February 14, 2018

Comment on 1801.10488

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Abstract

We perform a comparison between 1604.08167 and the recent paper 1801.10488. We show that the logic, methods and results are identical, apart from trivial convention and notation differences.

PACS numbers:

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The two papers 1604.08167 (DMP), [1] , and 1801.10488 (IP), [2] give highly accurate methods for calculating the neutrino oscillation probabilities in uniform matter, perturbatively. Here we compare the logic, methods and results of these two papers, emphasizing the differences.

These two papers use different conventions for the vacuum MNS mixing matrix which are related as follows:

$$\boldsymbol{U_{ip} = U^{\delta} U_{dmp} = U_{pdg} U^{\delta} \quad \text{using} \quad U^{\delta} \equiv \text{diag}(0, 0, e^{i\delta})} \tag{1}$$

where U_{ip}, U_{dmp} and U_{pdg} are mixing matrices in [1] (eqn 2.1.4) , [2] (eqn 12) and [3] (eqn 14.6), respectively. Since all three matrices have equal Jarlskog invariants, these three mixing matrices give identical physics.

Both papers perform three rotations; first in (23) subspace, second in (13) subspace[1] and third in (12) subspace. The combined rotations, denoted by U^m_{dmp} and U^m_{ip}, are

$$\boldsymbol{U^m_{dmp} = U_{23}(\theta_{23}, \delta) U_{13}(\phi) U_{12}(\psi) \quad \text{and} \quad U^m_{ip} = U^{\delta}\, U_{23}(\theta_{23}, \delta) U_{13}(\theta^m_{13}) U_{12}(\theta^m_{12}).} \tag{2}$$

These are eqn 2.5.2 of [1] and eqn 24 of [2] , with $U_{23}(\theta_{23}, \delta) \equiv (U^{\delta})^* \mathcal{O}_{23}(\theta_{23}) U^{\delta}$, etc . Simple algebra, see Appendix[2], gives that the variables of [2] for these rotations, θ^m_{13} and θ^m_{12}, are exactly identical to ϕ and ψ of [1] , respectively, i.e.

$$\boldsymbol{\theta^m_{13} = \phi \quad \text{and} \quad \theta^m_{12} = \psi, \quad \text{and therefore} \quad U^m_{ip} = U^{\delta}\, U^m_{dmp}.} \tag{3}$$

U^m_{dmp} and U^m_{ip} are the mixing matrices in matter, and are *identical* apart from the U^{δ}, which is just a reflection of the different MNS phase convention, as eq. (1) and (3), must agree in vacuum.

After these three identical rotations, the Hamiltonian in the matter basis is *identical* in both papers, given by

$$U^{\dagger}_{12}(\psi) U^{\dagger}_{13}(\phi) \left\{ U_{13}(\theta_{13}) U_{12}(\theta_{12}) \begin{pmatrix} 0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31} \end{pmatrix} U^{\dagger}_{12}(\theta_{12}) U^{\dagger}_{13}(\theta_{13}) + \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix} \right\} U_{13}(\phi) U_{12}(\psi), \tag{4}$$

with $a = 2\sqrt{2} G_F N_e E$. Again, it is simple algebra to show that difference of the diagonal elements of this Hamiltonian, $\Delta\lambda_{jk}$ in [1] and $2E(\mathcal{H}_j - \mathcal{H}_k)$ in [2] are identical[2],

$$\boldsymbol{2E\,(\mathcal{H}_2 - \mathcal{H}_1) = \Delta\lambda_{21} \quad \text{and} \quad 2E\,(\mathcal{H}_3 - \mathcal{H}_1) = \Delta\lambda_{31}.} \tag{5}$$

Since at zeroth order in this approximation, all that one needs to calculate the oscillation probabilities in matter are $\phi = \theta^m_{13}$, $\psi = \theta^m_{12}$, $\Delta\lambda_{21} = 2E(\mathcal{H}_2 - \mathcal{H}_1)$ and $\Delta\lambda_{31} = 2E(\mathcal{H}_3 - \mathcal{H}_1)$, as well as θ_{23} and δ, the zeroth order oscillation probabilities are also identical.

