Stephen Parke, Fermilab 1/31/2018

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Compact Perturbative Expressions For Neutrino Oscillations in Matter

Peter B. Denton[a,b] Hisakazu Minakata[c,d] Stephen J. Parke[a]

Addendum to "Compact Perturbative Expressions for Neutrino Oscillations in Matter"

1801.06514

Peter B. Denton,[a] Hisakazu Minakata,[b] Stephen J. Parke[c]

Exact Analytic Solution Issue:

- Solve Cubic Characteristic Eqn.

$$
\begin{aligned}
\lambda^3 - \left(a + \Delta m^2_{21} + \Delta m^2_{31}\right)\lambda^2 \\
+\left[\Delta m^2_{21}\Delta m^2_{31} + a\left\{(c^2_{12} + s^2_{12}s^2_{13})\Delta m^2_{21} + c^2_{13}\Delta m^2_{31}\right\}\right]\lambda \\
- c^2_{12}c^2_{13}a\Delta m^2_{21}\Delta m^2_{31} \qquad &= 0
\end{aligned}
$$

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IF

- $a = 0$
- or $\Delta m^2_{21} = 0$
- or $\sin\theta_{12} = 0$
- or $\sin\theta_{13} = 0$

THEN characteristic Eqn

FACTORIZES !

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FACTORIZES !

BUT

See Zaglauer & Schwarzer, Z. Phys. C 1988

$$\lambda_1 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}[u+\sqrt{3(1-u^2)}],$$

$$\lambda_2 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}[u-\sqrt{3(1-u^2)}],$$

$$\lambda_3 = \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2-3t},$$

$$s = \Delta_{21} + \Delta_{31} + a,$$

$$t = \Delta_{21}\Delta_{31} + a[\Delta_{21}(1-s^2_{12}c^2_{13}) + \Delta_{31}(1-s^2_{13})],$$

$$u = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c^2_{12}c^2_{13}}{2(s^2-3t)^{3/2}}\right)\right],$$

here $\Delta_{ij} \equiv \Delta m^2_{ij}$

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DOES NOT

TRIVIALLY SIMPLIFY !

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DOES NOT TRIVIALLY SIMPLIFY !

2 flavor mixing in matter

$$ax^2 + bx + c = 0$$

simple, intuitive, useful

2 flavor mixing in matter

$$ax^2 + bx + c = 0$$

simple, intuitive, useful

3 flavor mixing in matter

$$ax^3 + bx^2 + cx + d = 0$$

complicated, counter intuitive, ...

Neutrino Evolution in Matter:

$$i\frac{d}{dx}\nu = H\nu \quad \text{with} \quad \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$(2E)\, H = U_{PMNS} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} U^\dagger_{PMNS} + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a = 2\sqrt{2} G_F N_e E$$

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$$U_{PMNS} \equiv U_{23}(\theta_{23}, 0)\, U_{13}(\theta_{13}, -\delta)\, U_{12}(\theta_{12}, 0) :=: U_{23}(\theta_{23}, \delta)\, U_{13}(\theta_{13}, 0)\, U_{12}(\theta_{12}, 0)$$

$:=:$ means equal after multiplying by a diagonal phase matrix on the left and/or right hand side.

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$$i\frac{d}{dx}\,\nu' = \boxed{U^\dagger_{23}(\theta_{23},\delta)\,H\,U_{23}(\theta_{23},\delta)}\,\nu' \qquad \text{with } \nu' = U^\dagger_{23}(\theta_{23},\delta)\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Neutrino Evolution in Matter (conti):

$$U_{23}^{\dagger}(\theta_{23},\delta)\, H\, U_{23}(\theta_{23},\delta) = H_D + H_{OD}$$

D=diagonal   OD= off-diagonal

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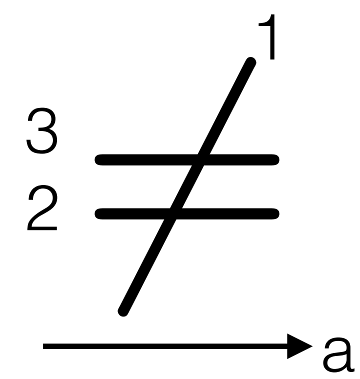

$$\Delta m_{ee}^2 \equiv \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2$$

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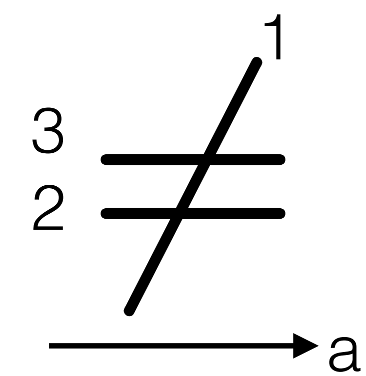

$$\Delta m_{ee}^2 \equiv \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2$$

$$(2E)\,H_{OD}/\Delta m_{ee}^2 = s_{13}c_{13}\begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix}$$

$$+\, c_{13}\; s_{12}c_{12}\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)\begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

$$-\, s_{13}\; s_{12}c_{12}\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)\begin{bmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{bmatrix}$$

Neutrino Evolution in Matter (conti):

$$U_{23}^{\dagger}(\theta_{23},\delta)\, H\, U_{23}(\theta_{23},\delta) = H_D + H_{OD}$$

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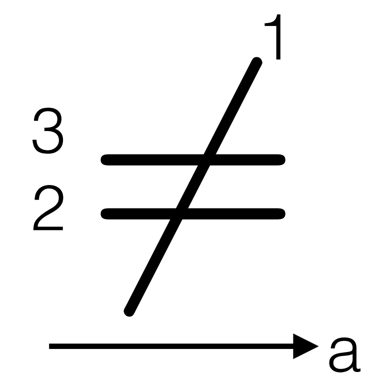

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0.15

0.015

0.002

Rotation by $U_{13}(\widetilde{\theta}_{13})$

$$a = 2\sqrt{2}G_F N_e E$$

$$\cos 2\widetilde{\theta}_{13} = \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}$$

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then $U_{12}(\widetilde{\theta}_{12})$

$$a' \equiv a \, \cos^2 \widetilde{\theta}_{13} + \Delta m^2_{ee} \sin^2(\widetilde{\theta}_{13} - \theta_{13})$$

$$\cos 2\widetilde{\theta}_{12} = \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12} \cos^2(\widetilde{\theta}_{13} - \theta_{13})}}$$

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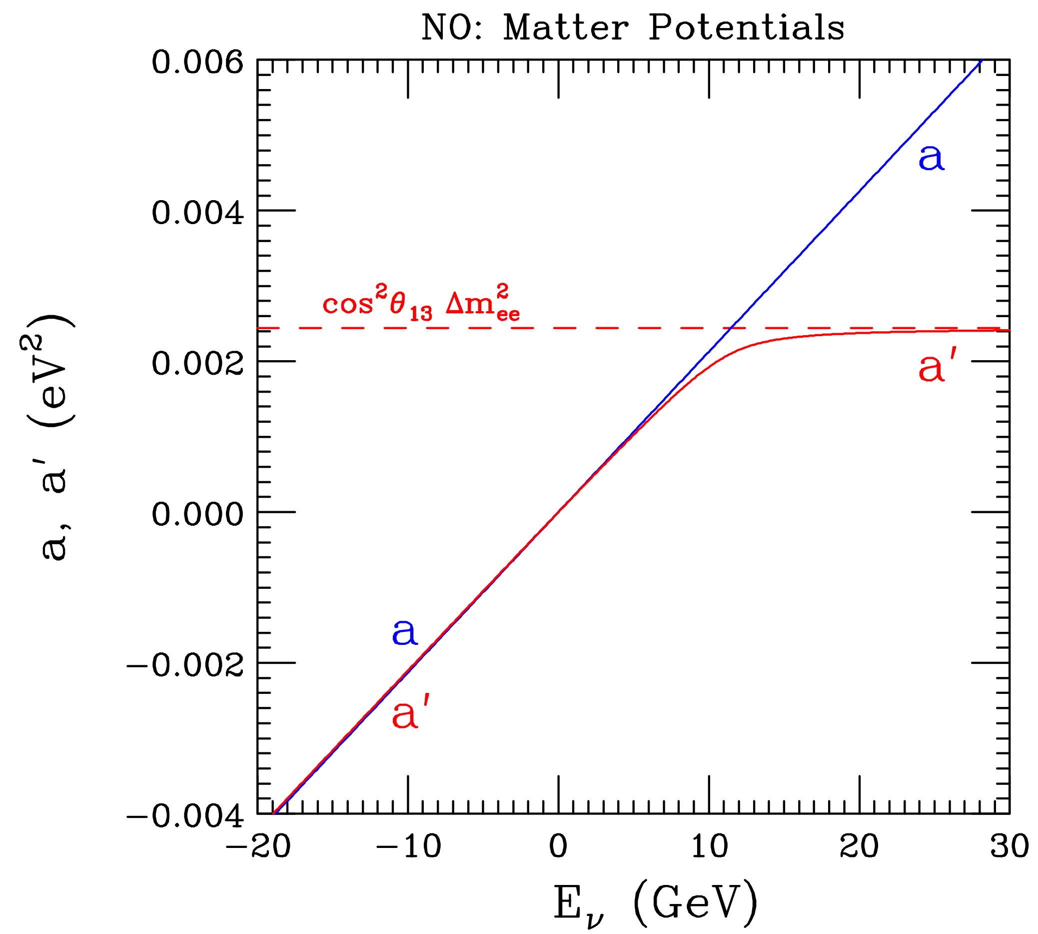

