{13} \sin^2 2\theta_{12}}} \quad (26)$$

$$\frac{\Delta m_{21}^2}{2E} = \frac{\Delta m_\odot^2}{2E}\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta'_{13} \sin^2 2\theta_{12}}, \quad (27)$$

$$\begin{aligned}\frac{\Delta m_{31}^2}{2E} &= \frac{\Delta m_{ee}^2}{2E}\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} \\ &-\frac{1}{4}\frac{\Delta m_{ee}^2}{2E}\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} \\ &+\frac{1}{4}[\frac{\Delta m_{ee}^2}{2E} + V] + \frac{1}{4E}(\Delta m_{21}^2 - \Delta m_\odot^2 \cos 2\theta_{12})\end{aligned} \quad (28)$$

$$\theta_{13}^m = \theta_{13} + \theta'_{13} \quad (29)$$

$$\epsilon_\odot = \frac{2EV}{\Delta m_\odot^2}(\cos^2 \theta_{13}^m + \frac{\sin^2 \theta'_{13}}{\epsilon_a})\ , \qquad \epsilon_a = \frac{2EV}{\Delta m_{ee}^2} \quad (30)$$

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_a^2 + s_{12}^2(\Delta m_a^2 - \Delta m_\odot^2) = \Delta m_a^2 - s_{12}^2 \Delta m_\odot^2 \quad (31)$$

In summary

The ocillation probabilities $P_{\nu_\alpha \to \nu_\beta}$ $(\alpha, \beta = e, \mu, \tau)$ have the same forms as for the vacuum oscillations with following replacements

$\Delta m^2_\odot \to \Delta m^2_{21}$
$\Delta m^2_a \to \Delta m^2_{31}$
$\theta_{12} \to \theta^m_{12}$
$\theta_{13} \to \theta^m_{13}$
$\theta^m_{23} \equiv \theta_{23}$
$\delta^m \equiv \delta$

This semi-analytic approximate solution is valid for all energies. For anti-neutrino oscillations $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$, V$\to$ -V and $\delta \to -\delta$. For normal mass hierarchy Δm^2_a is positive and for inverted mass hierarchy it is negative.

The results of this work have been presented as private communication to the members of the T2HKK collaboration in December 2017; similar results have been recently obtained in B. Denton, H. Minakata and S. J. Parke, arXiv:1801.06514 [hep-ph].

THANK YOU

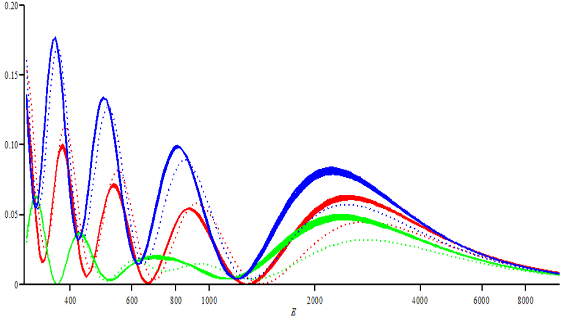

Figure: Normal mass hierarchy, $\delta_{cp} = 0$ (red), $\delta_{cp} = \frac{\pi}{2}$ (green), $\delta_{cp} = -\frac{\pi}{2}$ (blue). Thickness of the plots are from varying constant/uniform matter density 2.5 - 3 g/cm^3. Dotted plots are for vacuum oscillations

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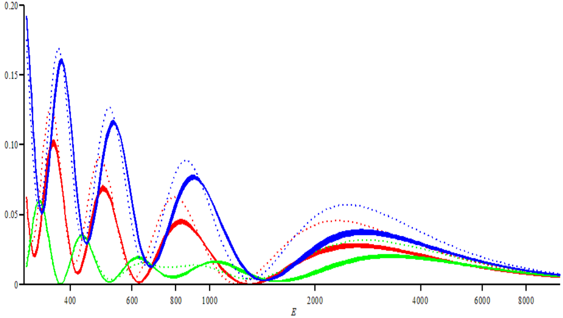