The off-diagonal elements of the above Hamiltonian are identical. Therefore, all higher orders in the perturbative expansions are identical. Only zeroth is given completely in

[1] Paper [2] performs two rotations in the (13) space, a θ_0 and then immediately follows with $\theta_m - \theta_0$ rotation. We combine these rotations here, as in [2] , eqn. 24.

[2] Eqn 26, 27, 21 & 23 of [2] (v2) are identical to eqn. 2.3.5, 2.4.9, 2.4.7 & 2.4.7 of [1] , respectively.

[2] with a sketch of first order, whereas zeroth, first and second order are given in explicit detail in ref. [1] .

We conclude that the logic, methods and results of 1801.10488, [2] , are identical to those of 1604.08167, [1] , apart from a trivial phase convention difference in the MNS matrix and minor differences in notation. An even more intuitive notation is given in an Addendum to 1604.08167, see [4], using the following variables for the mixing angles and Δm^2's in matter:

$$\widetilde{\theta}_{13}, \quad \widetilde{\theta}_{12}, \quad \widetilde{\Delta m^2}_{31} \text{ and } \widetilde{\Delta m^2}_{21},$$

as well as the unmodified variables θ_{23} and δ.

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[1] P. B. Denton, H. Minakata and S. J. Parke, "Compact Perturbative Expressions For Neutrino Oscillations in Matter," JHEP **1606**, 051 (2016) doi:10.1007/JHEP06(2016)051 [arXiv:1604.08167 [hep-ph]].

[2] A. Ioannisian and S. Pokorski, "Three Neutrino Oscillations in Matter," arXiv:1801.10488 [hep-ph]. Equation numbers used are for the v3 version of this paper.

[3] C. Patrignani *et al.* [Particle Data Group], "Review of Particle Physics," Chin. Phys. C **40**, no. 10, 100001 (2016). doi:10.1088/1674-1137/40/10/100001

[4] P. B. Denton, H. Minakata and S. J. Parke, "Addendum to 'Compact Perturbative Expressions for Neutrino Oscillations in Matter'," arXiv:1801.06514 [hep-ph].

I. APPENDIX: SIMPLE ALGEBRA

The key results of 1801.10488v3 , [2] are contained in equations (21), (23), (26) and (27). Here, we show how to trivially derive these equations by using the results of 1604.08167 , [1], and changing the notation, i.e. we show that the key variables

$$\theta^m_{13}, \ \theta^m_{12}\ , \ 2E(\mathcal{H}_2 - \mathcal{H}_1) \text{ and } 2E(\mathcal{H}_3 - \mathcal{H}_1) \text{ of 1801.10488v3}$$

are algebraically, and hence numerically, IDENTICAL to

$$\phi, \ \psi, \ \Delta\lambda_{21} \text{ and } \Delta\lambda_{31} \text{ of 1604.08167v1,}$$

respectively, **for all values of the matter potential**.

A. $\theta_{13}^m = \phi$ **: Start from 1604.08167/eqn 2.3.5:**

$$\cos 2\phi = \frac{(\lambda_c - \lambda_a)}{\Delta\lambda_{+-}} \quad \text{where} \quad \Delta\lambda_{+-} \equiv \lambda_+ - \lambda_- = \sqrt{(\lambda_c - \lambda_a)^2 + (\sin 2\theta_{13}\Delta m_{ee}^2)^2}$$

$$\lambda_c - \lambda_a = \Delta m_{ee}^2 \cos 2\theta_{13} - a = \Delta m_{ee}^2(\cos 2\theta_{13} - \epsilon_a) \quad \text{with } \epsilon_a \equiv a/\Delta m_{ee}^2$$

$$\text{using} \quad \lambda_a = a + s_{13}^2\Delta m_{ee}^2 + s_{12}^2\Delta m_{21}^2, \quad \lambda_c = c_{13}^2\Delta m_{ee}^2 + s_{12}^2\Delta m_{21}^2$$

$$\Delta\lambda_{+-} = \Delta m_{ee}^2\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} \qquad \text{used later in 2E} (\mathcal{H}_3 - \mathcal{H}_1)$$

$$\cos 2\phi = \frac{(\cos 2\theta_{13} - \epsilon_a)}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}} = \cos 2\theta_{13}^m \quad \Rightarrow \text{1801.10488v3/eqn (26).}$$