Rotation by $U_{13}(\widetilde{\theta}_{13})$

then $U_{12}(\widetilde{\theta}_{12})$

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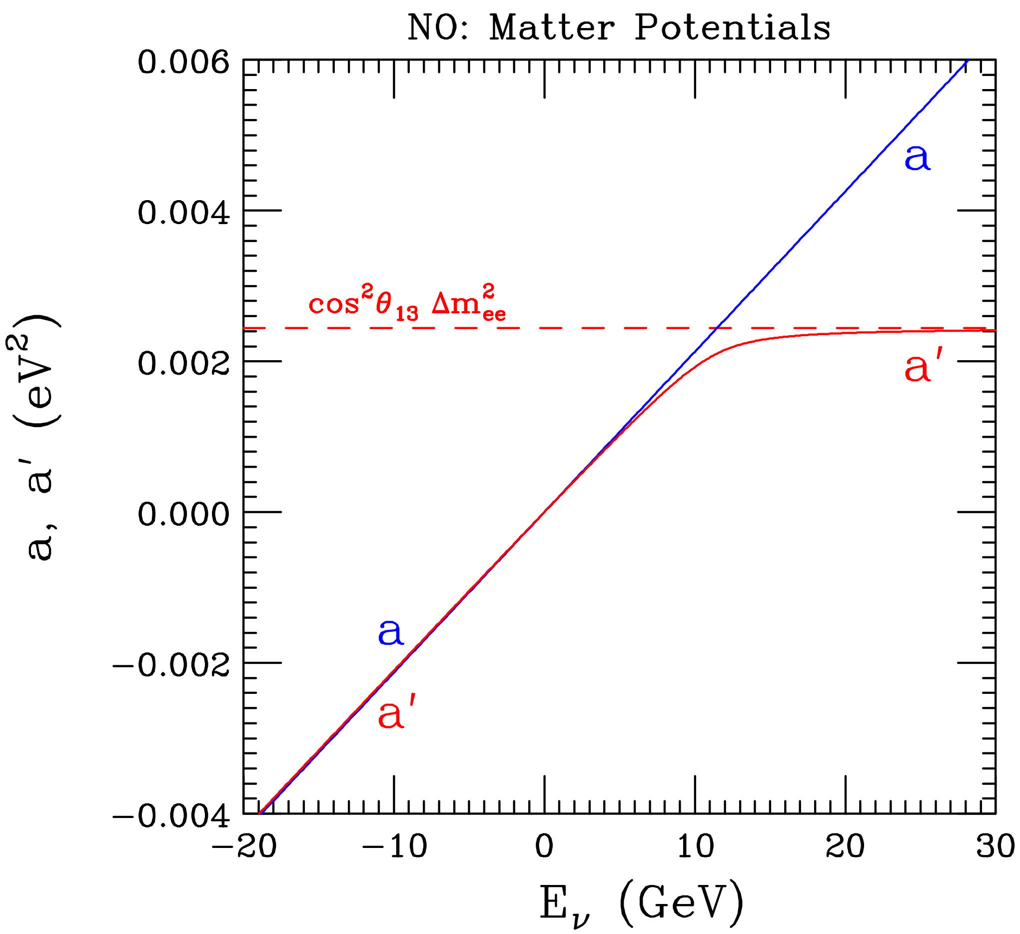