Figure: Inverted mass hierarchy, $\delta_{cp} = 0$ (red), $\delta_{cp} = \frac{\pi}{2}$ (green), $\delta_{cp} = -\frac{\pi}{2}$ (blue). Thickness of the plots are from varying constant/uniform matter density 2.5 - 3 g/cm^3. Dotted plots are for vacuum oscillations

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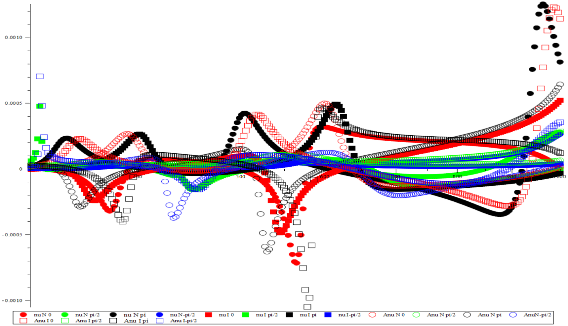

Figure: relative error for T2HKK, $0.4 \leq E \leq 1$ GeV

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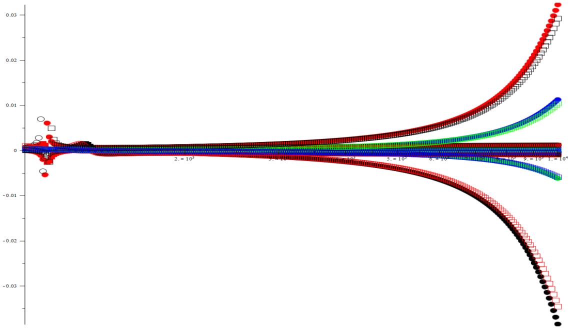

Figure: Relative error. 1≤ E≤10 GeV

The (23), (32) elements at energies below the second resonance $(23) \approx s_{12} c_{12} s_{13} \frac{\Delta m^2_\odot}{\Delta m^2_a} V \simeq 1.1 \cdot 10^{-13} eV$ and for DUNE(T2K) distance, L=1285 (295)km , $(23) \cdot L \simeq 1.2 \cdot 10^{-3} (3 \cdot 10^{-4})$. $VL \simeq 0.7$ for DUNE and it is 0.17 for T2K.

$$
\begin{aligned}
S &= S_0 + S_1 + \ldots \\
S_0 &= U_m \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\phi_{21}} & 0 \\ 0 & 0 & e^{-i\phi_{31}} \end{pmatrix} U_m^\dagger \\
S_1 &= \sin\theta'_{13} \frac{\sin 2\theta_{12}}{2} \frac{\Delta m^2_\odot L}{2E} U_m \begin{pmatrix} 0 & 0 & -\sin\theta^m_{12} \frac{e^{-i\phi_{31}} - 1}{\phi_{31}} \\ 0 & 0 & \cos\theta^m_{12} \frac{e^{-i\phi_{31}} - e^{-i\phi_{21}}}{\phi_{31} - \phi_{21}} \\ -\sin\theta^m_{12} \frac{e^{-i\phi_{31}} - 1}{\phi_{31}} & \cos\theta^m_{12} \frac{e^{-i\phi_{31}} - e^{-i\phi_{21}}}{\phi_{31} - \phi_{21}} & 0 \end{pmatrix}
\end{aligned}
$$

For normal hierarchy and for neutrinos $\sin\theta'_{13} \frac{\sin 2\theta_{12}}{2} \frac{\Delta m^2_\odot L}{2E \; \phi_{31}} \sin\theta^m_{12}$ reached its maximal value 0.4 % at second resonance. at E=6.5 GeV it is about 0.1%. Other term proportional to $\sin\theta^m_{12}$ is much smaller for all energies.

$$P_{\nu_\alpha \to \nu_\beta} = \sum_{i,j} U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j} e^{-i \frac{\Delta m^2_{ij}}{2E} L} \quad (35)$$

$$= \sum_{i,j} \Re[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] \cos \frac{\Delta m^2_{ij}}{2E} L + 2 \sum_{i>j} \Im[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] \sin \frac{\Delta m^2_{ij}}{2E} L \quad (36)$$

$$\phi_{ij} = \frac{\Delta m^2_{ij}}{2E} L \quad i, j = 1, 2, 3 \quad (37)$$

$$\sum_{i,j} \Re[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] \cos \phi_{ij} = \delta_{\alpha\beta} - 4 \sum_{i>j} \Re[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] \sin^2 \frac{\phi_{ij}}{2} \quad (38)$$