B. $2E\,(\mathcal{H}_2 - \mathcal{H}_1) = \Delta\lambda_{21}$ **then** $\theta_{12}^m = \psi$ **: Start from 1604.08167/eqn 2.4.5 & 2.4.9:**

$$\Delta\lambda_{21} = \sqrt{(\lambda_0 - \lambda_-)^2 + (\cos(\phi - \theta_{13})\sin 2\theta_{12}\Delta m_{21}^2)^2}$$

$$\lambda_0 - \lambda_- = \Delta m_{21}^2 \cos 2\theta_{12} - (ac_\phi^2 + \Delta m_{ee}^2 \sin^2(\phi - \theta_{13})) = \Delta m_{21}^2(\cos 2\theta_{12} - \epsilon_\odot)$$

$$\text{with } \epsilon_\odot \equiv (ac_\phi^2 + \Delta m_{ee}^2 \sin^2(\phi - \theta_{13}))/\Delta m_{21}^2$$

$$\text{using} \quad \lambda_0 = \lambda_b = c_{12}^2\Delta m_{21}^2, \quad \lambda_- = \lambda_a c_\phi^2 + \lambda_c s_\phi^2 - 2\Delta m_{ee}^2\, s_{13}c_{13}\, s_\phi c_\phi$$

$$= ac_\phi^2 + \Delta m_{ee}^2 \sin^2(\phi - \theta_{13}) + s_{12}^2\Delta m_{21}^2$$

$$\Delta\lambda_{21} = \Delta m_{21}^2\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2(\phi - \theta_{13})\sin^2 2\theta_{12}} = 2E\,(\mathcal{H}_2 - \mathcal{H}_1)$$

$\Rightarrow$ 1801.10488v3/eqn (21)

$$\cos 2\psi = \frac{(\lambda_0 - \lambda_-)}{\Delta\lambda_{21}} = \frac{(\cos 2\theta_{12} - \epsilon_\odot)}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2(\phi - \theta_{13})\sin^2 2\theta_{12}}} = \cos 2\theta_{12}^m$$

$\Rightarrow$ 1801.10488v3/eqn (27)

C. $2E\,(\mathcal{H}_3 - \mathcal{H}_1) = \Delta\lambda_{31}$ **: Starting from 1604.08167/eqn 2.4.5:**

$$\Delta\lambda_{31} = \lambda_3 - \lambda_1 = \lambda_+ - \frac{1}{2}(\lambda_0 + \lambda_- - \Delta\lambda_{21}) \qquad \text{then using} \quad \lambda_\pm = \frac{1}{2}(\lambda_a + \lambda_c \pm \Delta\lambda_{+-})$$

$$= \frac{3}{4}\Delta\lambda_{+-} + \frac{1}{4}(\lambda_a + \lambda_c - 2\lambda_0) + \frac{1}{2}\Delta\lambda_{21} \qquad \text{with } \Delta\lambda_{+-},\ \Delta\lambda_{21} \text{ given above,}$$

$$= \frac{3}{4}\Delta\lambda_{+-} + \frac{1}{4}(\Delta m_{ee}^2 + a) + \frac{1}{2}(\Delta\lambda_{21} - \Delta m_{21}^2 \cos 2\theta_{12}) = 2E\,(\mathcal{H}_3 - \mathcal{H}_1)$$

$\Rightarrow$ 1801.10488v3/eqn (23)

D. Appendix A of 1801.10488

Eqn (30) of Appendix A in 1801.10488v3 is equivalent to eqn 3.2.5 of 1604.08167 .