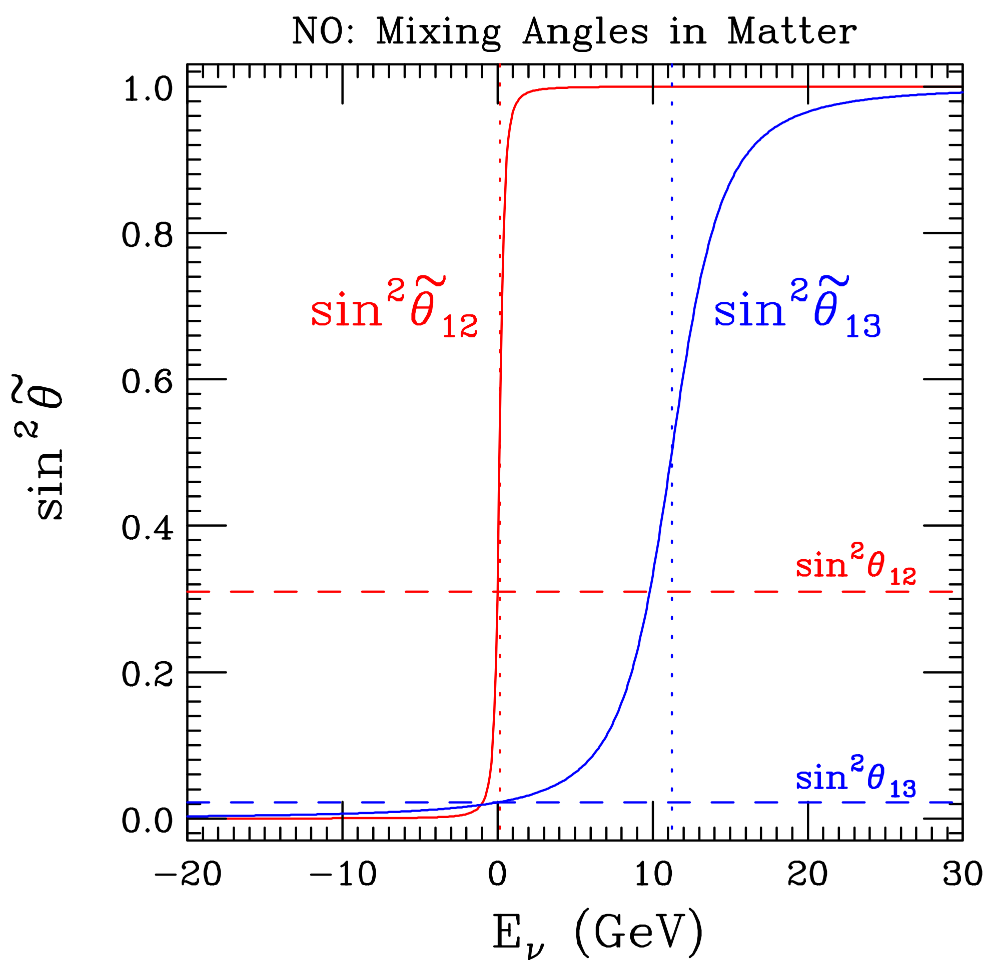

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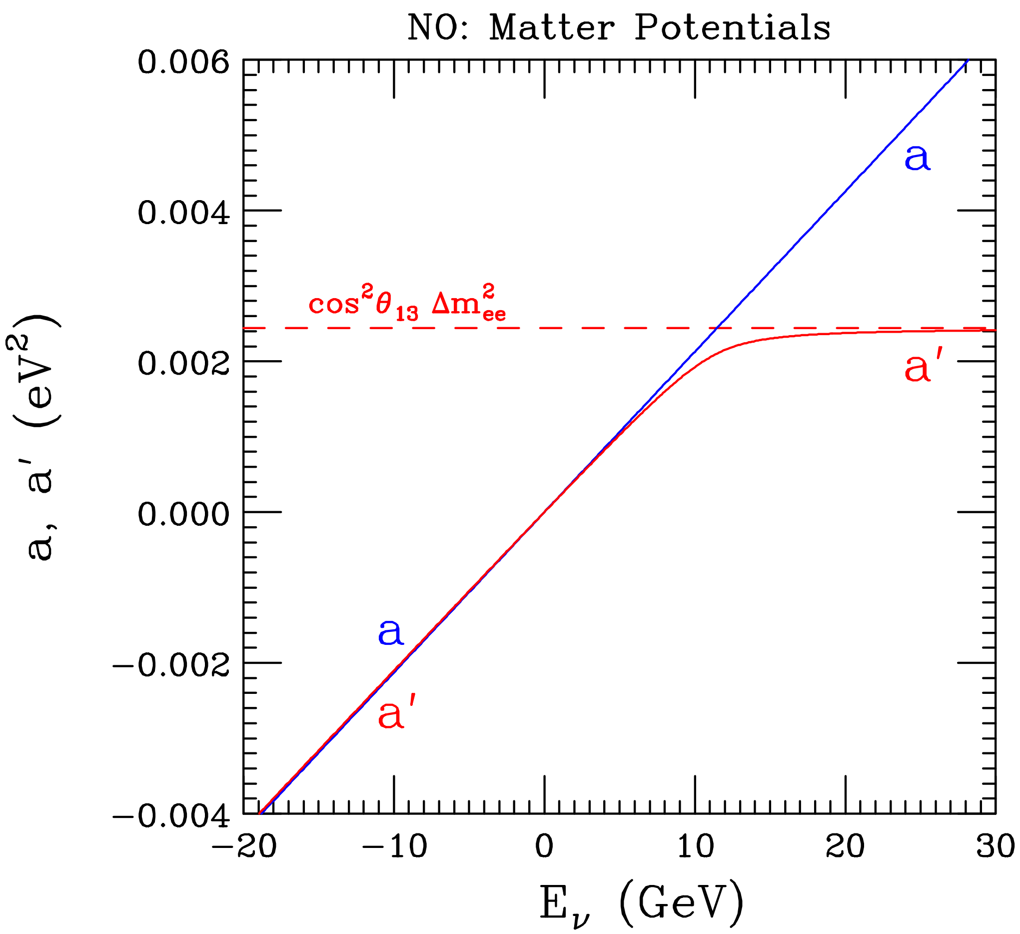
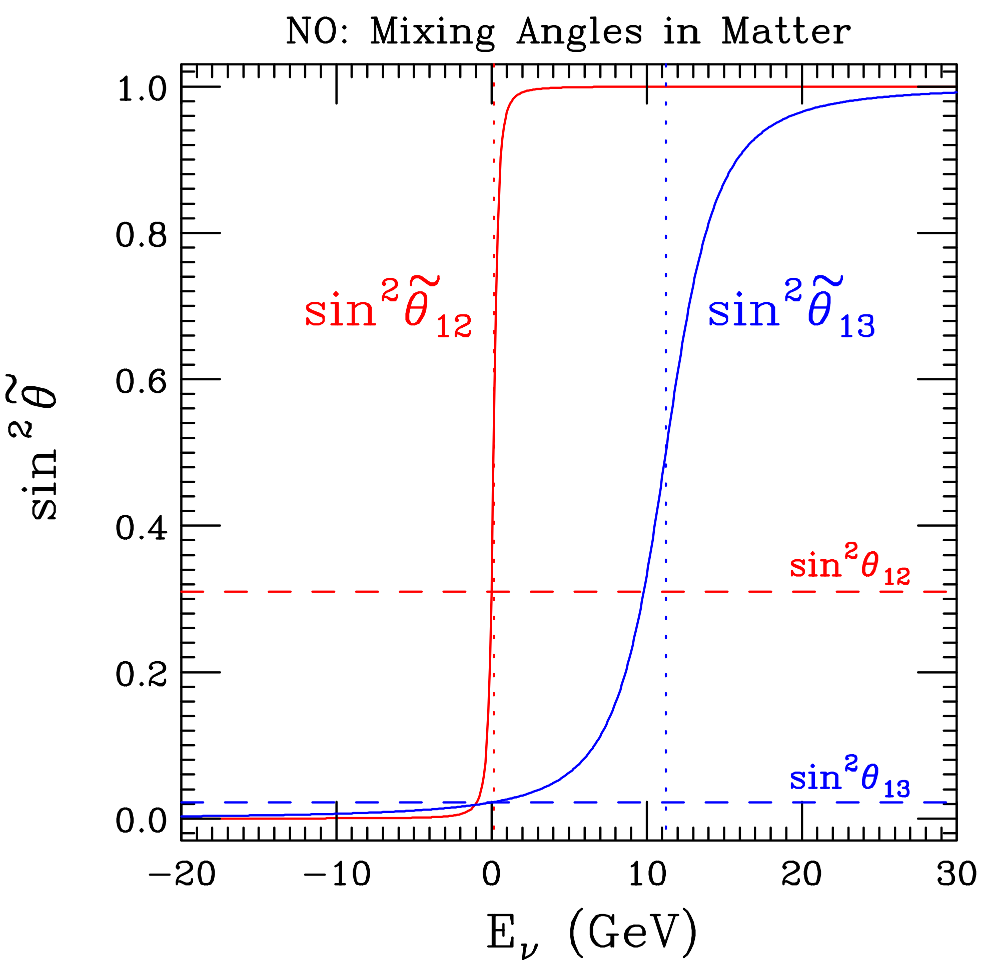
≈ 1





Masses Squared:

$$(2E)\,H_D = \text{diag}(\ \widetilde{m^2}_1,\ \widetilde{m^2}_2,\ \widetilde{m^2}_3\ )$$

$$\widetilde{m_1^2} = \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a'\,)$$

$$\widetilde{m_2^2} = \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a'\,)$$

$$\widetilde{m_3^2} = \Delta m^2_{31} + (\,a - a'\,)$$

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$$\text{when} \quad |a'| \gg \Delta m^2_{21}$$

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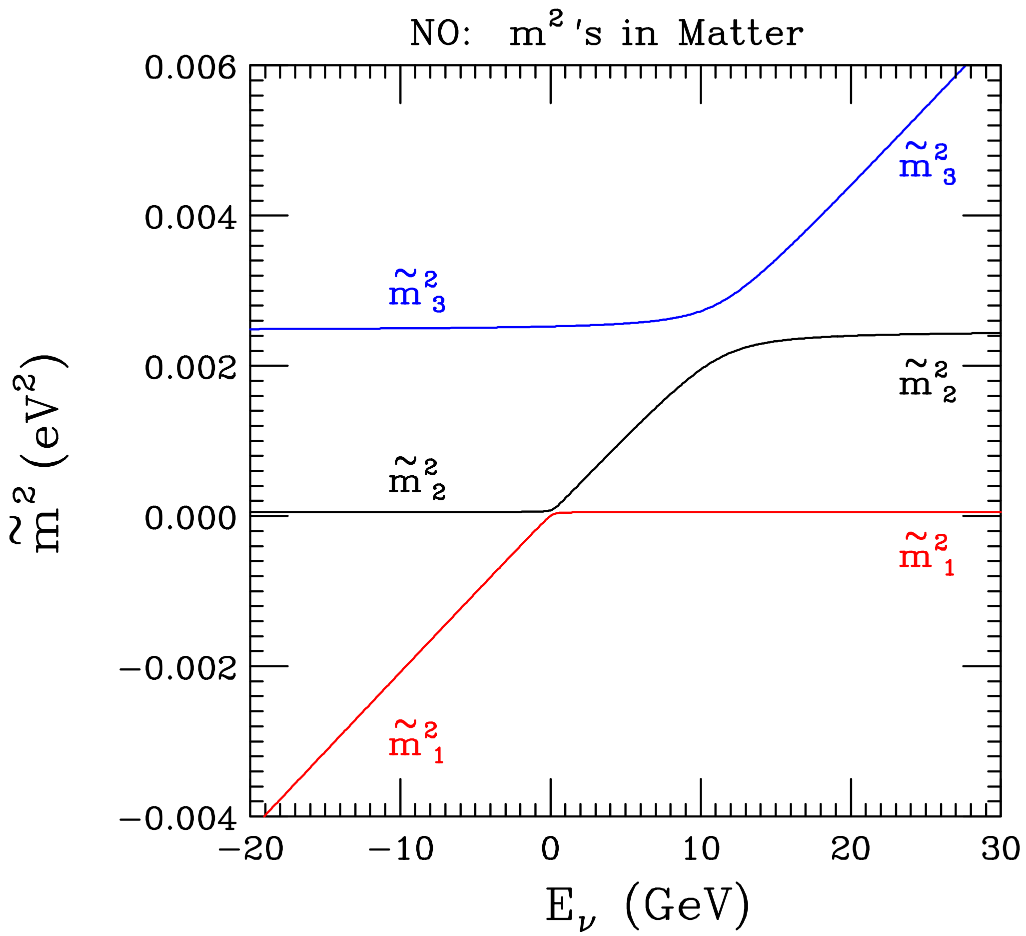