Jarskog invariant, $\mathcal{J}$,

$$\Im[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] = \pm \mathcal{J} \quad (39)$$

$$\mathcal{J} = c^2_{13} s_{13} c_{12} s_{12} c_{23} s_{23} \sin \delta \equiv \frac{1}{8} c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin \delta \quad (40)$$

$$\sin \phi_{32} - \sin \phi_{31} + \sin \phi_{12} = 4 \sin \frac{\phi_{21}}{2} \sin \frac{\phi_{31}}{2} \sin \frac{\phi_{32}}{2} \quad (\phi_{32} = \phi_{31} - \phi_{21}) \quad (41)$$

$$2 \sum_{i>j} \Im[U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j}] \sin \frac{\Delta m^2_{ij}}{2E} L = -8 \mathcal{J} \sin \frac{\phi_{21}}{2} \sin \frac{\phi_{31}}{2} \sin \frac{\phi_{32}}{2} \quad (\alpha = \mu, \beta = e) \quad (42)$$

$$P_{\nu_e\to\nu_e} = 1 - c_{13}^2 \sin^2 2\theta_{12} \sin^2 \frac{\phi_{21}}{2} - \sin^2 2\theta_{13}(c_{12}^2 \sin^2 \frac{\phi_{31}}{2} + s_{12}^2 \sin^2 \frac{\phi_{32}}{2}) \quad (43)$$

$$P_{\nu_\mu\to\nu_\mu} = 1 - 4|U_{\mu 1}|^2|U_{\mu 2}|^2 \sin^2 \frac{\phi_{21}}{2} - 4|U_{\mu 1}|^2|U_{\mu 3}|^2 \sin^2 \frac{\phi_{31}}{2} - 4|U_{\mu 2}|^2|U_{\mu 3}|^2 \sin^2 \frac{\phi_{32}}{2} \quad (44)$$

$$P_{\nu_\mu\to\nu_e} = -8\mathcal{J} \sin \frac{\phi_{21}}{2} \sin \frac{\phi_{31}}{2} \sin \frac{\phi_{32}}{2}$$
$$-4\Re[U_{e2}U_{\mu 2}^* U_{e1}^* U_{\mu 1}] \sin^2 \frac{\phi_{21}}{2} - 4\Re[U_{e3}U_{\mu 3}^* U_{e1}^* U_{\mu 1}] \sin^2 \frac{\phi_{31}}{2} - 4\Re[U_{e3}U_{\mu 3}^* U_{e2}^* U_{\mu 2}] \sin^2 \frac{\phi_{32}}{2} \quad (45)$$

$$|U_{\mu 1}|^2 = s_{12}^2 c_{23}^2 + c_{12}^2 s_{23}^2 s_{13}^2 + 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos\delta \quad (46)$$
$$|U_{\mu 2}|^2 = c_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 - 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos\delta \quad (47)$$
$$|U_{\mu 3}|^2 = c_{13}^2 s_{23}^2 \quad (48)$$

$$-4\Re[U_{e2}U_{\mu 2}^* U_{e1}^* U_{\mu 1}] = -c_{13}^2 \sin^2 2\theta_{12}(c_{23}^2 - s_{23}^2 s_{13}^2) - c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\delta \quad (49)$$
$$-4\Re[U_{e3}U_{\mu 3}^* U_{e1}^* U_{\mu 1}] = \sin^2 2\theta_{13} s_{23}^2 c_{12}^2 + \frac{1}{2} c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\delta \quad (50)$$
$$-4\Re[U_{e3}U_{\mu 3}^* U_{e2}^* U_{\mu 2}] = \sin^2 2\theta_{13} s_{23}^2 s_{12}^2 - \frac{1}{2} c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\delta \quad (51)$$