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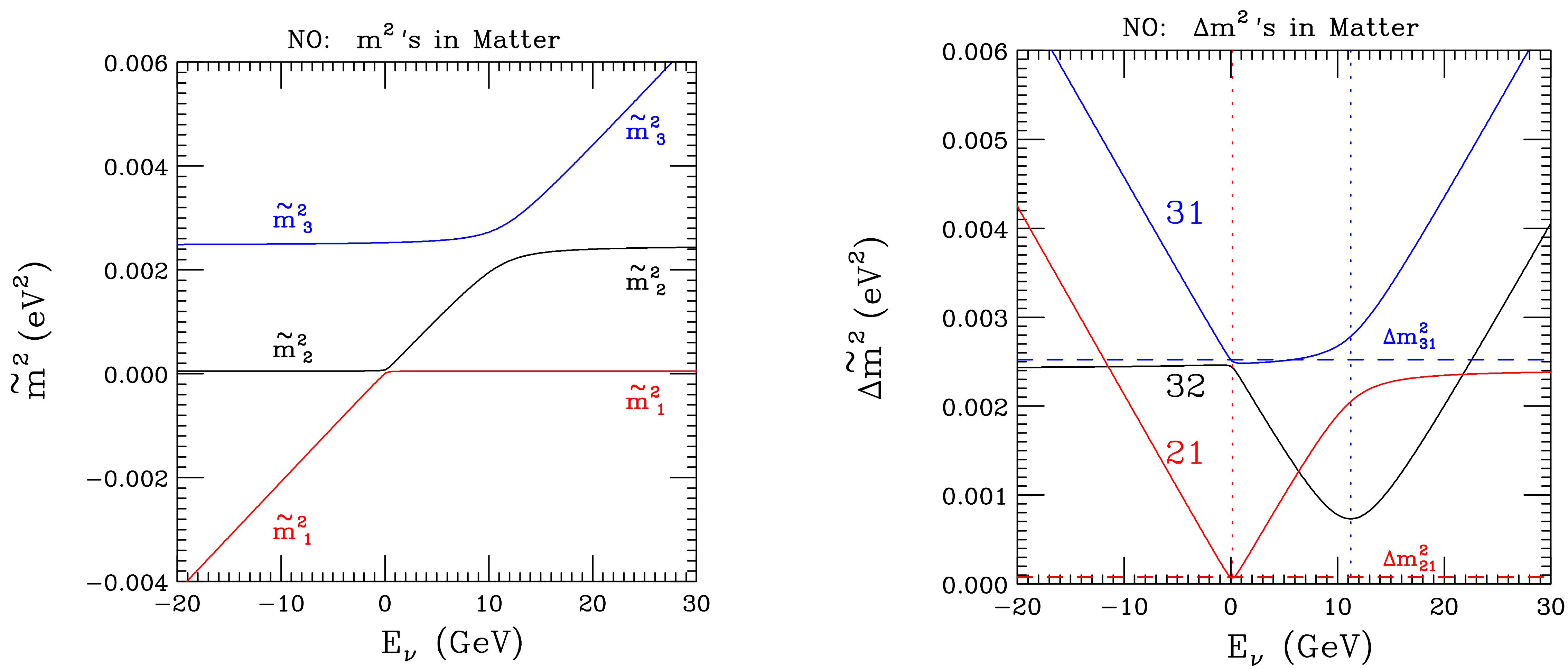
when $|a'| \gg \Delta m^2_{21}$



vacuum $\Rightarrow$ matter

$$\begin{aligned}
\Delta m^2_{jk} &\rightarrow \Delta \widetilde{m^2}_{jk} \\
\theta_{13} &\rightarrow \widetilde{\theta}_{13} \\
\theta_{12} &\rightarrow \widetilde{\theta}_{12} \\
\theta_{23} &\rightarrow \theta_{23} \\
\delta &\rightarrow \delta
\end{aligned}$$

$$P^{vac}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta m^2_{31}, \Delta m^2_{21}, \theta_{13}, \theta_{12}, \theta_{23}, \delta)$$

$$\Rightarrow P^{mat}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta \widetilde{m^2}_{31}, \Delta \widetilde{m^2}_{21}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

0th order !

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0th order !

What about H_{OD} ?

$$\Delta m^2_{jk} \rightarrow \Delta \widetilde{m^2}_{jk}$$
$$\theta_{13} \rightarrow \widetilde{\theta}_{13}$$
$$\theta_{12} \rightarrow \widetilde{\theta}_{12}$$
$$\theta_{23} \rightarrow \theta_{23}$$
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$$P^{vac}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta m^2_{31}, \Delta m^2_{21}, \theta_{13}, \theta_{12}, \theta_{23}, \delta)$$

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0th order !

What about H_{OD} ?

$\widetilde{s}_{12} \equiv \sin\widetilde{\theta}_{12}$, etc

$$(2E)\, H_{OD}/\Delta m^2_{ee} = \sin(\widetilde{\theta}_{13} - \theta_{13})\; s_{12}c_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right) \begin{bmatrix} & & -\widetilde{s}_{12} \\ & & \widetilde{c}_{12} \\ -\widetilde{s}_{12} & \widetilde{c}_{12} & \end{bmatrix}$$

vacuum $\Rightarrow$ matter

$$\Delta m^2_{jk} \rightarrow \Delta \widetilde{m^2}_{jk}$$
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What about H_{OD} ?

$\widetilde{s}_{12} \equiv \sin \widetilde{\theta}_{12}$, etc

$$(2E)\, H_{OD}/\Delta m^2_{ee} = \sin(\widetilde{\theta}_{13} - \theta_{13})\; s_{12}c_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right) \begin{bmatrix} & & -\widetilde{s}_{12} \\ & & \widetilde{c}_{12} \\ -\widetilde{s}_{12} & \widetilde{c}_{12} & \end{bmatrix}$$

$$\sin(\widetilde{\theta}_{13} - \theta_{13}) \approx s_{13}c_{13}\left(\frac{a}{\Delta m^2_{ee}}\right)$$

4×10^{-4}

for $E = 2$ GeV and $\rho = 3$ g.cm^{-3}

$$\Delta m^2_{jk} \rightarrow \Delta \widetilde{m^2}_{jk}$$
$$\theta_{13} \rightarrow \widetilde{\theta}_{13}$$
$$\theta_{12} \rightarrow \widetilde{\theta}_{12}$$
$$\theta_{23} \rightarrow \theta_{23}$$
$$\delta \rightarrow \delta$$

$$P^{vac}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta m^2_{31}, \Delta m^2_{21}, \theta_{13}, \theta_{12}, \theta_{23}, \delta)$$

$$\Rightarrow P^{mat}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta \widetilde{m^2}_{31}, \Delta \widetilde{m^2}_{21}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

0th order !

What about H_{OD} ?

$\widetilde{s}_{12} \equiv \sin\widetilde{\theta}_{12}$, etc

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Perturbation Theory !!!

1st order:

- $\Delta \widetilde{m^2}_{jk}$ are unchanged since $[\,H_{OD}\,]_{jj} \equiv 0$

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defn: $U^M_{PMNS} \equiv U_{23}(\theta_{23}, \delta)\, U_{13}(\widetilde{\theta}_{13}) U_{12}(\widetilde{\theta}_{12})$

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where:

$$W_1 = \sin(\widetilde{\theta}_{13} - \theta_{13})\; s_{12}c_{12}\; \Delta m^2_{21} \begin{pmatrix} 0 & 0 & -\widetilde{s}_{12}/\Delta \widetilde{m^2}_{31} \\ 0 & 0 & +\widetilde{c}_{12}/\Delta \widetilde{m^2}_{32} \\ +\widetilde{s}_{12}/\Delta \widetilde{m^2}_{31} & -\widetilde{c}_{12}/\Delta \widetilde{m^2}_{32} & 0 \end{pmatrix}$$

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$$0.002 \leq |W_1| \leq \; 0.01$$

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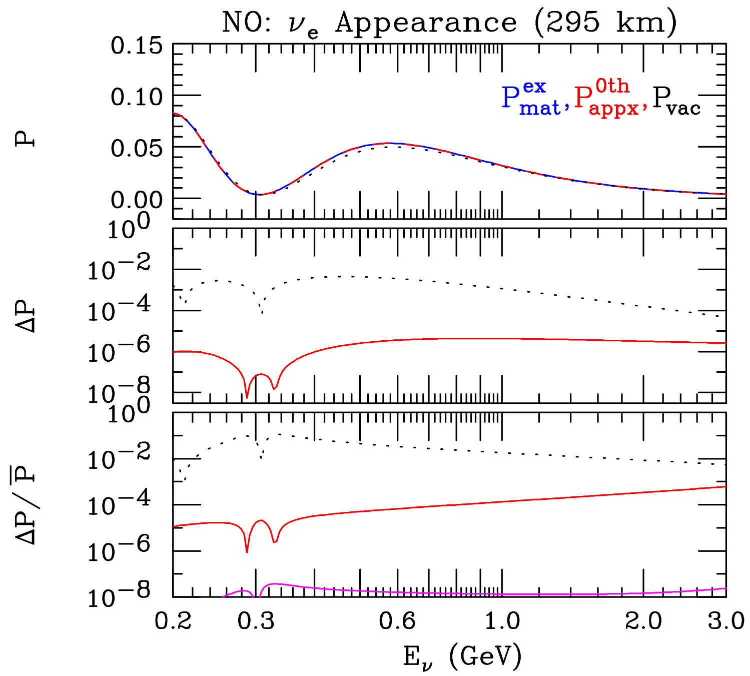
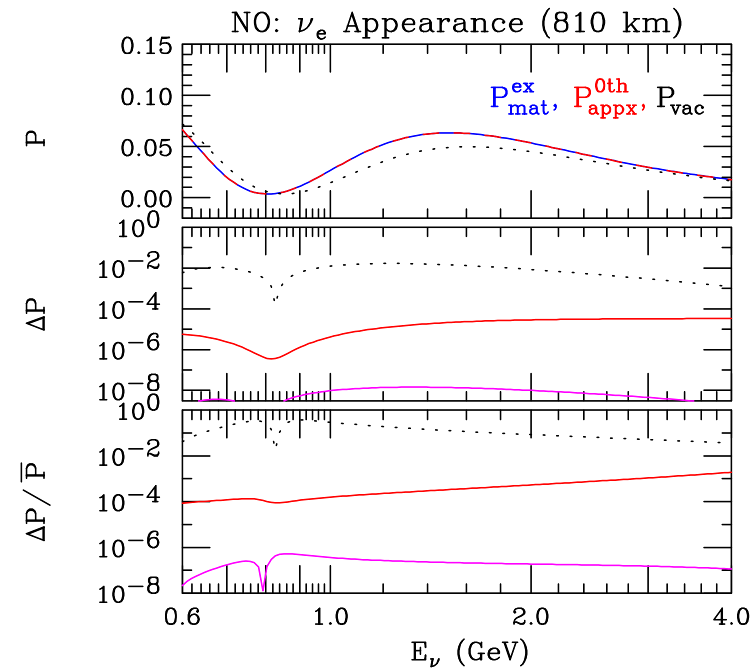
2nd order: see paper

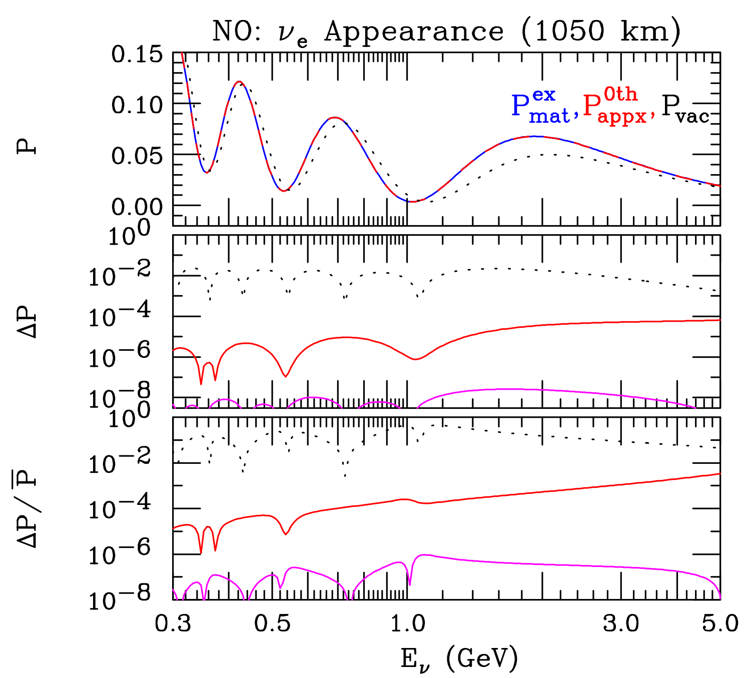
T2K/HK



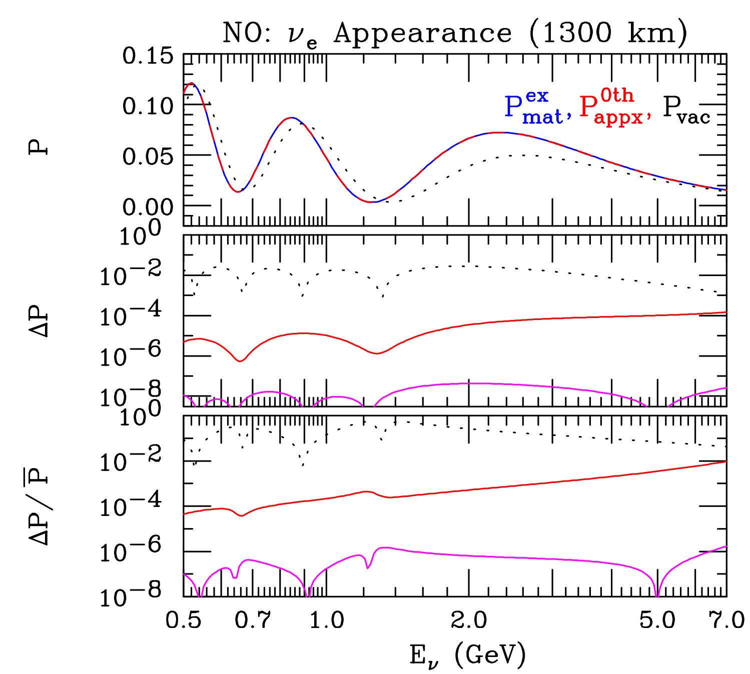
NOvA



T2HKK



DUNE



Top panel: P^{ex}_{mat}, P^{0th}_{appx} and P_{vac}

Middle and bottom panels:

black dotted lines	red solid lines	magenta solid lines
$\Delta P = \lvert P^{ex}_{mat} - P_{vac} \rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{0th}_{appx} \rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{1st}_{appx} \rvert$
$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P_{vac})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{0th}_{appx})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{1st}_{appx})$

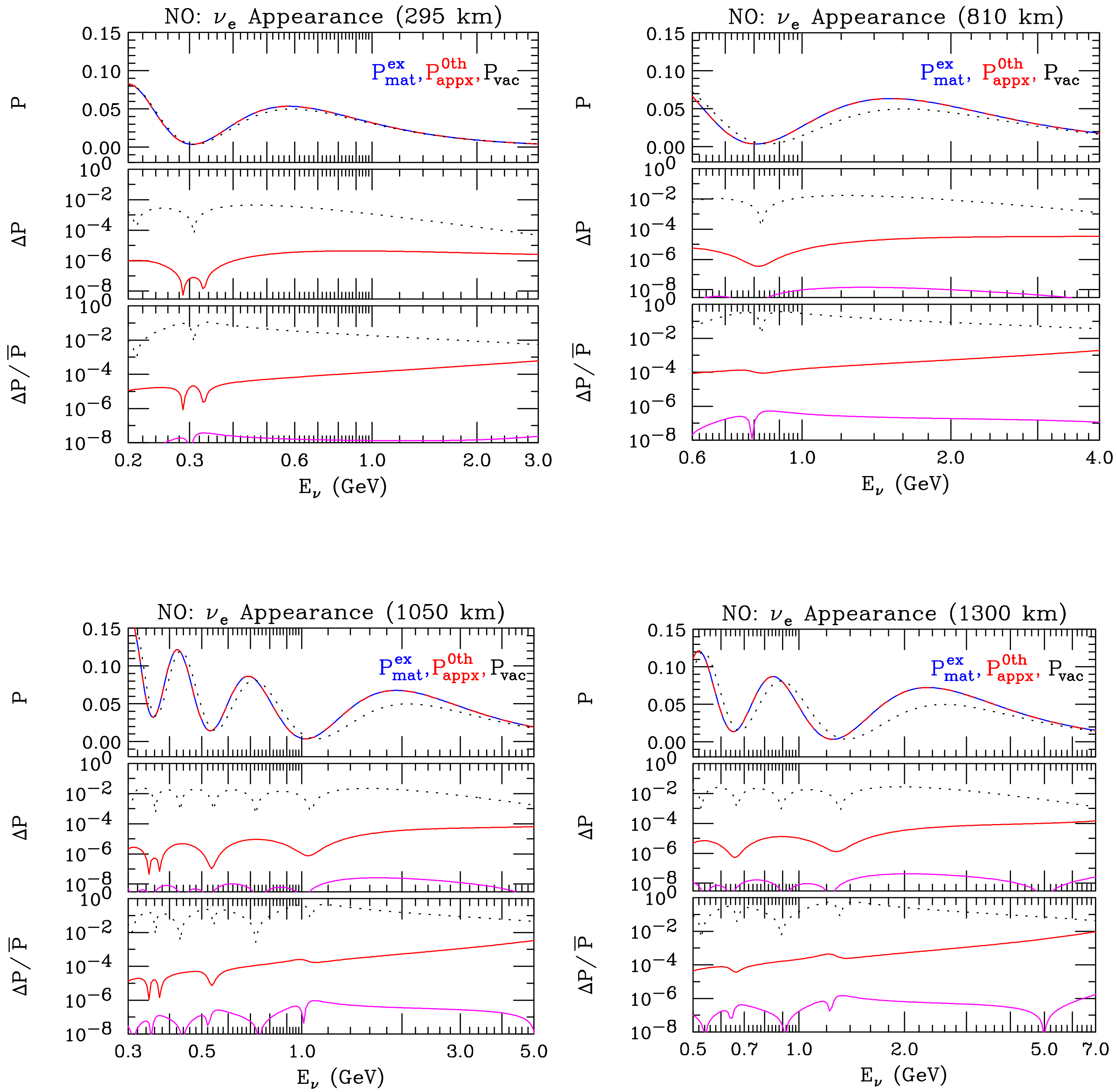
T2K/HK

NOvA

0th order

T2HKK

DUNE



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T2K/HK

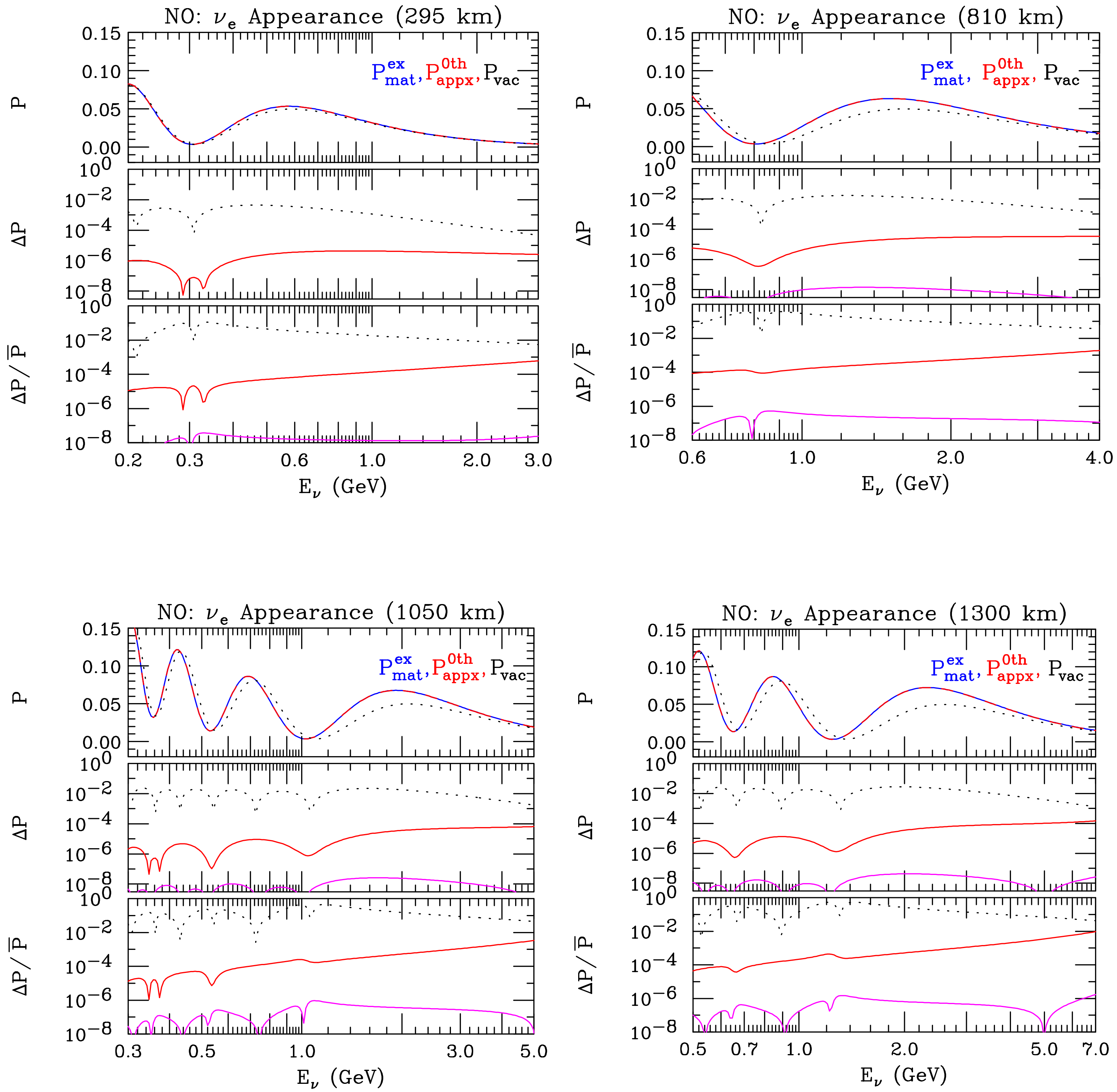
NOvA

0th order

1st order

T2HKK

DUNE



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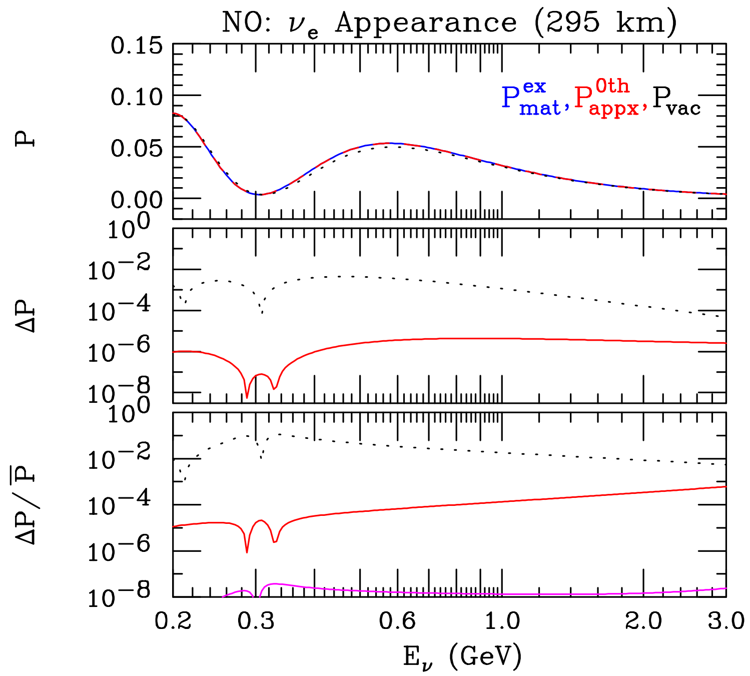

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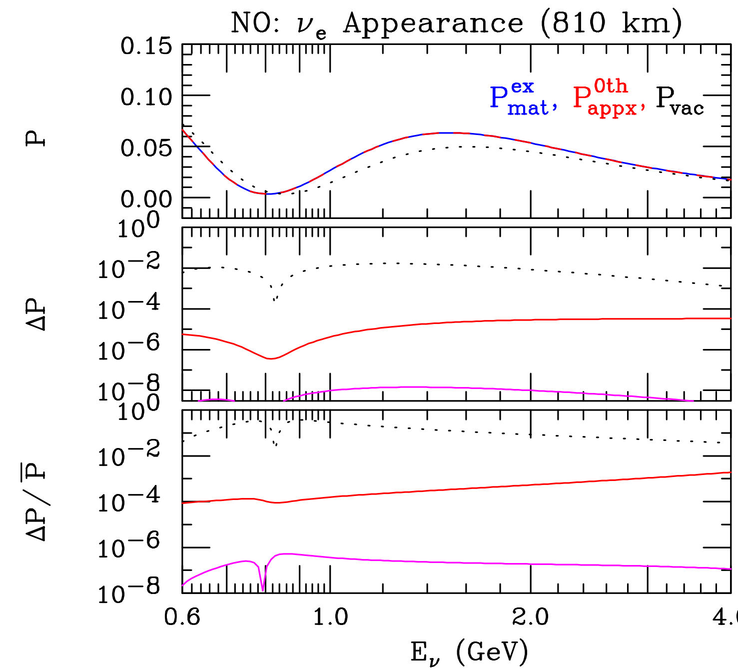
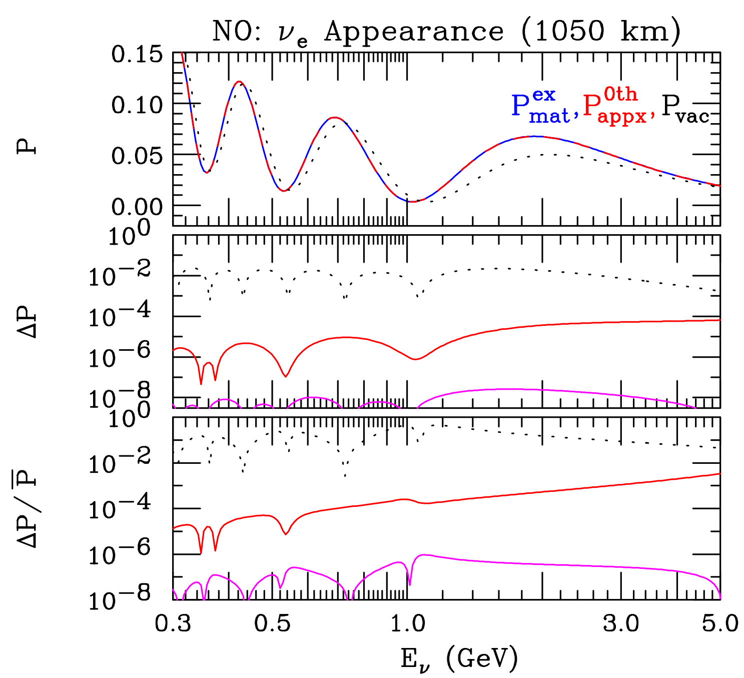


0th order

1st order

2nd order

T2HKK



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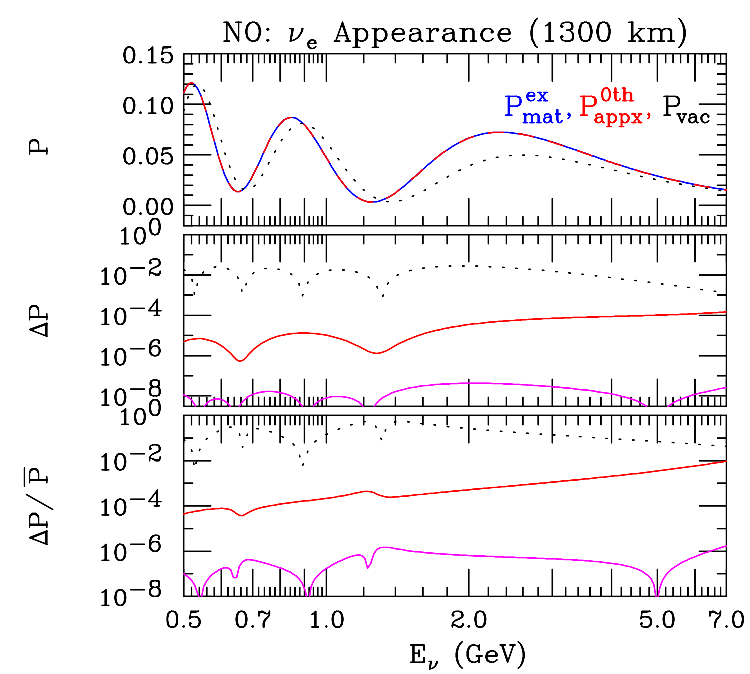

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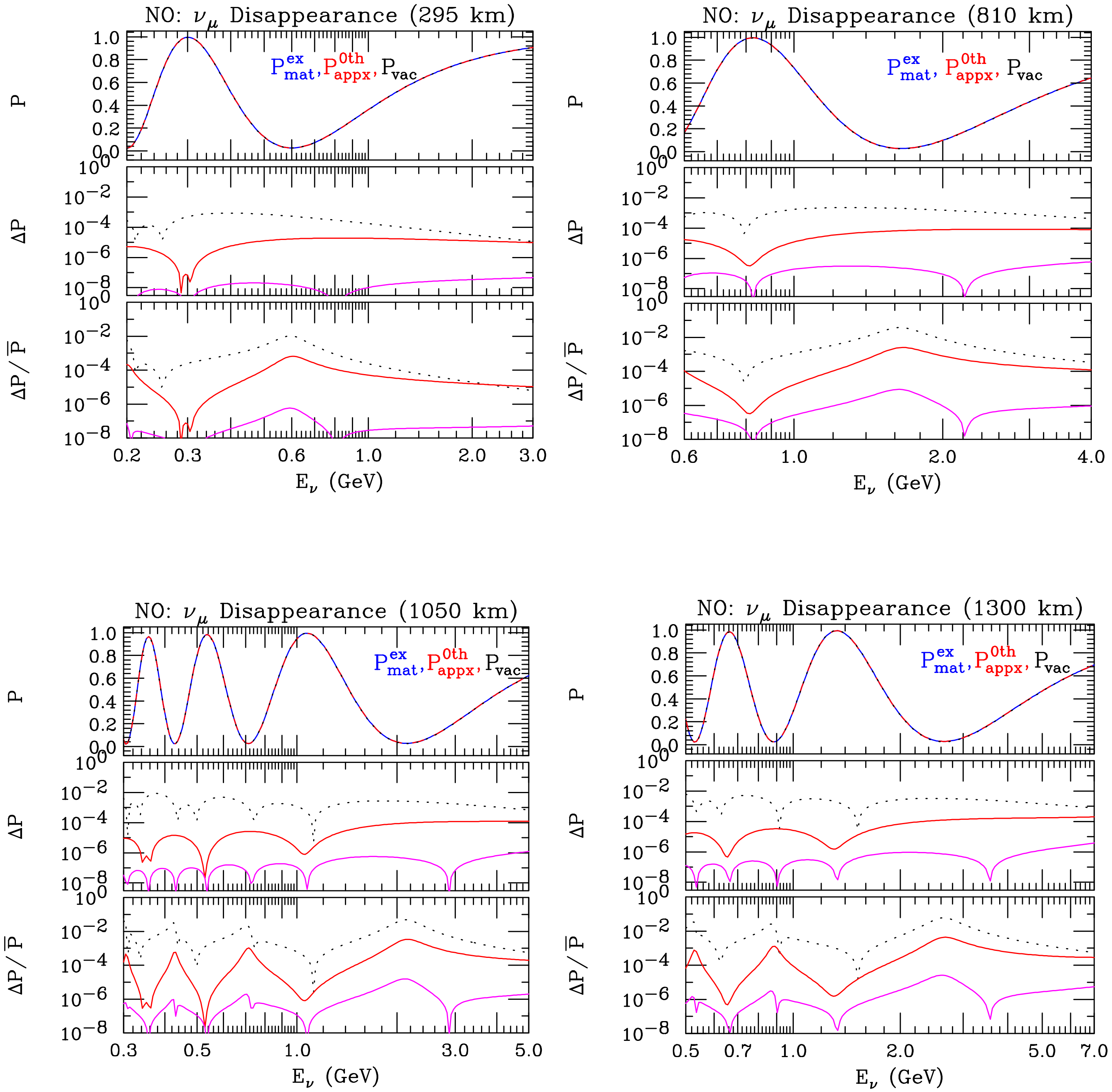

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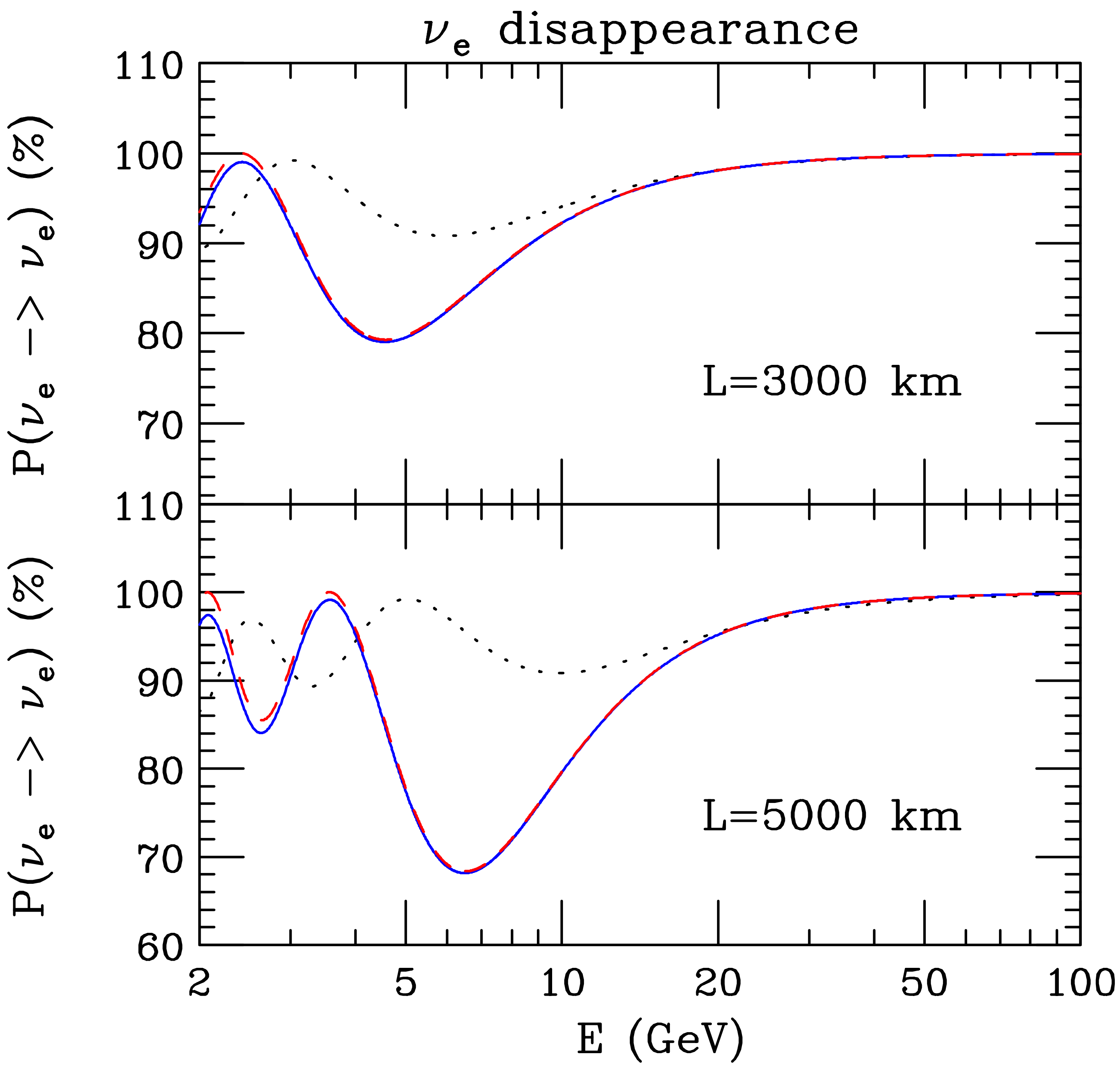
arXiv:1505.01826

ν_e Survival Probability:

$\widetilde{\theta}_{13}$

$$P(\nu_e \to \nu_e) \;=\; 1 - \sin^2 2\widetilde{\theta}_{13} \, \sin^2 \frac{\Delta \widetilde{m^2}_{ee} L}{4E} - \ldots$$



$$\sin 2\widetilde{\theta}_{13} = \frac{\sin^2 2\theta_{13}}{[(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}]}$$

$$\Delta \widetilde{m^2}_{ee} = \Delta m^2_{ee} \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

depth of first minimum

$$\sin 2\theta_{13} \quad \to \quad \sin 2\widetilde{\theta}_{13}$$

energy at first minimum

$$\frac{\Delta m^2_{ee} L}{2\pi} \quad \to \quad \frac{\Delta \widetilde{m^2}_{ee} L}{2\pi}$$

exact - approx - vaccum

Summary:

- DMP (1604.08167) gives us a SYSTEMATIC EXPANSION for the oscillation probabilities in matter

- 0th order is VERY SIMPLE and SUFFICIENT for most accelerator experiments

- the expansion parameter is SMALL: $\sin(\widetilde{\theta}_{13} - \theta_{13})\, s_{12}c_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right) \sim 0.0004$ for $E = 2$ GeV and $\rho = 3$ g.cm^{-3} and proportional to ρE

- by construction:
  - the 0th order REPRODUCES vacuum oscillation probabilities exactly
  - at 1st order ONLY the mixing matrix is modified, implying that at 0th order the neutrino masses in matter are very accurate

- at high orders, BOTH neutrino masses in matter and mixing matrix in matter are modified

- this DMP perturbative expansion gives an ENHANCED understanding of oscillation probabilities in matter backup

Translator: 1604.08167 to Addendum

13 sector:

$$\lambda_c - \lambda_a(a=0) = \Delta m^2_{ee}\cos 2\theta_{13}$$
$$\lambda_a - \lambda_a(a=0) = a$$
$$\lambda_c - \lambda_a = \Delta m^2_{ee}\cos 2\theta_{13} - a$$
$$\lambda_+ - \lambda_- = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$
$$eq:(2.3.5) \Rightarrow \quad \cos 2\phi = \cos 2\widetilde{\theta}_{13} \quad \Rightarrow \phi = \widetilde{\theta}_{13}$$

12 sector:

$$\lambda_0 - \lambda_-(a=0) = \Delta m^2_{21}\cos 2\theta_{12}$$
$$\lambda_- = \lambda_a c^2_\phi - 2 s_\phi c_\phi s_{12} c_{12}\Delta m^2_{ee} + \lambda_c s^2_\phi$$
$$\lambda_- - \lambda_-(a=0) = a c^2_\phi + \Delta m^2_{ee}\sin^2(\phi - \theta_{13}) = a'$$
$$\lambda_0 - \lambda_- = \Delta m^2_{21}\cos 2\theta_{12} - a'$$
$$\lambda_2 - \lambda_1 = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}\cos^2(\phi - \theta_{13})}$$
$$\equiv \Delta\widetilde{m^2}_{21}$$
$$\cos 2\psi = \cos 2\widetilde{\theta}_{12} \quad \Rightarrow \psi = \widetilde{\theta}_{12} \quad eq:(2.4.9)$$

$\widetilde{m^2}_j$:

$$\lambda_+ + \lambda_- = \lambda_a + \lambda_c = \Delta m^2_{31} + a + s^2_{12}\Delta m^2_{21}$$
$$\lambda_2 + \lambda_1 = \lambda_0 + \lambda_- = \Delta m^2_{21} + a'$$
$$\lambda_2 - \lambda_1 = \Delta\widetilde{m^2}_{21}$$
$$\lambda_1 \equiv \widetilde{m^2}_1 = \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a')$$
$$\lambda_2 \equiv \widetilde{m^2}_2 = \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a')$$
$$\lambda_3 = \lambda_+ \equiv \widetilde{m^2}_3 = \Delta m^2_{31} + (a - a')$$

DMP Summary:

The mixing angles in matter, which we denote by a $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ here, can also be calculated in the following way, using $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, as follows, see Addendum[5]:

$\widetilde{\theta}_{13}$

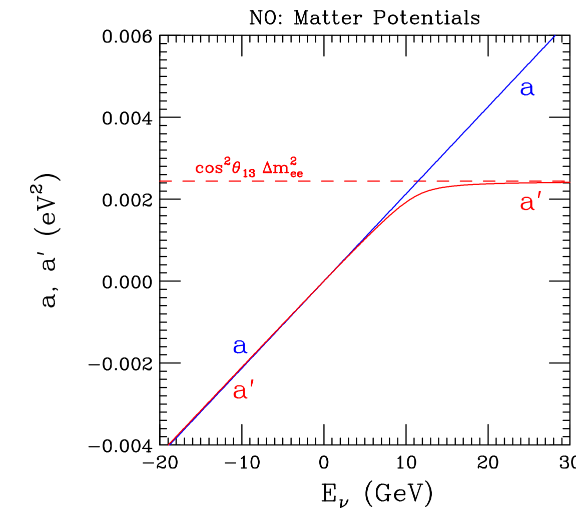
$$\cos 2\widetilde{\theta}_{13} = \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}, \tag{6}$$

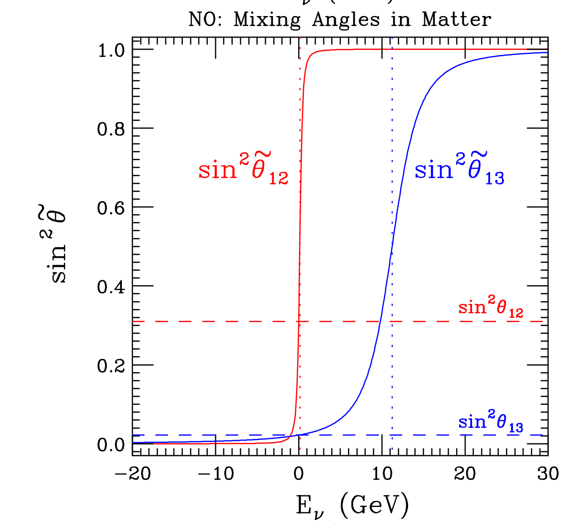
where $\boxed{a \equiv 2\sqrt{2} G_F N_e E_\nu}$ is the standard matter potential, and

$\widetilde{\theta}_{12}$

$$\cos 2\widetilde{\theta}_{12} = \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \ + \ \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}}, \tag{7}$$

where $\boxed{a' \equiv a\cos^2\widetilde{\theta}_{13} + \Delta m^2_{ee}\sin^2(\widetilde{\theta}_{13} - \theta_{13})}$ is the θ_{13}-modified matter potential for the 1-2 sector. In these two flavor rotations, both $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ are in range $[0, \pi/2]$.

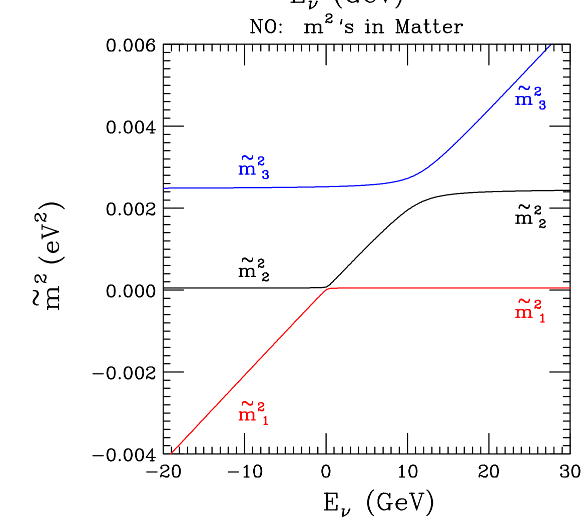




θ_{23} and δ are unchanged in matter for this approximation.

From the neutrino mass squared eigenvalues in matter, given by

$$\begin{aligned}
\widetilde{m^2_3} &= \Delta m^2_{31} + (a - a'), \\
\widetilde{m^2_2} &= \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a'), \\
\widetilde{m^2_1} &= \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a'),
\end{aligned} \tag{8}$$



it is simple to obtain the neutrino mass squared differences in matter, i.e. the Δm^2_{jk} in matter, which we denote by $\Delta\widetilde{m^2}_{jk}$, which are given by

$$\begin{aligned}
\Delta\widetilde{m^2}_{21} &= \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \ + \ \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}\ , \\
\Delta\widetilde{m^2}_{31} &= \Delta m^2_{31} + (a - \frac{3}{2}a') + \frac{1}{2}\left(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}\right), \\
\Delta\widetilde{m^2}_{32} &= \Delta\widetilde{m^2}_{31} - \Delta\widetilde{m^2}_{21}.
\end{aligned} \tag{9}$$